EXISTENCE OF TWO WEAK SOLUTIONS FOR ELLIPTIC EQUATIONS INVOLVING A GENERAL OPERATOR IN DIVERGENCE FORM

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In this paper, we establish the existence of at least two distinct weak solutions for a class of elliptic equations involving a general operator in divergence form, subject to Dirichlet boundary conditions in a smooth bounded domain in $\mathbb{R}^N$. A critical point result for differentiable functionals is exploited, in order to prove that the problem admits at least two distinct non-trivial weak solutions.

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1. INTRODUCTION

The purpose of this paper is to establish the existence of at least two distinct weak solutions for the following elliptic Dirichlet problem

$$\begin{cases} -\mathrm{div}(a(x,\nabla u)) = \lambda k(x) f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 2)$ with smooth boundary $\partial\Omega$, $p > N$, $a : \bar{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ is a suitable continuous map of gradient type, and λ is a positive real parameter. Further, $f : \mathbb{R} \to \mathbb{R}$ and $k : \bar{\Omega} \to \mathbb{R}^+$ are two continuous functions.

The operator $-\mathrm{div}(a(x,\nabla u))$ arises, for example, from the expression of the p-Laplacian in curvilinear coordinates. We refer to the overview papers [9, 11, 30, 39, 42] for the investigation on Dirichlet problems involving a general operator in divergence form. For example, De Nápoli and Mariani in [9] studied the existence of solutions to equations of p-Laplacian type. They proved the existence of at least one solution, and under further assumptions, the existence of infinitely many solutions. In order to apply mountain pass results, they introduced a notion of uniformly convex functional that generalizes the notion

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of uniformly convex norm. Duc and Vu in [11] studied the non-uniform case. The authors in [42] established the existence and multiplicity of weak solutions of a problem involving a uniformly convex elliptic operator in divergence form. They discussed the existence of one nontrivial solution by the mountain pass lemma, when the nonlinearity has a $(p-1)$-superlinear growth at infinity, and two nontrivial solutions by minimization and mountain pass when the nonlinear term has a $(p-1)$-sublinear growth at infinity. In [6], Colasuonno, Pucci and Varga studied different and very general classes of elliptic operators in divergence form looking at the existence of multiple weak solutions. Their contributions represent a nice improvement, in several directions, of the results obtained by Kristály *et al.* in [25] in which a uniform Dirichlet problem with parameter is investigated.

In [29] Molica Bisci and Rădulescu, applying mountain pass results studied the existence of solutions to nonlocal equations involving the p-Laplacian. More precisely, they proved the existence of at least one nontrivial weak solution, and under additional assumptions, the existence of infinitely many weak solutions. In [28], they also by using an abstract linking theorem for smooth functionals established a multiplicity result on the existence of weak solutions for a nonlocal Neumann problem driven by a nonhomogeneous elliptic differential operator.

Recently, motivated by this large interest, Molica Bisci and Repovš in [30, Theorem 3.5] studied the existence of at least three weak solutions for the following elliptic Dirichlet problem

$$\begin{cases} -\mathrm{div}(a(x,\nabla u)) = \lambda k(x) f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 2)$ with smooth boundary $\partial\Omega$, $p > N$, $a : \bar{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ is a suitable continuous map of gradient type, and λ is a positive real parameter. Further, $f : \mathbb{R} \to \mathbb{R}$ and $k : \bar{\Omega} \to \mathbb{R}^+$ are two continuous functions, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$(\mathrm{f}_1) \qquad |f(t)| \leq a_1 + a_2 |t|^{q-1}, \qquad \forall t \in R,$$

for some non-negative constants a_1, a_2, where $q \in]1, \frac{pN}{N-p}[$, if $p < N$ and $1 < q < +\infty$ if $p \geq N$. In addition,as customary, the symbol

$$p^* := \begin{cases} \dfrac{pN}{N-p} & \text{if } 1 < p < N, \\ \infty & \text{if } p \geq N, \end{cases}$$

denotes the critical Sobolev exponent of p. In this work, our goal is to obtain the existence of two distinct weak solutions for problem (1.1).

A special case of our main result reads as follows.

THEOREM 1.1. *Let $f:\mathbb{R}\to\mathbb{R}$ be a non-negative continuous function with $f(0)\neq 0$, satisfying a $(q-1)$-sublinear growth at infinity for some $q\in]p,p^*[$, i.e.,*

$$\lim_{|t|\to\infty}\frac{f(t)}{|t|^{q-1}}=0.$$

Moreover, assume that there exist $\theta>p$ and $M>0$ such that

$$0<\theta F(t)\leq tf(t),$$

for each $t\in\mathbb{R}$ and $|t|\geq M$. Then, there exists $\lambda^\star>0$, such that, for any $\lambda\in]0,\lambda^\star[$ the problem (1.1), admits two positive weak solutions.

For completeness, we recall that a careful and interesting analysis of elliptic problems was developed in the monographs [21, 34] as well as the papers [14, 18, 24, 26, 31] and references therein.

2. AUXILIARY RESULTS

Assume that Ω is a bounded domain in $\mathbb{R}^N (N\geq 2)$ with smooth boundary $\partial\Omega$. Further, denote by X the space $W_0^{1,p}(\Omega)$ endowed with the norm

$$\|u\|:=\left(\int_\Omega|\nabla u(x)|^p\,\mathrm{d}x\right)^{1/p},$$

and X^* the topological dual of X.

By the compact embedding $X\hookrightarrow L^q(\Omega)$ for each $q\in[1,p^*[$, there exists a positive constant c_q such that

$$\|u\|_{L^q(\Omega)}\leq c_q\|u\|,\qquad(\forall u\in X)\tag{2.1}$$

where c_q is the best constant of the embedding.

The functional $I_\lambda:X\to\mathbb{R}$ associated with (1.1) is introduced as following:

$$I_\lambda(u):=\Phi(u)-\lambda\Psi(u),$$

for every $u\in X$, where

$$\Phi(u):=\int_\Omega A(x,\nabla u(x))\mathrm{d}x,$$

and

$$\Psi(u):=\int_\Omega k(x)F(u(x))\mathrm{d}x,$$

for every $u\in X$, where $k:\bar{\Omega}\to\mathbb{R}^+$ is a positive and continuous function, and

$$F(s)=\int_0^s f(t)\mathrm{d}t,$$

for every $s \in \mathbb{R}$. By standard arguments, Φ is Gâteaux differentiable and sequentially weakly lower semicontinuous and its Gâteaux derivative is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)(v) := \int_\Omega a(x, \nabla u(x))\nabla v(x)\mathrm{d}x,$$

for every $v \in X$. Moreover, Ψ is a Gâteaux differentiable sequentially weakly upper continuous functional whose Gâteaux derivative is given by

$$\Psi'(u)(v) := \int_\Omega k(x)f(u(x))v(x)\mathrm{d}x,$$

for every $v \in X$. Fixing the real parameter λ, a function $u : \Omega \to \mathbb{R}$ is said to be a weak solution of (1.1) if $u \in X$ and

$$\int_\Omega a(x, \nabla u(x))\nabla v(x)\mathrm{d}x - \lambda \int_\Omega k(x)f(u(x))v(x)\mathrm{d}x = 0,$$

for every $v \in X$. Therefore, the critical points of I_λ are exactly the weak solutions of (1.1).

Definition 2.1. A Gâteaux differentiable function I satisfies the Palais-Smale condition (in short (PS)-condition) if any sequence $\{u_n\}$ such that

(a) $\{I(u_n)\}$ is bounded,

(b) $\lim_{n\to+\infty} \|I'(u_n)\|_{X^*} = 0$,

has a convergent subsequence.

Definition 2.2. Let X be a reflexive real Banach space. The operator $T : X \to X^*$ is said to satisfy the (S_+) condition if the assumptions $u_n \rightharpoonup u_0$ in X and $\limsup_{n\to+\infty}\langle T(u_n) - T(u), u_n - u_0\rangle \leq 0$ imply $u_n \to u_0$ in X.

Our main tool is the following critical point theorem.

THEOREM 2.3 (see [1, Theorem 3.2]). *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ such that $\sup_{\{\Phi(u)<r\}} < +\infty$ and assume that, for each $\lambda \in \left]0, \frac{r}{\sup_{\{\Phi(u)<r\}}\Psi(u)}\right[$, the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies* (PS)*-condition and it is unbounded from below. Then, for each $\lambda \in \left]0, \frac{r}{\sup_{\{\Phi(u)<r\}}\Psi(u)}\right[$, the functional I_λ admits two distinct critical points.*

3. MAIN RESULTS

In this section we formulate our main results. Let $p \geq 1$ and let $\Omega \subseteq \mathbb{R}^N$ be a bounded Euclidean domain, where $N \geq 2$. Further, let $A : \bar{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ and let $A = A(x, \xi)$ be a continuous function in $\bar{\Omega} \times \mathbb{R}^N$, with continuous gradient $a(x, \xi) := \nabla_\xi A(x, \xi) : \bar{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$, and assume that the following conditions hold:
(α_1) $A(x, 0) = 0$ for all $x \in \Omega$;
(α_2) A satisfies $\Lambda_1|\xi|^p \leq A(x, \xi) \leq \Lambda_2|\xi|^p$ for all $x \in \bar{\Omega}$, $\xi \in \mathbb{R}^N$, where Λ_1 and Λ_2 are positive constants.
(α_3) a satisfies the growth condition $|a(x, \xi)| \leq c(1 + |\xi|^{p-1})$ for all $x \in \Omega$, $\xi \in \mathbb{R}^N$, $c > 0$;
(α_4) A is p-uniformly convex, that is

$$A(x, \frac{\xi + \eta}{2}) \leq \frac{1}{2}A(x, \xi) + \frac{1}{2}A(x, \eta) - k|\xi - \eta|^p,$$

for every $x \in \bar{\Omega}$, $\xi, \eta \in \mathbb{R}^N$ and some $k > 0$.
(α_5) a satisfies the strictly monotonicity condition, that is

$$(a(x, \xi_1) - a(x, \xi_2))\,(\xi_1 - \xi_2) > 0$$

for all $\xi_1, \xi_2 \in \mathbb{R}^N$ with $\xi_1 \neq \xi_2$.

PROPOSITION 3.1. *The operator $T : X \to X^*$ defined by*

$$T(u)(v) := \int_\Omega a(x, \nabla u(x))\nabla v(x)\mathrm{d}x,$$

for every $u, v \in X$, is strictly monotone.

Proof. Taking into account (α_5), the operator T is strictly monotone. □

We recall that c_q is the constant of the embedding $X \hookrightarrow L^q(\Omega)$ for each $q \in [1, p^*[$, and c_1 stands for c_q with $q = 1$; see (2.1). Now, we establish the main abstract result of this paper.

THEOREM 3.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that condition* (f_1) *holds. Moreover, assume that*

(f_2) *there exist $\theta > p$ and $M > 0$ such that*

$$0 < \theta F(t) \leq t f(t),$$

for each $t \in \mathbb{R}$ *and* $|t| \geq M$*. Then, for each* $\lambda \in]0, \lambda^\star[$*, problem* (1.1) *admits at least two distinct weak solutions, where*

$$\lambda^\star := \frac{1}{\|k\|_\infty \left(\frac{a_1 c_1}{\Lambda_1^{1/p}} + \frac{a_2 c_q^q}{q \Lambda_1^{q/p}} \right)}.$$

Proof. Our aim is to apply Theorem 2.3 to problem (1.1) in the case $r = 1$ to the space $X := W_0^{1,p}(\Omega)$ with the norm

$$\|u\| := \left(\int_\Omega |\nabla u(x)|^p \mathrm{d}x \right)^{1/p},$$

and to the functionals $\Phi, \Psi : X \to \mathbb{R}$ be defined by

$$\Phi(u) := \int_\Omega A(x, \nabla u(x))\, \mathrm{d}x \quad \text{and} \quad \Psi(u) := \int_\Omega k(x) F(u(x))\, \mathrm{d}x, \text{ for all } u \in X.$$

The functional Φ is in $C^1(X, \mathbb{R})$ and $\Phi' : X \to X^*$ is strictly monotone (see Proposition 3.1). Now we prove that Φ' is a mapping of type (S_+). Let $u_n \rightharpoonup u$ in X and $\limsup_{n\to+\infty} \langle \Phi'(u_n) - \Phi'(u), u_n - u \rangle \leq 0$. Then $u_n \to u$ in X (see Theorem 3.1 of [12]). So, Φ' is a mapping of type (S_+). By Theorem 3.1 from [12], we get that $\Phi' : X \to X^*$ is a homeomorphism. Moreover, thanks to condition (f_1) and to the compact embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for each $q \in [1, p^*[$, the functional Ψ is in $C^1(X, \mathbb{R})$ and has compact derivative and

$$\Psi'(u)(v) = \int_\Omega k(x) f(u(x)) v(x) \mathrm{d}x,$$

for every $v \in X$. Now we prove that $I_\lambda = \Phi - \lambda\Psi$ satisfies (PS)-condition for every $\lambda > 0$. Namely, we will prove that any sequence $\{u_n\} \subset X$ satisfying

$$(3.1) \qquad d := \sup_n I_\lambda(u_n) < +\infty, \quad \|I'_\lambda(u_n)\|_{X^*} \to 0, \quad \text{as } n \to +\infty$$

contains a convergent subsequence. For n large enough, we have by (3.1)

$$d \geq I_\lambda(u_n) = \int_\Omega A(x, \nabla u_n(x))\, \mathrm{d}x - \lambda \int_\Omega k(x) F(u_n(x))\, \mathrm{d}x,$$

and

$$\begin{aligned} I_\lambda(u_n) &\geq \int_\Omega A(x, \nabla u_n(x))\, \mathrm{d}x - \frac{\lambda}{\theta} \int_\Omega k(x) f(u_n(x)) u_n(x)\, \mathrm{d}x \\ &= \left(1 - \frac{1}{\theta}\right) \int_\Omega A(x, \nabla u_n(x))\, \mathrm{d}x \\ &\quad + \frac{1}{\theta} \left(\int_\Omega A(x, \nabla u_n(x))\, \mathrm{d}x - \lambda \int_\Omega k(x) f(u_n(x)) u_n(x)\, \mathrm{d}x \right) \end{aligned}$$

$$\geq \left(1-\frac{1}{\theta}\right)\Lambda_2\|u_n\|^p+\frac{1}{\theta}\langle I'_\lambda(u_n),u_n\rangle.$$

Due to (3.1), we can actually assume that $|\frac{1}{\theta}\langle I'_\lambda(u_n),u_n\rangle|\leq\|u_n\|$. Thus,

$$d+\|u_n\|\geq I_\lambda(u_n)-\frac{1}{\theta}\langle I'_\lambda(u_n),u_n\rangle\geq\left(1-\frac{1}{\theta}\right)\Lambda_2\|u_n\|^p.$$

It follows from this quadratic inequality that $\{\|u_n\|\}$ is bounded. By the Eberlian-Smulyan theorem, passing to a subsequence if necessary, we can assume that $u_n\rightharpoonup u$. Then $\Psi'(u_n)\to\Psi'(u)$ because of compactness. Since $I'_\lambda(u_n)=\Phi'(u_n)-\lambda\Psi'(u_n)\to 0$, then $\Phi'(u_n)\to\lambda\Psi'(u)$. Since Φ' is a homeomorphism, then $u_n\to u$ and so I_λ satisfies (PS)-condition.
From (f_2), by standard computations, there is a positive constant C such that

$$F(t)\geq C|t|^\theta \tag{3.2}$$

for all $t\in\mathbb{R}$ and $|t|>M$. In fact, setting $a:=\min_{|\xi|=M}F(\xi)$ and

$$\varphi_t(s):=F(st),\quad\forall s>0, \tag{3.3}$$

by (f_2), for every $t\in\Omega$ and $|t|>M$ one has

$$0<\theta\varphi_t(s)=\theta F(st)\leq st\cdot f(st)=s\varphi'_t(s),\quad\forall s>\frac{M}{|t|}.$$

Therefore,

$$\int_{M/|t|}^1\frac{\varphi'_t(s)}{\varphi_t(s)}\mathrm{d}s\geq\int_{M/|t|}^1\frac{\theta}{s}\mathrm{d}s,$$

then

$$\varphi_t(1)\geq\varphi_t\Big(\frac{M}{|t|}\Big)\frac{|t|^\theta}{M^\theta}.$$

Taking into account of (3.3), we obtain

$$F(t)\geq F\Big(\frac{M}{|t|}t\Big)\frac{|t|^\theta}{M^\theta}\geq a(x)\frac{|t|^\theta}{M^\theta}\geq C|t|^\theta,$$

where $C>0$ is a constant. Thus, (3.2) is proved.

Fixed $u_0\in X\backslash\{0\}$, for each $t>1$ one has

$$I_\lambda(tu_0)\leq\Lambda_2t^p\|u_0\|^p-\lambda Ct^\theta\int_\Omega|u_0(x)|^\theta\mathrm{d}x.$$

Since $\theta>p$, this condition guarantees that I_λ is unbounded from below. Fixed $\lambda\in\,]0,\lambda^\star[$, from condition ($\alpha_2$) it follows that

$$\|u\|<(\frac{r}{\Lambda_1})^{1/p}, \tag{3.4}$$

for each $u \in X$ such that $u \in \Phi^{-1}(]-\infty, 1[)$. Moreover, the compact embedding $X \hookrightarrow L^1(\Omega)$, (f_1), (3.4) and the compact embedding $X \hookrightarrow L^q(\Omega)$ imply that, for each $u \in \Phi^{-1}(]-\infty, 1[)$, we have

$$\begin{aligned}\Psi(u) &\leq \|k\|_\infty \left(a_1 \|u\|_{L^1(\Omega)} + \frac{a_2}{q} \|u\|^q_{L^q(\Omega)} \right) \\ &\leq \|k\|_\infty \left(a_1 c_1 \|u\| + \frac{a_2}{q} (c_q \|u\|)^q \right) \\ &< \|k\|_\infty \left(\frac{a_1 c_1 r^{1/p}}{\Lambda_1^{1/p}} + \frac{a_2 c_q^q r^{q/p}}{q \Lambda_1^{q/p}} \right),\end{aligned}$$

hence, by choosing $r = 1$, one has

$$\sup_{\Phi(u)<1} \Psi(u) \leq \|k\|_\infty \left(\frac{a_1 c_1}{\Lambda_1^{1/p}} + \frac{a_2 c_q^q}{q \Lambda_1^{q/p}} \right) = \frac{1}{\lambda^\star} < \frac{1}{\lambda}. \tag{3.5}$$

From (3.5) one has

$$\lambda \in]0, \lambda^\star[\subseteq \left] 0, \frac{1}{\sup_{\{\Phi(u)<1\}} \Psi(u)} \right[.$$

So all hypotheses of Theorem 2.3 are verified. Therefore, for each $\lambda \in]0, \lambda^\star[$, the functional I_λ admits two distinct critical points that are weak solutions of problem (1.1). □

Remark 3.3. Theorem 1.1 is an immediately consequence of Theorem 3.2.

Remark 3.4. We observe that, if f is non-negative and $f(0) \neq 0$ in Ω, then Theorem 3.2 ensures the existence of two positive weak solutions for problem (1.1) (see, e.g., [32, Theorem 11.1]).

Remark 3.5. Thanks to Talenti's inequality, it is possible to obtain an estimate of the embedding's constants c_1, c_q. By the Sobolev embedding theorem there exists a positive constant c such that

$$\|u\|_{L^{p^*}(\Omega)} \leq c \|u\|, \qquad (\forall\ u \in X), \tag{3.6}$$

see [33, Proposition B.7]. The best constant that appears in (3.6) is

$$c := \frac{1}{N\sqrt{\pi}} \left(\frac{N! \Gamma(\frac{N}{2})}{2\Gamma(\frac{N}{p}) \Gamma(N + 1 - \frac{N}{p})} \right)^{1/N} \eta^{1-1/p}, \tag{3.7}$$

where

$$\eta := \frac{N(p-1)}{N-p},$$

see, for instance, [38].

Due to (3.7), as a simple consequence of Hölder's inequality, it follows that

$$c_q \le \frac{\mathrm{meas}(\Omega)^{\frac{p^*-q}{p^*q}}}{N\sqrt{\pi}} \left(\frac{N!\Gamma(\frac{N}{2})}{2\Gamma(\frac{N}{p})\Gamma(N+1-N/p)} \right)^{1/N} \eta^{1-1/p},$$

where "meas(Ω)" denotes the Lebesgue measure of the set Ω.

In conclusion we present the concrete examples of application of Theorem 3.2 whose construction is motivated by [2, Example 4.1].

Example 3.6. We consider the function f defined by

$$f(t) := \begin{cases} c + dqt^{q-1}, & \text{if } t \ge 0, \\ c - dq(-t)^{q-1}, & \text{if } t < 0. \end{cases}$$

for each $t \in \mathbb{R}$, where $1 < p < q < p^*$ and c, d are two positive constants. For fixed $p < \theta < q$ and

$$r > \max\left\{ \left[\frac{(\theta-1)c}{d(q-\theta)}\right]^h, \left[\frac{c}{d}\right]^h \right\}, \tag{3.8}$$

with $h = \frac{1}{q-1}$, we prove that f verifies the assumptions requested in Theorem 3.2. Condition (f_1) of Theorem 3.2 is easily verified. We observe that

$$F(t) = ct + d|t|^q,$$

for each $t \in \mathbb{R}$. Taking (3.8) into account, condition (f_2) is verified (see Example 4.1 of [2]) and clearly $f(0) \neq 0$ in Ω,. Therefore, problem (1.1) has at least two non-trivial weak solutions for every $\lambda \in]0, \lambda^\star[$, where $\lambda^\star$ is the constant introduced in the statement of Theorem 3.2.

Example 3.7. Thanks to Theorem 1.1, the problem

$$\begin{cases} -\mathrm{div}(a(x, \nabla u)) = \lambda(u^3 + 1), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

admits two positive weak solutions for each $\lambda \in]0, \lambda^*[$, where

$$\begin{aligned} \lambda^* &= \frac{1}{\|k\|_\infty \left(\frac{c_1}{\Lambda_1^{1/p}} + \frac{c_4^4}{4\Lambda_1^{4/p}} \right)} \\ &\ge N\sqrt{\pi} \left(\frac{2\Gamma(\frac{N}{p})\Gamma(N+1-\frac{N}{p})}{N!\Gamma(\frac{N}{2})} \right)^{1/N}. \end{aligned}$$

In fact, it is enough to observe that f satisfies

$$\lim_{|t|\to\infty} \frac{f(t)}{|t|^3} = 0,$$

and $0 < 3F(\xi) \le \xi f(\xi)$ for all $|\xi| \ge 2$.

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