UNIFORM CONVERGENCE ANALYSIS OF MONOTONE HYBRID SCHEME FOR CONVECTION-DIFFUSION PROBLEMS ON LAYER ADAPTED MESHES

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In this paper, we present a monotone hybrid numerical scheme to compute the solution and the normalized flux for a singularly perturbed convection-diffusion problem on different kind of nonuniform meshes. This scheme which involves weight parameters is different from the usual hybrid scheme. The error analysis for the proposed scheme is carried out. A second order parameter uniform error bound is obtained for approximating the solution and the weighted derivative of the solution. Further, the idea is extended for a system of singularly perturbed convection-diffusion problems and the error estimate for the system is also calculated. Some numerical experiments are presented to illustrate the efficiency and effectiveness of the proposed scheme.

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1. INTRODUCTION

Differential equations with a small parameter 'ε' multiplying the highest order derivative terms are usually known to be singularly perturbed. *Singular perturbation problems* (SPPs) arise in many physical phenomena like a fluid flow with high Reynolds number, semiconductor device modeling, mathematical biology etc. The numerical analysis of singularly perturbed problems has always been far from trivial. As is well known, the numerical approximation of convectiondiffusion equations requires some special treatment in order to obtain good results when the problem is convection dominated due to the presence of boundary or interior layers. Boundary layers are nothing but thin regions where the solution varies very rapidly while away from these, the solution behaves regularly. This makes the solution of the SPPs multi-scale in character which causes severe numerical difficulties. Standard numerical methods for solving such problems are unstable and fail to give accurate results when the perturbation parameter approaches to zero [8]. Therefore, it is important to develop suitable numerical methods for these problems, whose accuracy does not depends on the parameter value. For the various approaches on the numerical solution of differential equations with steep, continuous solutions, one may refer the recent books [20, 24, 27] and the references therein.

Until now, several numerical methods for linear and nonlinear singularly perturbed convection-diffusion problems on layer-adapted meshes such as Shishkin type meshes and adaptive grids have been developed by many researchers [8, 16, 19, 20, 23, 27]. The limitation of such approaches is that most of these methods are of first order convergent. In order to get higher order accuracy, Andreev and Kopteva [1] proposed a monotone three-point finite difference scheme and proved the second order uniform convergent on Bakhvalov mesh and the second order uniform convergent up to a logarithmic limit on Shishkin mesh for the convection-diffusion problem. In [14], first and second-order difference schemes for a singularly perturbed quasilinear two-point boundary value problem were studied on arbitrary nonuniform meshes. Linß [16] provided sufficient conditions for uniform convergence on layer-adapted grids for quasilinear convection-diffusion problems. The hybrid scheme derived in [28] combines a central difference scheme in the layer region and a midpoint upwind scheme in the outer region. Uniformly convergent class of hybrid schemes for the solution and the derivatives of quasilinear singularly perturbed problems were studied by Zheng *et al.* [30].

In recent years, there is an increasing interest towards the system of SPPs as these problems appear in various applications [18]. Few numerical methods have been developed to tackle these problems on different kind of nonuniform meshes. In [4], a numerical method was constructed for a system of weakly coupled convection-diffusion problems on Shishkin mesh. The convergence analysis of the upwind scheme for a coupled system of equations on various nonuniform meshes was provided in [17]. From the literature, it is evident that the use of Shishkin mesh is more frequent for a uniformly convergent method. But, to implement this one should have enough information about the location and the width of the boundary layers, which is not available in all cases. In recent years, the idea of the adaptive grid to obtain the parameter uniform numerical approximation has gain tremendous attention of some researchers. This mesh detects the layer automatically having the advantage over Shishkin type meshes discussed earlier. The adaptive mesh is obtained from the idea of the equidistribution of a positive monitor function, which involves the derivatives of the solution. Thus, it detects the abrupt change in the solution and automatically locates the boundary layer. Beckett and Mackenzie [2] applied adaptive mesh generated by equidistributing a monitor function for convection-diffusion problems. One can also refer Beckett *et al.* [3] for one-dimensional parabolic

PDEs. This work concerned with the derivation of posteriori error estimators for convection-diffusion problems using adaptive algorithms were established in [5, 9, 12, 29]. An optimal error estimate using mesh equidistribution techniques for a singularly perturbed system of reaction-diffusion boundary-value problems was provided by Das and Natesan [6]. Gowrisankar and Natesan applied the adaptive mesh obtained by equidistribution of a monitor function for the reaction-diffusion parabolic problem in [10] and convection-diffusion parabolic problem in [11]. Recently, Shakti and Mohapatra developed adaptive mesh based numerical methods for parameterized SPPs in [25] and for system of nonlinear singularly perturbed problems in [26].

In this work, we propose a monotone hybrid scheme which is a combination of the midpoint and the central difference scheme with variable weights on the nonuniform meshes for convection-diffusion problems to obtain second order convergence. The weights are so chosen that the scheme automatically switches from the midpoint scheme to the central difference scheme as the mesh goes from coarse to fine and so, having an advantage for no *apriori* information about the layer and the outer region. Furthermore, we have extended this idea to system of convection diffusion-problems.

The rest of the paper is organized as follows: In Section 2, we define the continuous problems and their properties. Section 3 and Section 4 is devoted to the construction of a monotone hybrid scheme and description of the layer adapted meshes for the scalar and the system of convection-diffusion problems respectively. Moreover, in Section 5 we carried out a convergence analysis and obtained the parameter uniform bound of the scheme. To validate the theoretical results, numerical experiments with some test problems are carried out in Section 6 and the results are shown in the shape of tables and figures. Henceforth, 'C' denotes a generic positive constant independent of both the perturbation parameters and the mesh parameter N which can take different values at different places. Throughout this paper, we assume $\varepsilon \leq CN^{-1}$, $\varepsilon_1 \leq CN^{-1}$, and $\varepsilon_2 \leq CN^{-1}$, where ε, ε_1, and ε_2 are the perturbation parameters, as is generally the case in practise. Here, we denote $||\gamma|| = \max |\gamma(x)|$, $||\mathbf{\Gamma}|| = \max_k \|\gamma_k(x)\|$ where $\mathbf{\Gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_k)^T$ and $g(x_i) = g_i, \quad g_{i-1/2} = (g(x_{i-1}) + g(x_i))/2$.

2. CONTINUOUS PROBLEM & ITS PROPERTIES

In this section, we first describe the properties of a scalar convection-diffusion problem and then the properties of system of convection diffusion problems.

2.1. SCALAR CONVECTION DIFFUSION PROBLEM

Consider the following singularly perturbed boundary value problem(BVP):

$$
\begin{cases} L_\varepsilon u \equiv -\varepsilon u''(x) - (a(x)u(x))' = f(x) \quad x \in \Omega = (0,1), \\ u(0) = s_0, \quad u(1) = s_1, \end{cases} \tag{1}
$$

where $0 < \varepsilon \ll 1$, $0 < \alpha \leq a(x) \leq \alpha^*$ and $s_0,\ s_1$ are given constants. It has a boundary layer in neighborhood of $x = 0$. The following lemma shows that the operator Ł$_\varepsilon$ satisfies the maximum principle.

Lemma 2.1. *Let $v(x)$ be a smooth function satisfying $v(0) \geq 0$, $v(1) \geq 0$ and $L_\varepsilon v(x) \geq 0$ for every $x \in \Omega$ then $v(x) \geq 0$ for all $x \in \overline{\Omega}$.*

Proof. Let $x^* \in \Omega$ be such that $v(x^*) = \min_{x \in \overline{\Omega}} v(x)$ and assume that $v(x^*) < 0$. Clearly $x^* \neq 0, 1$ and $v'(x^*) = 0$. So, $L_\varepsilon v(x^*) \equiv -\varepsilon v''(x^*) + (a(x^*)v(x^*))' < 0$, which is a contradiction. Hence, $v(x) \geq 0$ for every $x \in \overline{\Omega}$. □

Lemma 2.2. *The solution $u(x)$ of the BVP (1) and its derivatives satisfy the following bounds:*

$$
|u^{(k)}(x)| \leq C(1 + \varepsilon^{-k} \exp(-\alpha x/\varepsilon)), \quad k = 0, 1, 2, 3. \tag{2}
$$

Proof. Using the idea discussed in Lemma 2.3 page 1027 of [13], one can prove the above lemma. □

Let us decompose the solution of BVP (1) into a smooth component $v(x)$ and a singular component $w(x)$ as: $u(x) = v(x) + w(x)$.

Now, $v(x) = v_0(x) + \varepsilon v_1(x) + \varepsilon^2 v_2(x)$, where v_0, v_1 and v_2 are the solution of the following problems:

$$
\begin{cases} -a v_0'(x) = f(x), \quad v_0(1) = s_1, \\ -a v_1'(x) = -v_0''(x), \quad v_1(1) = 0, \\ -L_\varepsilon v_2(x) = v_1''(x), \quad v_2(0) = 0,\ v_2(1) = 0. \end{cases}
$$

The regular component $v(x)$ satisfies the BVP:

$$
L_\varepsilon v(x) = f(x) \quad v(0) = v_0(0) + \varepsilon v_1(0),\ v(1) = s_1, \tag{3}
$$

and the singular component $w(x)$ satisfies

$$
L_\varepsilon w(x) = 0, \quad |w(0)| \leq C,\ w(1) = 0, \tag{4}
$$

where $w(0)$ depends on $v(x)$ and its derivatives which are bounded uniformly in ε.

The following lemma provides the bounds for the components v and w.

LEMMA 2.3. *The smooth component $v(x)$ and the singular component $w(x)$ of the solution $u(x)$ of BVP (1) satisfy the following bounds:*

$$\begin{aligned} |v^{(k)}(x)| &\leq C, \\ |w^{(k)}(x)| &\leq C\varepsilon^{-k}\exp(-\alpha x/\varepsilon), \quad k = 0,1,2,3. \end{aligned}$$

Proof. Using the idea given in Lemma 3.3 of [8], one can prove the above bounds. □

2.2. SYSTEM OF CONVECTION DIFFUSION PROBLEMS

Next, consider the following coupled system of convection-diffusion BVPs:

$$(5)\qquad \begin{cases} \mathcal{L}_1 u_1 \equiv -\varepsilon_1 u_1''(x) - (a_1(x)u_1(x))' = f_1(x), \quad u_1(0) = u_1(1) = 0, \\ \mathcal{L}_2 u_2 \equiv -\varepsilon_2 u_2''(x) - (a_2(x)u_2(x))' - (b_1(x)u_1)' = f_2(x), \\ u_2(0) = u_2(1) = 0, \end{cases}$$

where $0 \leq \varepsilon_1 \leq \varepsilon_2 \ll 1$ are the small parameters and $0 < \alpha_k \leq a_k \leq \alpha_k^*$ for $x \in \Omega$, $k = 1,2$. The system (6) is a coupled system of convection-diffusion problems, when $b_1(x) \neq 0$. The solution u_1 exhibits a boundary layer of width $O(\varepsilon_1)$ near $x = 0$. The boundary layer occurring in the solution u_2 depends on the relative values of two small parameters. One may refer introduction section of [4] for more details.

The following theorem gives the derivative estimate of the solution component u_1 and u_2.

THEOREM 2.4. *The solution $\mathbf{u} = (u_1, u_2)$ of the system (6) and its derivatives satisfy the following bound for $x \in \overline{\Omega}$:*

$$(6)\qquad \begin{cases} |u_1^{(k)}(x)| \leq C(1 + \varepsilon_1^{-k}\exp(-\alpha_1 x/\varepsilon_1)), \quad k = 0,1,2,3, \\ |u_2(x)| \leq C, \\ |u_2^{(k)}(x)| \leq C(1 + \varepsilon_2^{-(k-1)}(\varepsilon_1^{-1}\exp(-\alpha_1 x/\varepsilon_1) + \varepsilon_2^{-1}\exp(-\alpha_2 x/\varepsilon_2)), \\ \qquad\qquad\qquad k = 1,2,3. \end{cases}$$

Proof. The bounds of the solution $\mathbf{u}$ can be obtained by using the idea given in Theorem 3 of Bellew [4] and in Section 3 of Linß [17]. □

3. SCALAR CONVECTION DIFFUSION PROBLEM

This section is devoted for the discretization of the scalar convection-diffusion problems using a monotone hybrid difference scheme and the description of certain nonuniform meshes.

3.1. DISCRETE PROBLEM

We consider a monotone hybrid finite difference scheme to find $U^N = \{U_i^N\}_{i=0}^N$ on an arbitrary nonuniform mesh $\Omega^N = \{0 = x_0 < x_1 < \dots x_{N-1} < x_N = 1\}$, which is defined as follows:

$$\text{(7)} \quad \begin{cases} L_\varepsilon^N U \equiv -\dfrac{[A^N U^N]_{i+1} - [A^N U^N]_i}{\hbar_{\sigma,i}} = f_{\sigma,i-1/2}, \quad \text{for } i = 1, ..., N-1 \\ U_0^N = s_0, \quad U_N^N = s_1, \end{cases}$$

where,

$$\begin{aligned} [A^N U]_i &= \varepsilon \frac{U_i^N - U_{i-1}^N}{h_i} + \sigma_i a_i U_i^N + (1-\sigma_i) a_{i-1} U_{i-1}^N, \\ \hbar_{\sigma,i} &= (1-\sigma_i) h_i + \sigma_{i+1} h_{i+1}, \quad h_i = x_i - x_{i-1}, \\ f_{\sigma,i-1/2} &= (f(x_{i-1} + h_i \sigma_i) + f(x_i + h_{i+1}\sigma_{i+1}))/2. \end{aligned} \tag{8}$$

The σ_i is chosen differently by different researchers [1, 16, 30]. The idea behind the different choice of σ_i is that it should satisfies the discrete maximum principle. Here, we have chosen σ_i as:

$$\sigma_i = \begin{cases} \frac{1}{2}, & \text{if } \frac{1}{2} \geq 1 - \varepsilon / h_i a_{i-1}, \\ 1, & \text{if } \frac{1}{2} < 1 - \varepsilon / h_i a_{i-1}. \end{cases} \tag{9}$$

The derivative at $x_{i-1/2}$ is computed as:

$$D^- U_i^N = \frac{U_i^N - U_{i-1}^N}{h_i}, \quad i = 1, 2 \dots N. \tag{10}$$

After rearranging the terms in (7), the following form is obtained:

$$\begin{cases} r_i^- U_{i-1}^N + r_i^c U_i^N + r_i^+ U_{i+1}^N = f_{\sigma,i-1/2}, \quad \text{for } i = 1, ..., N-1, \\ U_0^N = s_0, \quad U_N^N = s_1, \end{cases} \tag{11}$$

where,

$$\begin{aligned} r_i^- &= -\frac{1}{\hbar_{\sigma,i}} \left(\frac{\varepsilon}{h_i} - (1-\sigma_i) a_{i-1} \right), \quad r_i^+ = -\frac{1}{\hbar_{\sigma,i}} \left(\frac{\varepsilon}{h_{i+1}} + \sigma_{i+1} a_{i+1} \right), \\ r_i^c &= \frac{1}{\hbar_{\sigma,i}} \left(\frac{\varepsilon}{h_i} + \frac{\varepsilon}{h_{i+1}} + \sigma_i a_i - (1-\sigma_{i+1}) a_i \right). \end{aligned}$$

The tri-diagonal system (11) has the following properties:

$$r_i^- < 0, \quad r_j^+ < 0, \quad r_i^c > 0 \quad \text{for } i = 1, ..., N-1. \tag{12}$$

This matrix has diagonal predominance with respect to the columns. Therefore, the tri-diagonal matrix (11) is a monotone matrix.

3.2. GENERATION PF THE GRID

Here, in this section we describe the construction of nonuniform meshes specially the Shishkin type meshes and the adaptive grid.

3.2.1. SHISHKIN TYPE MESHES(S-TYPE MESHES)

Without loss of generality, assume N is an even integer for the discretization of $\overline{\Omega}$. Define $\tau = \min\left\{\frac{1}{2}, \frac{2\varepsilon}{\alpha}\ln N\right\}$ as the transition point. Since the layer is on the left side, the mesh is obtained by dividing each of the subdomains $[0,\tau]$ and $[\tau,1]$ of Ω^N into $N/2$ subintervals. In $[0,\tau]$, the mesh will be graded and in $[\tau,1]$ the mesh will be coarse. On $[0,\tau]$, the mesh is given by the piecewise continuously differentiable and monotonic increasing function ϕ such that $\phi(0)=0$ and $\phi(1/2)=\ln N$. Now the mesh points are,

$$x_i = \begin{cases} \dfrac{2\varepsilon}{\alpha}\phi(t_i), & \text{for } t_i = \frac{i}{N},\ i = 0,1,\dots,N/2, \\ 1-\left(1-\dfrac{2\varepsilon}{\alpha}\ln N\right)\dfrac{2(N-i)}{N}, & \text{for } i = N/2+1,\dots,N. \end{cases} \tag{13}$$

To define the Shishkin mesh (S-mesh) and the Bakhvalov-Shishkin mesh (B-S-mesh), let us consider a new monotonically decreasing function 'ψ' that is closely related to ϕ, defined by $\phi = -\ln\psi$ which satisfies $\psi(0)=1$ and $\psi(\frac{1}{2}) = N^{-1}$, then

$$\psi(t) = e^{(-2\ln N)t}, \qquad (\text{ S-mesh}) \tag{14}$$

$$\psi(t) = 1-2(1-N^{-1})t, \qquad (\text{B-S-mesh}) \tag{15}$$

For S-type meshes, we have the following results:

LEMMA 3.1. *For piecewise differentiable mesh generating function ϕ satisfying the condition*

$$\max_{[0,1/2]}\phi'(x) = \max_{[0,1/2]}\frac{|\psi'|}{\psi} \le CN, \tag{16}$$

we have

$$\max_i \int_{x_{i-1}}^{x_i}(1+\varepsilon^{-1}\exp(-\alpha x/2\varepsilon))\mathrm{d}x \le C\left\{\varepsilon + (N^{-1}+N^{-\tau/2})\max_{x\in[0,1/2]}|\psi'(x)|\right\}, \tag{17}$$

where

$$\max_{x\in[0,1/2]}|\psi'(x)| \le C, \quad \textit{(B-S-mesh)}$$

$$\max_{x\in[0,1/2]}|\psi'(x)| \le C\ln N, \quad \textit{(S-mesh).}$$

Proof. One can refer Lemma 3 page no. 251 of [16] for the proof. □

3.2.2. ADAPTIVE GRID

In order to show the parameter uniform bounds, the adaptive grid approach has attracted many researchers in last few years [3, 6, 15, 25]. The idea of equidistribution principle is a commonly used technique to generate the adaptive grid. A mesh Ω^N is said to be equidistributed if

$$\int_{x_{j-1}}^{x_j} M\big(u(s), s\big)\mathrm{d}s = \int_{x_j}^{x_{j+1}} M\big(u(s), s\big)\mathrm{d}s, \quad j = 1, \dots, N-1, \tag{18}$$

where $M\big(u(x), x\big) > 0$ is called the monitor function. Generally, the monitor function is a measure of the computational error or the solution variation. Equivalently,

$$\int_{x_{j-1}}^{x_j} M\big(u(s), s\big)\mathrm{d}s = \frac{1}{N}\int_0^1 M\big(u(s), s\big)\mathrm{d}s, \quad j = 1, \dots, N-1. \tag{19}$$

The advantage of using the adaptive grid is that it does not require any *apriori* information about solution and it can detect the locations and width of the layers accurately. Here, we have used $M(u(x), x) = 1+|u''(x)|^{1/2}$ as the monitor function.

The proposed idea of using the monotone hybrid scheme using the adaptive grid technique involves two steps: firstly the adaptive mesh has to be determined by a mesh generation algorithm and thereafter the difference scheme is to be computed using the adaptively generated nonuniform mesh. We consider the well-known de-Boor algorithm to generate the appropriate adaptive mesh. Many researchers have used the adaptive mesh generation algorithm to solve the various kind of SPPs. Mackenzie [19] analyzed the uniform convergence of an upwind method for convection diffusion problem on an adaptive grid. Uniform convergence of finite difference approximation for SPP on an adaptive grid was proved in [2]. Kopteva and Stynes [14] constructed an adaptive method for solving a quasi linear one dimensional convection diffusion problem. Das and Natesan [6] consider the adaptive mesh technique to solve the system of reaction diffusion problem. The adaptive mesh can be obtained by using the following algorithm:

ADAPTIVE MESH GENERATION ALGORITHM

Step 1: Let us consider the initial mesh $\{x_i^0 : 0, 1/N, 2/N, \dots, 1\}$ as the uniform mesh.

Step 2: For $k = 0, 1, \ldots$ assuming the mesh $\{x_i^k\}$ is given, compute the discrete solution from the discrete problem.

Step 3: Find the discretized monitor function M_i^k. Compute

$$l_i^{(k)} = h_i^{(k)} \left(\frac{M_{i-1}^{(k)} + M_i^{(k)}}{2} \right)$$

for $i = 1, \ldots, N$, and set $M_0^{(k)} = M_1^{(k)}$, $M_N^{(k)} = M_{N-1}^{(k)}$. Denote $L_0 = 0$ and $L_i := \sum_{i=1}^{N} l_i^{(k)}$.

Step 4: Let C_0 be the user chosen constant, where $C_0 > 1$. If, $\dfrac{\max l_i^{(k)}}{L_N} \le \dfrac{C_0}{N}$, then go to Step 6, otherwise continue to Step 5.

Step 5: Set $Y_i = iL_N/N$ for $i = 0, 1, \ldots, N$. Interpolate the points (L_i, x_i). Generate the new mesh $\{x_i^{k+1}\}$ by evaluating this interpolate at Y_i for $i = 0, 1, \ldots, N$.

Step 6: Set x_i^k as the final mesh and compute the difference approximation on the final mesh. Stop.

LEMMA 3.2. *The mesh width generated by the equidistribution of the monitor function $M(u(x), x)$ satisfies $h_i \le CN^{-1}$ for $i = 0, 1, ..., N$.*

Proof. It is clear that $M(u(x), x) \ge 1$. So, by the equidistribution principle and the bounds of derivatives of the solution given in Lemma 2.2

$$h_i \le \int_{x_{i-1}}^{x_i} M(u(x), x)\mathrm{d}x = \frac{1}{N} \int_0^1 M(u(x), x)\mathrm{d}x \le CN^{-1}. \tag{20}$$

Thus, we have the desired inequality. One can refer [2, 19] for more details. □

The following result allows an easy analyzation of the uniform convergence of the numerical solution and the derivative obtained by the proposed monotone hybrid scheme on the adaptive grid.

LEMMA 3.3. *For an appropriate choice of the monitor function, we have*

$$\max_i \int_{x_{i-1}}^{x_i} (1 + \varepsilon^{-1} \exp(-\alpha x/2\varepsilon))\mathrm{d}x \le CN^{-1}. \tag{21}$$

Proof. From the solution decomposition, $|w''(x)| \le |u''(x)| + |v''(x)| \le C[1 + u''(x)]$ and from the equidistribution principle, we have

$$\max_i \int_{x_{i-1}}^{x_i} \left(1 + \varepsilon^{-1} \exp(-\alpha x/2\varepsilon)\right)\mathrm{d}x \;\le\; \max_i \int_{x_{i-1}}^{x_i} \left(1 + |w''(x)|^{1/2}\right)\mathrm{d}x,$$

$$\leq \max_i \int_{x_{i-1}}^{x_i} \left(1 + |u''(x)|^{1/2}\right) \mathrm{d}x$$

$$\leq \frac{1}{N} \int_0^1 M(u(x), x) \mathrm{d}x \leq CN^{-1}.$$

Thus, we have the desired estimate. □

4. SYSTEM OF CONVECTION DIFFUSION PROBLEMS

In this section, we extend the idea of the monotone hybrid scheme for the system of convection diffusion problems.

4.1. DISCRETE PROBLEM

Here, we develop a monotone finite difference method for (6) on a nonuniform grid Ω^N. Here, we have chosen two appropriate variable weights which automatically switches the scheme from the central difference scheme to the midpoint upwind scheme as nodal points moves from the layer region to the outer region. Let $\mathbf{U}$ be the discrete approximation of the continuous solution $\mathbf{u}$. Then our scheme is:

$$(22) \qquad \begin{cases} \text{Find } \ \mathbf{U} = [U_1, U_2] \ \text{ such that} \\ [\mathbf{L^N U}]_\mathbf{i} = \mathbf{f_i}, \quad \text{for } i = 1, 2, \ldots, N, \\ \mathbf{U_0} = \mathbf{U_N} = \mathbf{0}, \end{cases}$$

i.e. find $U_{k,1} \ldots, U_{k,N-1}$ for $k = 1, 2$ satisfying

$$(23) \qquad \begin{cases} L_1^N U_{1,i} \equiv -\dfrac{[A_1U_1]_{i+1} - [A_1U_1]_i}{\hbar_{\sigma_1,i}} = f_{1,\sigma_1,i-1/2}, \\ L_2^N U_{2,i} \equiv -\dfrac{[A_2U_2]_{i+1} - [A_2U_2]_i}{\hbar_{\sigma_2,i}} = f_{2,\sigma_2,i-1/2}, \end{cases}$$

where,

$$[A_1U_1]_i = \varepsilon_1 \frac{U_{1,i} - U_{1,i-1}}{h_i} + \sigma_{1,i}a_{1,i}U_{1,i} + (1 - \sigma_{1,i})a_{1,i-1}U_{1,i-1},$$

$$\begin{aligned} [A_2U_2]_i = \varepsilon_2 \frac{U_{2,i} - U_{2,i-1}}{h_i} \quad &+ \quad \sigma_{2,i}a_{2,i}U_{2,i} + (1 - \sigma_{2,i})a_{2,i-1}U_{2,i-1} \\ &+ \quad \sigma_{2,i}b_{1,i}U_{1,i} + (1 - \sigma_{2,i})b_{1,i-1}U_{1,i-1}, \end{aligned}$$

$\hbar_{\sigma_k,i} = (1 - \sigma_{k,i})h_i + \sigma_{k,i+1}h_{i+1}$, with increasing variable weights given by

$$(24) \qquad \sigma_{k,i} = \begin{cases} \frac{1}{2}, & \text{if } \frac{1}{2} \geq 1 - \varepsilon_k / h_i a_{k,i-1}, \\ 1, & \text{if } \frac{1}{2} < 1 - \varepsilon_k / h_i a_{k,i-1}. \end{cases}$$

For $\sigma_{k,i} = \frac{1}{2}$, we obtain the central difference scheme while for $\sigma_{k,i} = 1$ the midpoint scheme is obtained. After rearranging the terms in (23), we get the following form of the difference scheme:

$$(25) \quad \begin{cases} r^-_{1,i} U^N_{1,i-1} + r^c_{1,i} U^N_{1,i} + r^+_{1,i} U^N_{1,i+1} = f_{1,\sigma_1,i-1/2}, & \text{for } i = 1, ..., N-1 \\ U^N_0 = s_0, \quad U^N_N = s_1, \end{cases}$$

where,

$$r^-_{1,i} = -\frac{1}{\hbar_{\sigma_1,i}}\left(\frac{\varepsilon_1}{h_i} - (1-\sigma_{1,i})a_{1,i-1}\right), \quad r^+_{1,i} = -\frac{1}{\hbar_{\sigma_1,i}}\left(\frac{\varepsilon_1}{h_{i+1}} + \sigma_{1,i+1}a_{1,i+1}\right),$$
$$r^c_{1,i} = \frac{1}{\hbar_{\sigma_1,i}}\left(\frac{\varepsilon_1}{h_i} + \frac{\varepsilon_1}{h_{i+1}} + \sigma_{1,i}a_{1,i} - (1-\sigma_{1,i+1})a_{1,i}\right).$$

Again,

$$(26) \quad \begin{cases} r^-_{2,i} U^N_{2,i-1} + r^c_{2,i} U^N_{2,i} + r^+_{2,i} U^N_{2,i+1} = g_{\sigma_2,i-1/2}, & \text{for } i = 1, ..., N-1 \\ U^N_0 = s_0, \quad U^N_N = s_1, \end{cases}$$

where,

$$\begin{aligned} r^-_{2,i} &= -\frac{1}{\hbar_{\sigma_2,i}}\left(\frac{\varepsilon_2}{h_i} - (1-\sigma_{2,i})a_{2,i-1}\right), \\ r^+_{2,i} &= -\frac{1}{\hbar_{\sigma_2,i}}\left(\frac{\varepsilon_2}{h_{i+1}} + \sigma_{2,i+1}a_{2,i+1}\right), \\ r^c_{2,i} &= \frac{1}{\hbar_{\sigma_2,i}}\left(\frac{\varepsilon_2}{h_i} + \frac{\varepsilon_2}{h_{i+1}} + \sigma_{2,i}a_{2,i} - (1-\sigma_{2,i+1})a_{2,i}\right) \\ g_{\sigma_2,i-1/2} &= f_{2,\sigma_1,i-1/2} + \frac{1}{\hbar_{\sigma_2,i}}\Big(\sigma_{2,i+1}b_{1,i+1}U_{1,i+1} + (1-\sigma_{2,i+1})b_{1,i}U_{1,i} \\ &\quad -\sigma_{2,i}b_{1,i}U_{1,i} - (1-\sigma_{2,i})b_{1,i-1}U_{1,i-1}\Big). \end{aligned}$$

It is clear that tri-diagonal systems (25) and (26) has the following properties:

$$(27) \qquad \mathbf{r}^-_i < 0, \quad \mathbf{r}^+_j < 0, \quad \mathbf{r}^c_i > 0 \quad \text{for } i = 1, ..., N-1.$$

This matrix has the diagonal predominance with respect to the columns. Therefore, the tri-diagonal matrix (25) and (26) are monotone matrices.

4.2. MESH GENERATION

4.2.1. SHISHKIN MESH

To construct the Shishkin mesh, we divide the domain $\overline{\Omega}$ to three subdomains as $[0, \tau_1]$, $[\tau_1, \tau_2]$ and $[\tau_2, 1]$. On the subinterval $[\tau_2, 1]$, where the solution behaves regularly, the mesh is coarse; on the other two subintervals $[0,\tau_1]$ and $[\tau_1,\tau_2]$ the mesh is fine. Let N be the mesh parameter which is divisible by 4. Define the transition parameters τ_1 and τ_2 as

$$\tau_2=\min\left\{\frac{1}{2},\kappa_0\frac{\varepsilon_2}{\alpha_2}\ln N\right\}\quad\text{and}\quad \tau_1=\min\left\{\frac{1}{4},\frac{\tau_2}{2},\kappa_1\frac{\varepsilon_2}{\alpha_1}\ln N\right\},\quad \kappa_0,\ \kappa_1\geq 1.$$

Then the mesh widths are

$$h_i=\begin{cases} H_1=\frac{4\tau_1}{N} & 1\leq i\leq \frac{N}{4},\\ H_2=\frac{4(\tau_2-\tau_1)}{N} & \frac{N}{4}<i\leq\frac{N}{2},\\ H_3=\frac{2(1-\tau_2)}{N} & \frac{N}{2}<i\leq N.\end{cases}\tag{28}$$

We consider the case $\tau_1\leq\dfrac{\tau_2}{2}$, since when $\tau_1=\dfrac{\tau_2}{2}$ then $\varepsilon_1=O(\varepsilon_2)$ and the results can be easily obtained. Further, we assume that $\tau_2=\kappa_0\frac{\varepsilon_2}{\alpha_2}\ln N$ and $\tau_1=\kappa_1\frac{\varepsilon_2}{\alpha_1}\ln N$, as otherwise N^{-1} is exponentially small compared with ε_1 and ε_2.

Lemma 4.1. *Let the nodes $0=x_0<x_1<\ldots<x_N=1$ of Ω^N generated with the step size (28) with $x_i-x_{i-1}=O(N^{-1})$, then*

$$\left[\int_{x_{i-1}}^{x_i}\left(1+\sum_{k=1}^{2}\varepsilon_k^{-1}\exp(-\alpha_k x/2\varepsilon_k)\right)\mathrm{d}x\right]^2\leq CN^{-2}\ln^2 N.\tag{29}$$

Proof. For $1\leq i\leq N/4$, using the mesh width estimates, we obtain

$$\begin{aligned}&\left[\int_{x_{i-1}}^{x_i}\left(1+\sum_{k=1}^{2}\varepsilon_k^{-1}\exp(-\alpha_k x/2\varepsilon_k)\right)\mathrm{d}x\right]^2\\ &\leq C\left[h_i-\frac{2}{\alpha_1}\exp(-\alpha_1 x/2\varepsilon_1)\Big|_{x_{i-1}}^{x_i}-\frac{2}{\alpha_2}\exp(-\alpha_2 x/2\varepsilon_2)\Big|_{x_{i-1}}^{x_i}\right]^2,\\ &\leq C\left[h_i-\frac{2}{\alpha_1}\exp(-\alpha_1 x_i/2\varepsilon_1)(1-\exp(-\alpha_1 h_i/2\varepsilon_1))-\right.\\ &\left.\frac{2}{\alpha_2}\exp(-\alpha_2 x/2\varepsilon_2)(1-\exp(-\alpha_2 h_i/2\varepsilon_2))\right]^2,\\ &\leq CN^{-2}\ln^2 N.\end{aligned}\tag{30}$$

Using the inequality $\varepsilon_1^{-1}\exp(-\alpha_1 x/2\varepsilon_1)\leq\varepsilon_2^{-1}\exp(-\alpha_2 x/2\varepsilon_2)$, for $x>\frac{2\varepsilon_1}{\alpha_1}$ and the mesh width estimates, we have

$$\left[\int_{x_{i-1}}^{x_i}\left(1+\sum_{k=1}^{2}\varepsilon_k^{-1}\exp(-\alpha_k x/2\varepsilon_k)\right)\mathrm{d}x\right]^2$$

$$
\begin{aligned}
&\le C\Big[\int_{x_{i-1}}^{x_i}\Big(1+\exp(-\alpha_2 x/2\varepsilon_2)\Big)\mathrm{d}x\Big]^2,\\
&\le C\Big[h_i - \frac{2}{\alpha_2}\exp(-\alpha_2 x_i/2\varepsilon_2)(1-\exp(-\alpha_2 h_i/2\varepsilon_2))\Big]^2,\\
&\le CN^{-2}\ln^2 N. \qquad (31)
\end{aligned}
$$

For $N/4 < i \le N/2$. Again, using the inequality (31) and the mesh width estimates for $N/2 < i \le N$, we have

$$
\begin{aligned}
&\Big[\int_{x_{i-1}}^{x_i}\Big(1+\sum_{k=1}^{2}\varepsilon_k^{-1}\exp(-\alpha_k x/2\varepsilon_k)\Big)\mathrm{d}x\Big]^2\\
&\le C\Big[\int_{x_{i-1}}^{x_i}\Big(1+\exp(-\alpha_2 x/2\varepsilon_2)\Big)\mathrm{d}x\Big]^2,\\
&\le C\Big[h_i + \frac{2}{\alpha_2}(\exp(-\alpha_2 x_i/2\varepsilon_2)-\exp(-\alpha_2 x_{i-1}/2\varepsilon_2)\Big]^2,\\
&\le C\Big[h_i + \frac{2}{\alpha_2}\exp(-\alpha_2 x_{N/2}/2\varepsilon_2)\Big]^2,\\
&\le CN^{-2}. \qquad (32)
\end{aligned}
$$

Combining (30), (31) and (32), we complete the proof. □

4.2.2. ADAPTIVE GRID

The adaptive grid generation algorithm for the system of SPPs has got much attention. Here, we have used the same adaptive algorithm which is described in Section 3. For the coupled system, we need such a monitor function which can measure the solution variation in both the components of the system simultaneously. Once such type of monitor function was discussed in [6]. The details about the monitor function is discussed in the next section in Remark 5.11.

5. ERROR ANALYSIS

Here, we carry out the error estimates in solving the scalar and the system of convection diffusion problems by the monotone hybrid scheme on the nonuniform meshes discussed above. Also, we derive the analysis for the scaled derivative of the solution.

5.1. SCALAR CONVECTION DIFFUSION PROBLEMS

The following result provides the stability of the monotone hybrid scheme for scalar convection diffusion problem.

THEOREM 5.1. *Let v_i is a solution of the monotone hybrid scheme with zero boundary condition then,*

$$\|v\| \le C \max_i \left| \sum_{j=i}^{N-1} \hbar_{\sigma,j} f_{\sigma,\, j-1/2} \right|, \tag{33}$$

and

$$\max_i \varepsilon |D^- v_i| \le C \max_i \left| \sum_{j=i}^{N-1} \hbar_{\sigma,j} f_{\sigma,\, j-1/2} \right|. \tag{34}$$

Proof. The proof of (33) is given in [1]. Now to prove (34), let us write the mesh function v as

$$v_i = bV_i + W_i, \tag{35}$$

where V_i and W_i are the solutions of the difference equations given as:

$$\begin{aligned} A^N V_i &= 1, \quad i = 1, 2, ..., N, \quad V_0 = 0, V_N = 0, \\ A^N W_i &= \sum_{j=i}^{N-1} \hbar_{\sigma,j} f_{\sigma,\, j-1/2}, \quad i = 1, 2, ..., N \quad W_0 = 0, W_1 = 0, \end{aligned}$$

and the constant b is

$$b = -\frac{[\varepsilon/h_N - (1-\sigma_N)a_{N-1}]W_{N-1}}{1 + [\varepsilon/h_N - (1 - \sigma_N a_{N-1})]V_{N-1}}.$$

Since the matrix associated with the above is a monotone matrix, then by the discrete maximum principle, we can prove that

$$0 \le V_i \le 2/\alpha \quad \text{and} \quad |W_i| \le V_i \max_i \left| \sum_{j=i}^{N-1} \hbar_{\sigma_j} f_{\sigma,\, j-1/2} \right|.$$

As $\varepsilon/h_N - (1-\sigma_N)a_{N-1}$ is non-negative, then we have

$$|b| \le \max_i \left| \sum_{j=i}^{N-1} \hbar_{\sigma,j} f_{\sigma,\, j-1/2} \right|.$$

Now,

$$A^N v_i = bA^N V_i + A^N W_i = b + \sum_{j=i}^{N-1} \hbar_{\sigma,j} f_{\sigma,\, j-1/2}. \tag{36}$$

So,

$$\max_i |A^N v_i| \le C \max_i \left| \sum_{j=i}^{N-1} \hbar_{\sigma,j} f_{\sigma,\, j-1/2} \right|. \tag{37}$$

Thus from (10), we obtain

$$\begin{aligned} \max_i \varepsilon |D^- v_i| &\le \max_i |A^N v_i| + \max_i |\sigma_i a_i v_i + (1-\sigma_i) a_{i-1} v_{i-1}| \\ &\le 2 \max_i \left| \sum_{j=i}^{N-1} \hbar_{\sigma,j} f_{\sigma,\, j-1/2} \right| + \alpha \|v\| \\ &\le C \max_i \left| \sum_{j=i}^{N-1} \hbar_{\sigma,j} f_{\sigma,\, j-1/2} \right|, \end{aligned}$$

which is the required bound. □

5.1.1. **ERROR ESTIMATES OF THE NUMERICAL SOLUTION**

THEOREM 5.2. *The error of the monotone hybrid difference scheme satisfies the following bound:*

$$\|u - U^N\| \le C \max_i h_i^2 + C\left(\max_i \int_{x_{i-1}}^{x_i} (1 + \varepsilon^{-1} \exp(-\alpha x / 2\varepsilon) \mathrm{d}x) \right)^2. \tag{38}$$

Proof. Let $z_i = U_i^N - u(x_i)$ be the error function, then

$$\|U^N - u\| \le C \max_i \left| \sum_{j=i}^{N-1} \hbar_{\sigma,j} (L_\varepsilon^N z)_j \right|. \tag{39}$$

Integrating the BVP (1) from $x_{\sigma,j}$ to $x_{\sigma,j+1}$, we get

$$(Au)(x_{\sigma,j+1}) - (Au)(x_{\sigma,j}) - \int_{x_{\sigma,j}}^{x_{\sigma,j+1}} f(x) \mathrm{d}x = 0, \tag{40}$$

where $(Au)(x) = \varepsilon u'(x) + a(x)u(x)$. Now

$$\hbar_{\sigma,j} [L_\varepsilon^N z]_j = \varepsilon \left[u'_{j+1} - \frac{u_{j+1} - u_j}{h_{j+1}} \right] - \varepsilon \left[u'_j - \frac{u_j - u_{j-1}}{h_j} \right] + \sigma_i a_j u_j +$$

$$(1-\sigma_i) a_{j-1} u_{j-1} - a(x_{\sigma,j-1/2}) u(x_{\sigma,j-1/2}) - \int_{x_{\sigma,j}}^{x_{\sigma,j+1}} f(x) \mathrm{d}x + \hbar_{\sigma,j} f_{\sigma,j-1/2}.$$

Combining the above equation with (39), we have

$$\|u - U^N\| \le C \max_i \left| [A^N u]_i - [Au](x_{\sigma,i}) \right| + C \max_i \left| \hbar_{\sigma,i} f_{\sigma,i-1/2} - \int_{x_{\sigma,i}}^{x_{\sigma,i+1}} f(x) \mathrm{d}x \right|. \tag{41}$$

From (8), we get

$$\|u-U^N\| \le C\max_i \Big|[A^N u]_i - [Au](x_{\sigma,i})\Big| + C\max_i h_i^2. \tag{42}$$

Here, we have two cases for σ_i.

Case 1. For $\sigma_i = 1/2$, we have

$$\begin{aligned}\Big|[A^N u]_i - [Au](x_{\sigma,i})\Big| &\le C\varepsilon \int_{x_{i-1}}^{x_i} |u'''(x)|(x-x_{i-1})\mathrm{d}x \\ &\quad + C\int_{x_{i-1}}^{x_i} |u''(x)|(x-x_{i-1})\mathrm{d}x.\end{aligned} \tag{43}$$

Differentiating once the BVP (1), we get $\varepsilon u''' = -f - a''u - 2a'u' - au''$ which implies that $|\varepsilon u'''| \le C(1+u'')$. So we have,

$$\begin{aligned}\Big|[A^N u]_i - [Au](x_{\sigma,i})\Big| &\le C\varepsilon \int_{x_{i-1}}^{x_i} (x-x_{i-1})\mathrm{d}x + C\int_{x_{i-1}}^{x_i} |u''(x)|(x-x_{i-1})\mathrm{d}x \\ &\le Ch_i^2 + C\int_{x_{i-1}}^{x_i} (1+\varepsilon^{-2}\exp(-\alpha x/\varepsilon))(x-x_{i-1})\mathrm{d}x \\ &\le Ch_i^2 + C\int_{x_{i-1}}^{x_i} \varepsilon^{-2}\exp(-\alpha x/\varepsilon)(x-x_{i-1})\mathrm{d}x.\end{aligned}$$

By using the inequality $\int_a^b \psi(t)(t-a)^{k-1}\mathrm{d}t \le \frac{1}{k}\Big[\int_a^b \psi(t)^{1/k}\mathrm{d}t\Big]^k$, we have

$$\begin{aligned}\Big|[A^N u]_i - [Au](x_{\sigma,i})\Big| &\le Ch_i^2 + C\Big[\int_{x_{i-1}}^{x_i} \varepsilon^{-1}\exp(-\alpha x/2\varepsilon)\mathrm{d}x\Big]^2 \\ &\le Ch_i^2 + C\Big[\int_{x_{i-1}}^{x_i} (1+\varepsilon^{-1}\exp(-\alpha x/2\varepsilon))\mathrm{d}x\Big]^2.\end{aligned}$$

Case 2. If $\sigma_i = 1$, then

$$\Big|[A^N u]_i - [Au](x_{\sigma,i})\Big| \le C\frac{\varepsilon}{(x_i - x_{i-1})}\int_{x_{i-1}}^{x_i} |u''(x)|(x-x_{i-1})\mathrm{d}x. \tag{44}$$

From the definition of σ_i, we have $\varepsilon/(x_i - x_{i-1}) \le \alpha^*/2$. Thus,

$$\begin{aligned}\Big|[A^N u]_i - [Au](x_{\sigma,i})\Big| &\le C\int_{x_{i-1}}^{x_i} (1+\varepsilon^{-2}\exp(-\alpha x/\varepsilon))(x-x_{i-1})\mathrm{d}x \\ &\le Ch_i^2 + C\int_{x_{i-1}}^{x_i} \varepsilon^{2}\exp(-\alpha x/\varepsilon)(x-x_{i-1})\mathrm{d}x.\end{aligned}$$

Proceeding as we have done for $\sigma_i = 1/2$, we obtain

$$\Big|[A^N u]_i - [Au](x_{\sigma,i})\Big| \le Ch_i^2 + C\Big[\int_{x_{i-1}}^{x_i} (1+\varepsilon^{-1}\exp(-\alpha x/2\varepsilon))\mathrm{d}x\Big]^2. \tag{45}$$

Thus we have the required estimate. □

5.1.2. **ERROR ESTIMATES OF THE NORMALIZED FLUX**

In many real life application, the approximation of the derivatives are desirable such as normal derivatives are required to compute the skin friction coefficients and to calculate the stress intensity factors. There are very few research articles available approximating the derivatives (refer[7, 21, 22]). The approximation of unweighted derivative outside layer was dealt in [7] and the approximated derivative in the entire domain on Shishkin mesh and Bakhvalov mesh in [15]. The uniform first order convergent upwind finite difference scheme using grid equidistribution to approximate the normalized flux was developed in [21]. Recently, a second order convergent scheme is proposed to compute the derivative on S-type mesh for quasilinear SPP in [30]. Here, we shall derive the bounds of ε-weighted derivative errors on the adaptive grid.

THEOREM 5.3. *There exists a constant C independent of ε and mesh points such that*

$$\begin{aligned} &\max_i \varepsilon|D^- U_i^N - u'(x_{i-1/2})| \leq \\ &\qquad C \max_i h_i^2 + C\left(\max_i \int_{x_{i-1}}^{x_i} (1+\varepsilon^{-1}\exp(-\alpha x/2\varepsilon))\mathrm{d}x \right)^2. \end{aligned} \tag{46}$$

Proof. By Taylor's series expansion, we have

$$\begin{aligned} \varepsilon|D^- U_i^N - u'(x_{i-1/2})| &\leq \varepsilon|D^- U_i^N - D^- u_i| + \varepsilon|D^- u_i - u'(x_{i-1/2})| \\ &\leq \varepsilon|D^- z_i| + \left| \frac{3\varepsilon}{2} \int_{x_{i-1}}^{x_i} u'''(x)(x - x_{i-1})\mathrm{d}x \right| \\ &\leq C \max_i \left| \sum_{j=i}^{N-1} \hbar_{\sigma,j} [L_\varepsilon^N z_j] \right| \\ &+ \max_i \int_{x_i}^{x_{i-1}} (1+\varepsilon^{-2}\exp(-\alpha x/\varepsilon))\mathrm{d}x. \end{aligned}$$

From the proof of Theorem 5.2, we have the desired bound. □

The main contribution of this paper is to derive a second order uniform convergence results for approximating the numerical solution and the weighted normalized flux by the proposed scheme on the adaptive grid. Here, we give an outline of the convergence results on S-type mesh.

Theorem 5.4. *Let $u = \{u_i\}_{i=0}^N$ and $U^N = \{U_i^N\}_{i=0}^N$ be the exact solution and the discrete solution obtained by the monotone hybrid method on S-type meshes respectively. Then,*

$$\begin{aligned} \|u - U^N\| &\leq CN^{-2}\ln^2 N, \quad (S\text{-}mesh) \\ \|u - U^N\| &\leq CN^{-2}, \quad (B\text{-}S\text{-}mesh) \end{aligned}$$

Proof. The detailed proof of the parameter uniform error estimates of the monotone hybrid method for BVP (1) on Shishkin mesh and Bakhvalov mesh was carried out by Andreev and Kopteva in [1]. □

Theorem 5.5. *The numerical derivative of BVP (1) on S-type meshes satisfies the following estimates:*

$$\max_i \varepsilon|D^- U_i^N - u'_{i-1/2}| \leq CN^{-2}\ln^2 N, \quad (S\text{-}mesh)$$

$$\max_i \varepsilon|D^- U_i^N - u'_{i-1/2}| \leq CN^{-2} \quad (B\text{-}S\text{-}mesh)$$

Proof. From Lemma 3.1 and Theorem 5.3, we get the desired estimates. □

Now, the convergence result of Theorem 5.2 and Theorem 5.3 is used to conclude the parameter uniform second order convergence for approximating the solution and the derivative on the adaptive grid.

Theorem 5.6. *Let $u = \{u_i\}_{i=0}^N$ and $U^N = \{U_i^N\}_{i=0}^N$ be the exact solution and the discrete solution obtained by the monotone hybrid method on adaptive grid respectively. Then, we have the following estimates:*

$$\|u - U^N\| \leq CN^{-2}, \tag{47}$$

and

$$\max_i \varepsilon|D^- U_i^N - u'_{i-1/2}| \leq CN^{-2}. \tag{48}$$

Proof. Combining Lemma 3.3 and Theorem 5.2 with Theorem 5.3, one can obtain the desired error estimates. □

5.1.3. CONVERGENCE OF THE GLOBAL SOLUTIONS

Theorem 5.7. *Let $u = \{u_i\}_{i=0}^N$ be the solution of problem (1) and $U^N = \{U_i^N\}_{i=0}^N$ be the solution obtained by the monotone hybrid scheme on the adaptive grid. Then for sufficiently large N, we have the following bound*

$$\|u - \widetilde{U}\| \leq CN^{-2},$$

where $\widetilde{U}$ is a piecewise linear interpolation of U^N on $[0, 1]$.

Proof. Let $\widetilde{U}(x)$ denotes the piecewise polynomial and is defined as $\widetilde{U}(x) = \sum_{i=0}^{N} U_i^N \rho_i(x)$, where $\rho_i(x)$ is the piecewise linear function denoted by $\rho_i(x_j) = \delta_{i,j}$ for $0 \le i, j \le N$. By using the triangle inequality,

$$\|u - \widetilde{U}\| \le \|\widetilde{u} - \widetilde{U}\| + \|\widetilde{u} - u\|, \tag{49}$$

As, $\widetilde{u} - \widetilde{U} = \sum_{i=0}^{N}(u_i - U_i^N)\rho_i(x)$, where $\rho_i(x) \ge 0$ and $\|\sum_{i=0}^{N} \rho_i(x)\| \le 1$, thus from Theorem 5.6

$$\|\widetilde{u} - \widetilde{U}\| \le CN^{-2}. \tag{50}$$

Now to bound $\|\widetilde{u} - u\|$, we use the error estimate of linear interpolation. Thus,

$$\|\widetilde{u} - u\| \le 2h_i \int_{x_i}^{x_{i-1}} |u''(x)|\mathrm{d}x.$$

From the bound of the solution, we deduce

$$\begin{aligned}
\|\widetilde{u} - u\| &\le Ch_i \int_{x_i}^{x_{i-1}} (1 + \varepsilon^{-2}\exp(-\alpha x/\varepsilon))\mathrm{d}x \\
&\le C\left(h_i^2 + h_i\varepsilon^{-2} \int_{x_i}^{x_{i-1}} (\exp(-\alpha x/2\varepsilon)\mathrm{d}x\right) \\
&\le C\left(N^{-2} + N^{-1}\varepsilon \int_{x_{i-1}}^{x_i} (1 + |u_{xx}(x)|^{1/2})\mathrm{d}x\right).
\end{aligned}$$

Using the equidistribution principle, we obtain

$$\|\widetilde{u} - u\| \le C\left(N^{-2} + \varepsilon N^{-2} \int_0^1 M(u(x), x)\mathrm{d}x\right) \le CN^{-2}.$$

Thus, we have desired bound. □

5.2. SYSTEM OF CONVECTION DIFFUSION PROBLEMS

THEOREM 5.8. *Let* $\mathbf{u}$ *be the solution of (6) and* $\mathbf{U}$ *be its approximation obtained by the scheme (23) on any arbitrary non uniform grid then*

$$\|\mathbf{u}-\mathbf{U}\| \le C\left\{\max_i h_i^2 + \left\{\max_i \int_{x_{i-1}}^{x_i} (1+\sum_{k=1}^{2} \varepsilon_1^{-k}\exp(-\alpha_k x/2\varepsilon_k))\mathrm{d}x\right\}^2\right\}. \tag{51}$$

Proof. We know from the stability inequality (refer Theorem 1 of Andreev and Kopteva [1], Lemma 1 of Linß[17]) that

$$\|v\| \le C \min_{V:V_x=\mathcal{L}_\mu v} \|V\|, \quad \mu = 1, 2, \qquad \text{for all } v \in \mathbb{R}_0^{N+1}. \tag{52}$$

Introducing the continuous operator, discrete operator and functions as:

$$\mathcal{B}_1 v_1 = \varepsilon_1 v_1' + a_1 v_1, \quad \mathcal{F}_1 = \int_{\cdot}^{1} f_1(s)\mathrm{d}s, \tag{53}$$

(54) $$\mathcal{B}_2 v_2 = \varepsilon_2 v_2' + a_2 v_2 + b_2 v_2, \quad \mathcal{F}_2 = \int_.^1 f_2(s)\mathrm{d}s,$$

$$\begin{aligned} B_1 v_{1,i} &= \varepsilon_1 \frac{v_{1,i} - v_{1,i-1}}{h_i} + \sigma_{1,i} a_{1,i} v_{1,i} + (1-\sigma_{1,i}) a_{1,i-1} v_{1,i-1}, \\ F_{1,\sigma_1,i-1/2} &= \sum_{j=i}^{N-1} \hbar_{\sigma_1,i-1/2} f_{1,\sigma_1,i-1/2}, \end{aligned}$$

$$\begin{aligned} B_2 v_{2,i} &= \varepsilon_2 \frac{v_{2,i} - v_{2,i-1}}{h_i} + \sigma_{2,i} a_{1,i} v_{2,i} + (1-\sigma_{2,i}) a_{2,i-1} v_{2,i-1} \\ &+ \sigma_{2,i} b_{1,i} v_{1,i} + (1-\sigma_{2,i}) b_{1,i-1} v_{1,i-1}, \\ F_{2,\sigma_2,i-1/2} &= \sum_{j=i}^{N-1} \hbar_{\sigma_2,i-1/2} f_{2,\sigma_2,i-1/2}. \end{aligned}$$

Note that $\mathcal{L}_k v_k = -(\mathcal{B}_k v_k)'$ and $f_k = \mathcal{F}_k'$ and $\mathcal{L}_k v_k = -D(B_k v_k)$ and $f_k = DF_k$ on Ω^N.

Thus $\mathcal{B}_k \mathbf{u} - \mathcal{F}_k = \xi$ and $B_k \widetilde{\mathbf{U}} - F_k = \zeta$ on Ω^N with constant ξ and ζ. Applying the stability inequality, we have

(55) $$\|\mathbf{u} - \mathbf{U}\| \leq C \min_{c \in \mathbb{R}} \|\mathcal{B}_k(\mathbf{u} - \mathbf{U}) + c\|.$$

Taking $c = \zeta - \xi$, where ζ and ξ are constants, we get

(56) $$\|\mathbf{u} - \mathbf{U}\| \leq C\|B_k \mathbf{u} - \mathcal{B}_k \mathbf{u} - F_k + \mathcal{F}_k\|,$$

Here, we have two cases for $\sigma_{k,i}$.

Case 1. For $\sigma_{k,i} = 1/2$, we have

$$\begin{aligned} &\max_i |B_1 u_{1,i} - (\mathcal{B}_1 u_1)(x_{\sigma_1,i})| \\ &\leq \frac{3\varepsilon_1}{2} \left| \int_{x_{i-1}}^{x_i} u_1'''(x)(x - x_{i-1})\mathrm{d}x \right| + C_1 \left| \int_{x_{i-1}}^{x_i} u_1''(x)(x - x_{i-1})\mathrm{d}x \right|, \\ &\leq C \int_{x_{i-1}}^{x_i} (1 + \varepsilon_1^{-2} \exp(-\alpha_1 x/\varepsilon_1))(x - x_{i-1})\mathrm{d}x. \end{aligned}$$

and

$$\begin{aligned} &\max_i |B_2 u_{2,i} - (\mathcal{B}_2 u_2)(x_{\sigma_2,i})| \\ &\leq \frac{3\varepsilon_2}{2} \left| \int_{x_{i-1}}^{x_i} u_2'''(x)(x - x_{i-1})\mathrm{d}x \right| + C_2 \left| \int_{x_{i-1}}^{x_i} u_2''(x)(x - x_{i-1})\mathrm{d}x \right| \\ &+ C_3 \left| \int_{x_{i-1}}^{x_i} u_1''(x)(x - x_{i-1})\mathrm{d}x \right|, \end{aligned}$$

$$\leq C \int_{x_{i-1}}^{x_i} (1 + \varepsilon_1^{-2} \exp(-\alpha_1 x/\varepsilon_1) + \varepsilon_2^{-2} \exp(-\alpha_2 x/\varepsilon_2))(x - x_{i-1}) \mathrm{d}x.$$

Case 2. If $\sigma_{k,i} = 1$, then

$$\max_i |B_k u_{k,i} - (\mathcal{B}_k u_k)(x_{\sigma_k,i})| \leq \frac{C_4 \varepsilon_k}{(x_i - x_{i-1})} \left| \int_{x_{i-1}}^{x_i} u_k''(x)(x - x_{i-1}) \mathrm{d}x \right|$$

From the choice of $\sigma_{k,i} = 1$, we know that $\dfrac{\varepsilon_k}{(x_i - x_{i-1})} \leq \dfrac{\alpha_k^*}{2}$. Thus

$$\begin{aligned}
&\max_i |B_k u_{k,i} - (\mathcal{B}_k u_k)(x_{\sigma_k,i})| \\
&\leq \frac{C_4 \varepsilon_k}{(x_i - x_{i-1})} \left| \int_{x_{i-1}}^{x_i} u_k''(x)(x - x_{i-1}) \mathrm{d}x \right| \\
&\leq C \int_{x_{i-1}}^{x_i} (1 + \varepsilon_1^{-2} \exp(-\alpha_1 x/\varepsilon_1) + \varepsilon_2^{-2} \exp(-\alpha_2 x/\varepsilon_2))(x - x_{i-1}) \mathrm{d}x
\end{aligned}$$

By using the inequality, $\int_a^b g(x)(x-a)^{k-1} \mathrm{d}x \leq \dfrac{1}{k} \left[\int_a^b g(x)^{1/k} \mathrm{d}x \right]^k$, which holds true for any positive monotonically decreasing function g on $[a, b]$.

$$\begin{aligned}
&\max_i |B_k u_{k,i} - (\mathcal{B}_k u_k)(x_{\sigma_k,i}) \\
&\leq C \left[\int_{x_{i-1}}^{x_i} (1 + \varepsilon_1^{-1} \exp(-\alpha_1 x/2\varepsilon_1) + \varepsilon_1^{-1} \exp(-\alpha_2 x/2\varepsilon_2)) \mathrm{d}x \right]^2. \qquad (57)
\end{aligned}$$

And for all i, we have

$$\begin{aligned}
|F_{k,\sigma_k,i-1/2} - \mathcal{F}(x_{\sigma_{k,i}})| &= \left| \sum_{i=1}^{N-1} \left\{ \int_{x_{\sigma_k,i}}^{x_{\sigma_k,i+1}} f_k(x) \mathrm{d}x - \hbar_{\sigma_2,i} f_{k,\sigma_{k,i-1/2}} \right\} \right| \\
&= O(\hbar_{\sigma_k}^2) \qquad (58)
\end{aligned}$$

Thus we get the desired estimate. □

The following theorem sates the main result of the proposed method.

THEOREM 5.9. *Let* **u** *be the solution of the system (6) and* **U** *be the numerical solution of the corresponding discretized problem (23), on the mesh piecewise uniform Shishkin mesh* Ω^N. *Then*

$$\|\mathbf{u} - \mathbf{U}\| \leq C N^{-2} \ln^2 N.$$

Proof. Combining the estimates given in Theorem (29) and (5.8), we obtain the desired estimate. □

THEOREM 5.10. *There exists a constant C such that*

(59) $$\|\mathbf{u}-\mathbf{U}\| \le C\Big\{ \max_i \max_{x\in[x_{i-1},x_i]} \sum_{k=1}^{2} h_i^2 \varepsilon_k |u_k'''(x)| + \max_i \max_{x\in[x_{i-1},x_i]} h_i^2 [1+\sum_{k=1}^{2} |u_k''(x)|] \Big\}.$$

Proof. For $\sigma_{k,i}=\frac{1}{2}$,

$$\begin{aligned} B_1u_{1,i}-(\mathcal{B}_1u_1)(x_{\sigma_1,i}) &= \frac{\varepsilon_1 h_i^2}{24}u_1'''(\xi_{1,i}) + \frac{h_i^2}{4}u_1''(\eta_{1,i}),\\ B_2u_{2,i}-(\mathcal{B}_2u_2)(x_{\sigma_2,i}) &= \frac{\varepsilon_2 h_i^2}{24}u_2'''(\xi_{2,i}) + \frac{h_i^2}{4}u_1''(\eta_{21,i}) + \frac{h_i^2}{4}u_2''(\eta_{22,i}). \end{aligned}$$

On the other hand if $\sigma_{k,i}=1$,

$$B_ku_{k,i}-(\mathcal{B}_ku_k)(x_{\sigma_k,i}) = \frac{\varepsilon_k}{h_i}\left(\frac{h_i^2}{2}u_k''(\chi_{k,i})\right), \text{for } k=1,2,$$

where $\xi_{1,i},\ \xi_{2,i},\ \eta_{1,i},\ \eta_{21,i},\ \chi_{1,i},\ \chi_{2,i} \in [x_{i-1},x_i]$. Now, we apply these relation and (58) to the stability inequality (56) and the desired result follows. □

Remark 5.11. The estimate (59) equivalently can be written as

$$\|\mathbf{u}-\mathbf{U}\| \le C\max_i h_i^2\left[1+\sum_{k=1}^{2}\left(\varepsilon_k^{1/2}|u_k'''(x)|^{1/2}+|u_k''(x)|^{1/2}\right)\right]^2.$$

Furthermore, differentiating (6) and taking the boundedness of $a_1(x),\ a_2(x)$ and $b_1(x)$, we obtain,

$$\|\mathbf{u}-\mathbf{U}\| \le C\max_i h_i^2\left[1+\sum_{k=1}^{2} C_k|u_k''(x)|^{1/2}\right]^2.$$

By choosing sufficient large 'C', we have

$$\|\mathbf{u}-\mathbf{U}\| \le C\max_i h_i^2\left[1+\sum_{k=1}^{2} |u_k''(x)|^{1/2}\right]^2.$$

Further, we make the reasonable assumption that the discrete analogues of the above inequality holds true and gives sharp bounds on the error in the computed solution. So,

(60) $$\|u_k-U_k\| \le C\max_i \mathbf{h_i^2}\left[1+\sum_{k=1}^{2} |\delta^2 u_{k,i}|^{1/2}\right]^2,$$

where $\delta^2 U_{k,i} = \dfrac{2}{h_{i+1}+h_i}\left[\dfrac{U_{k,i+1}-U_{k,i}}{h_{i+1}} - \dfrac{U_{k,i}-U_{k,i-1}}{h_i}\right]$. Thus, it is clear that which monitor function is needed to be taken into consideration. Also, from the equidistribution principle we achieve a second order accuracy on the adaptive grid. This provide the optimal rate of convergence for the proposed scheme. For computational purpose, we have chosen the monitor function as $M_i = 1 + \sum_{k=1}^{2} |\delta^2 u_{k,i}|^{1/2}$.

6. NUMERICAL RESULTS

In this section, we present the numerical results to illustrate the efficiency and accuracy of the proposed method described in this paper. We have verified the proposed scheme on three test problems. The maximum pointwise errors and the corresponding rates of convergence are presented through tables and figures.

Example 6.1. Consider the following SPP:

$$\begin{cases} -\varepsilon u'' - u' = (\frac{\varepsilon\pi^2 x}{2} - 2)\cos(\frac{\pi x}{2}) + \pi(2\varepsilon + x)\sin(\frac{\pi x}{2}), \quad x \in \Omega = (0,1)\,, \\ u(0) = 1, \quad u(1) = 0. \end{cases} \tag{61}$$

The exact solution is $u(x) = \dfrac{\exp(-x/\varepsilon) - \exp(-1/\varepsilon)}{1-\exp(-1/\varepsilon)} + 2x\cos(\pi x/2)$. Now, we calculate the maximum pointwise error and the corresponding rate of convergence by

$$E_\varepsilon^N = \max_i |u(x_i) - U_i^N|, \quad r_\varepsilon^N = \log_2\left(\frac{E_\varepsilon^N}{E_\varepsilon^{2N}}\right),$$

where $u(x_i)$ and U_i^N denotes the exact and numerical solution obtained on the mesh Ω^N with N number of mesh subintervals. Similarly, we can define the scaled error associated with normalized flux and the corresponding rate of convergence as:

$$D_\varepsilon^N = \max_i \varepsilon |D^- U_i^N - u'_{i-1/2}|, \quad p_\varepsilon^N = \log_2\left(\frac{D_\varepsilon^N}{D_\varepsilon^{2N}}\right).$$

Example 6.2. Consider the following test problem:

$$\begin{cases} \varepsilon u'' + (1 + x(1-x))u' = \exp(x), \quad x \in \Omega = (0,1)\,, \\ u(0) = 0, \quad u(1) = 1. \end{cases} \tag{62}$$

The exact solution is not available for BVP (63). In order to calculate the maximum pointwise error, the scaled error and the corresponding rate of convergence, we use the idea of interpolation. Define $\overline{U}_i^{2N}$ as the piecewise linear interpolate to U_i^N in Ω^N. For any value of N, the maximum pointwise error $\overline{E}_\varepsilon^N$ of the numerical solution and the scaled error $\overline{D}_\varepsilon^N$ of the normalized flux is calculated by $\overline{E}_\varepsilon^N = \max_i |U_i^N - \overline{U}_i^{2N}|$ and $\overline{D}_\varepsilon^N = \max_i \varepsilon |D^- U_i^N - D^- \overline{U}_i^{2N}|$. To verify the accuracy, the convergence rate of the numerical solution and the numerical derivative is computed by $\overline{r}_\varepsilon^N = \log_2\left(\frac{\overline{E}_\varepsilon^N}{\overline{E}_\varepsilon^{2N}}\right)$ and $\overline{p}_\varepsilon^N = \log_2\left(\frac{\overline{D}_\varepsilon^N}{\overline{D}_\varepsilon^{2N}}\right)$.

In order to show the efficiency of the proposed method, we have solved Example 6.1 and Example 6.2 both on S-type meshes and on the adaptive grid. The corresponding computational results are given in Table 1 and Table 2 respectively for different value of ε and N. One can easily compare the results and observe that the parameter uniform second order convergence of the proposed method. Further, to compare the theoretical estimates with numerical order of convergence, we have plotted the maximum pointwise errors in log-log scale for Example 6.2 in Figure 1, obtained by proposed method on adaptive grid along with the results on S-type meshes. This ensures the effectiveness of the proposed scheme on the adaptive grid over S-type meshes. We have also presented the numerical results of the normalized flux by the proposed method on the adaptive grid for Example 6.1 and 6.2 in Table 3 and 4 respectively. The errors on adaptive grid and B-S-mesh are second order convergent while on S-mesh it is less than second order convergent due to the logarithmic factor. The error behaviour of the computational results within layer and outer region of the numerical solution and the normalized flux on adaptive grid along with on S-type mesh for $\varepsilon = 1e-3$ with $N = 32$ are shown in Figure 2.

Our next problem is a system of convection diffusion problem, which is a well discussed example (refer [4]).

Example 6.3. Consider the following system of singularly perturbed problems:

(63)
$$\begin{cases} \varepsilon_1 u_1''(x) + 3u_1'(x) = 15x^4, \quad u_1(0) = 0, \quad u_1(1) = 0, \\ \varepsilon_2 u_2''(x) + 2u_2'(x) + 2.75u_2'(x) = 0.6\exp(x), \quad u_2(0) = 0, \quad u_2(1) = 0. \end{cases}$$

Note that $u_1(x) \approx x^5 - 1$, $u_2(x) \approx 0.3(\exp(x) - \exp(1)) - 1.375(x^5 - 1)$ and hence $\|u_1\| \approx 1$ and $\|u_2\| \approx 1$. We calculate the exact maximum pointwise errors E_{ε_k,u_k}^N for the solution component u_k, $k = 1, 2$ by

$$E_{\varepsilon_k,u_k}^N = \max_{0 \le i \le N} |u_k(x_i) - U_{k,i}^N|,$$

where $u_k(x_i)$ is the exact solution and $U_{k,i}^N$ is the numerical solution at the mesh point x_i obtained by using N number of mesh intervals in the domain Ω^N. The corresponding rates of convergence is calculated by the formula $r^N_{\varepsilon_k,u_k} = \log_2\left(\frac{E^N_{\varepsilon_k,u_k}}{E^{2N}_{\varepsilon_k,u_k}}\right)$.

For all our experiment shown in tables and figures, we take various values of ε_1, and fixed value of $\varepsilon_2 = 10^{-4}$ as done usually. In Tables 5 and 6, we present the maximum pointwise error and the corresponding rate of convergence for the solution components u_1 and u_2 of Example 6.3 respectively on Shishkin mesh. This clearly indicates that the proposed method is ε-uniform convergent and almost second order accurate. The maximum pointwise error and corresponding rate of convergence for the solution components u_1 and u_2 of Example 6.3 on adaptive grid is shown in Tables 7 and 8. From these tables one can clearly visualize the second order accuracy of the proposed method on adaptive grid which produces more accuracy than error produced on Shishkin mesh. In Figures 3 and 4, the maximum pointwise errors versus number of mesh intervals for Example 6.3 is plotted in logarithmic scale for $\varepsilon_1 = 2^{-35}$ for solution component u_1 and u_2 respectively. These figures also suggest that the computed errors are decreasing with rate $O(N^{-2}\ln^2 N)$ on Shishkin mesh and $O(N^{-2})$ on the adaptive grid as the number of interval N increases. In Figure 5, we display the error plot for Example 6.3 which shows that the maximum error occurs only in the boundary layer region.

7. CONCLUDING REMARKS

In this article, we present an analysis of the discretization of singularly perturbed convection-diffusion problems by using a monotone hybrid scheme with variable weights on layer adapted meshes. The numerical solution is obtained on various nonuniform meshes viz Shishkin type meshes and the adaptive grid. To generate Shishkin mesh, one should have *aprior* information of the width and location of the boundary layer. While the advantage of the mesh generated by the adaptive technique using the equidistribution principle does not require any such *apriori* knowledge about the location and width of the layer. Also from the analysis it is shown both theoretically and computationally that the adaptive grid technique leads to an optimal parameter uniform convergence corresponding to the monotone hybrid discretization. Numerical results shown confirm the effectiveness of the proposed scheme and support the theoretical error estimates.

Table 1 – E_ε^N and the corresponding r_ε^N for Example 6.1

Method	N	$\varepsilon = 1e-6$		$\varepsilon = 1e-8$		$\varepsilon = 1e-10$	
		E_ε^N	r_ε^N	E_ε^N	r_ε^N	E_ε^N	r_ε^N
S-mesh	16	5.9522e-3	1.7698	5.9523e-3	1.7688	5.9523e-3	1.7688
	32	1.7455e-3	1.8815	1.7455e-3	1.8815	1.7455e-3	1.8815
	64	4.7373e-4	1.9439	4.7374e-4	1.9439	4.7374e-4	1.9439
	128	1.2313e-4	1.6727	1.2313e-4	1.6727	1.2313e-4	1.6727
	256	3.8623e-5	1.5267	3.8623e-5	1.5267	3.8623e-5	1.5267
	512	1.3405e-5	1.6051	1.3405e-5	1.6051	1.3405e-5	1.6051
B-S-mesh	16	5.8052e-3	1.7338	5.8054e-3	1.7338	5.8054e-3	1.7338
	32	1.7454e-3	1.8815	1.7455e-3	1.8815	1.7455e-3	1.8815
	64	4.7372e-4	1.9440	4.7374e-4	1.9439	4.7374e-4	1.9439
	128	1.2312e-4	1.9727	1.2313e-4	1.9727	1.2313e-4	1.9727
	256	3.1368e-5	1.9865	3.1370e-5	1.9865	3.1370e-5	1.9865
	512	7.9155e-6	1.9933	1.9882e-6	1.9933	1.9882e-6	1.9933
Adaptive grid	16	9.7642e-3	2.0713	9.8051e-3	2.0769	9.8056e-3	2.0769
	32	2.3234e-3	2.0238	2.3241e-3	2.0239	2.3241e-3	2.0240
	64	5.7135e-4	1.9670	5.7146e-4	1.9669	5.7143e-4	1.9668
	128	1.4614e-4	1.9867	1.4618e-4	1.9846	1.4618e-4	1.9853
	256	3.6874e-5	2.0096	3.6936e-5	2.0249	3.6919e-5	2.0043
	512	9.1573e-6	1.9883	9.0757e-6	1.9749	9.2025e-6	1.9949

Table 2 – $\overline{E}_\varepsilon^N$ and the corresponding $\overline{r}_\varepsilon^N$ for Example 6.2

N	$\varepsilon = 1e-4$			$\varepsilon = 1e-8$		
	S-Mesh	B-S-mesh	Adaptive grid	S-mesh	B-S-mesh	Adaptive grid
16	5.9169e-3 1.4664	3.9954e-4 1.8614	1.0054e-2 2.0691	5.9113e-3 1.4677	3.8138e-4 1.9325	1.0502e-2 2.1454
32	2.1413e-3 1.5359	1.0996e-4 1.8322	2.3960e-3 2.1091	2.1373e-3 1.5932	9.9912e-5 1.9654	2.3738e-3 2.0879
64	7.3845e-4 1.5903	3.0881e-5 1.7479	5.5537e-4 2.0149	7.3618e-4 1.5932	2.5585e-5 1.9824	5.5837e-4 2.0167
128	2.4524e-4 1.6344	9.1945e-6 1.6124	1.3742e-4 1.9997	2.4399e-4 1.6389	2.5585e-5 1.9911	1.3799e-4 2.0206
256	7.8994e-5 1.6694	3.0072e-6 1.4478	3.4362e-5 1.9738	7.8346e-5 1.6767	1.6287e-6 1.9954	3.4008e-5 2.0023
512	2.4835e-5 1.6953	1.1024e-6 1.2912	8.7482e-6 1.9531	2.4506e-5 1.7079	4.0848e-7 1.9976	8.4885e-6 2.0055
1024	7.6688e-6	4.5044e-7	2.2593e-6	7.5015e-6	1.0229e-7	2.1141e-6

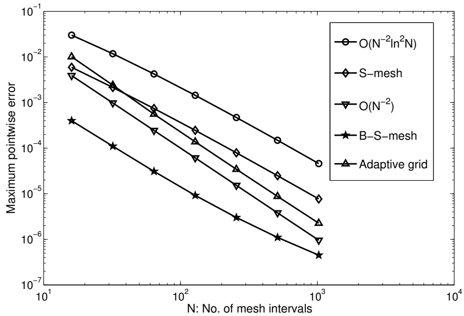


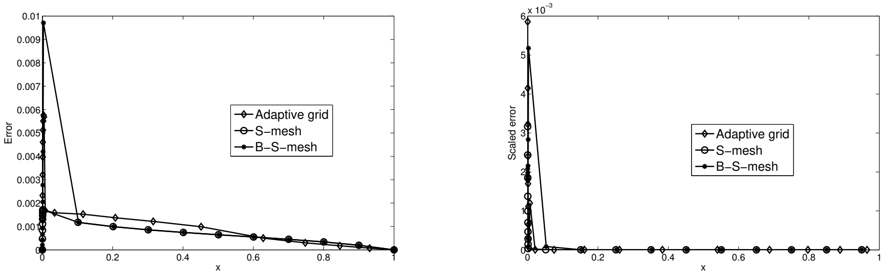
Fig. 1 – loglog plot of maximum point-wise errors for Example 6.2.

Table 3 – D_ε^N and the corresponding p_ε^N for Example 6.1 for $\varepsilon = 10^{-8}$

N	S-mesh		B-S-mesh		Adaptive grid	
	D_ε^N	p_ε^N	D_ε^N	p_ε^N	D_ε^N	p_ε^N
16	3.1202e-2	1.0023	7.7913e-3	1.9637	1.1057e-2	2.2162
32	1.5576e-2	1.0047	1.9975e-3	1.9826	2.3795e-3	2.0553
64	7.7628e-3	1.0094	5.0543e-4	1.9915	5.7251e-4	1.9703
128	3.8563e-3	1.0189	1.2710e-4	1.9958	1.4610e-4	1.9845
256	1.9031e-3	1.0384	3.1867e-5	1.9979	3.6919e-5	2.0246
512	9.2655e-4	1.0801	7.9781e-6	1.9989	9.0739e-6	1.9747

Table 4 – $\overline{D}_\varepsilon^N$ and the corresponding $\overline{p}_\varepsilon^N$ for Example 6.2 for $\varepsilon = 10^{-8}$

N	S-mesh		B-S-mesh		Adaptive grid	
	D_ε^N	p_ε^N	D_ε^N	p_ε^N	D_ε^N	p_ε^N
16	1.0744e-2	1.2652	6.7515e-3	1.8459	1.0462e-2	2.1370
32	4.4701e-3	1.3979	1.8781e-3	1.9210	2.3785e-3	2.0945
64	1.6963e-3	1.5041	4.9595e-4	1.9599	5.5691e-4	2.0007
128	5.9805e-4	1.5839	1.2748e-4	1.9799	1.3916e-4	2.0208
256	1.9949e-4	1.6434	3.2317e-5	1.9906	3.4292e-5	1.9983
512	6.3859e-5	1.6874	8.1319e-6	1.9998	8.5831e-6	2.0138



(a) Error *(b) Scaled error*

Fig. 2 – Error and scaled error for Example 6.2 with $\varepsilon = 1e-3$.

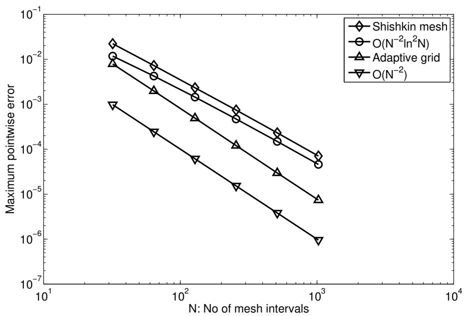


Fig. 3 – loglog plot of maximum point-wise errors for u_1 for Example 6.3 .

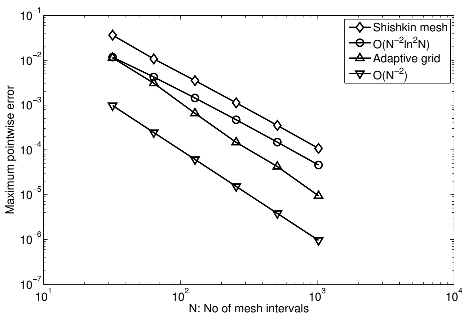


Fig. 4 – loglog plot of maximum point-wise errors for u_2 for Example 6.3.

Table 5 – *$E^N_{\varepsilon_1,u_1}$ and $r^N_{\varepsilon_1,u_1}$ for Example 6.3 on Shishkin mesh*

ε_1	Number of intervals N					
	32	64	128	256	512	1024
2^{-15}	2.2143e-2 1.6120	7.2440e-3 1.6450	2.3163e-3 1.6419	7.4222e-4 1.6850	2.3084e-4 1.7104	7.0540e-5
2^{-19}	2.2142e-2 1.6120	7.2434e-3 1.6450	2.3160e-3 1.6421	7.4202e-4 1.6853	2.3072e-4 1.7109	7.0479e-5
2^{-23}	2.2142e-2 1.6121	7.2433e-3 1.6451	2.3159e-3 1.6421	7.4200e-4 1.6853	2.3072e-4 1.7111	7.0470e-5
2^{-27}	2.2142e-2 1.6121	7.2433e-3 1.6451	2.3159e-3 1.6421	7.4200e-4 1.6853	2.3072e-4 1.7111	7.0479e-5
2^{-31}	2.2142e-2 1.6121	7.2433e-3 1.6451	2.3159e-3 1.6421	7.4200e-4 1.6853	2.3072e-4 1.7111	7.0479e-5
2^{-35}	2.2142e-2 1.6121	7.2433e-3 1.6451	2.3159e-3 1.6421	7.4200e-4 1.6853	2.3072e-4 1.7107	7.0486e-5
2^{-39}	2.2142e-2 1.6121	7.2433e-3 1.6451	2.3159e-3 1.6421	7.4200e-4 1.6853	2.3072e-4 1.7111	7.0470e-5

Table 6 – *$E^N_{\varepsilon_1,u_2}$ and $r^N_{\varepsilon_1,u_2}$ for Example 6.3 on Shishkin mesh*

ε_1	Number of intervals N					
	32	64	128	256	512	1024
2^{-15}	1.3619e-2 2.0009	3.4026e-3 1.9508	8.8013e-4 1.9911	2.2139e-4 1.6583	7.0141e-5 1.6926	2.1699e-5
2^{-19}	3.3769e-2 1.7795	9.8367e-3 1.6344	3.1685e-3 1.6782	9.9009e-4 1.7025	3.0421e-4 1.7351	9.1379e-5
2^{-23}	3.6184e-2 1.7623	1.0666e-2 1.6074	3.5005e-3 1.6525	1.1135e-3 1.6763	3.4839e-4 1.7105	1.0645e-4
2^{-27}	3.6342e-2 1.7612	1.0721e-2 1.6058	3.5225e-3 1.6507	1.1219e-3 1.6749	3.5137e-4 1.7089	1.0748e-4
2^{-31}	3.6352e-2 1.7612	1.0724e-2 1.6056	3.5239e-3 1.6505	1.1225e-3 1.6749	3.5156e-4 1.7090	1.0753e-4
2^{-35}	3.6352e-2 1.7612	1.0724e-2 1.6056	3.5240e-3 1.6505	1.1225e-3 1.6748	3.5157e-4 1.7088	1.0755e-4
2^{-39}	3.6352e-2 1.7612	1.0724e-2 1.6056	3.5240e-3 1.6505	1.1225e-3 1.6748	3.5157e-4 1.7088	1.0755e-4

Table 7 – $E^N_{\varepsilon_1,u_1}$ *and* $r^N_{\varepsilon_1,u_1}$ *for Example 6.3 on the adaptive grid*

ε_1	Number of intervals N					
	32	64	128	256	512	1024
2^{-15}	6.0084e-3 1.9004	1.6095e-3 2.1051	3.7411e-4 1.9976	9.3682e-5 1.9934	2.3528e-5 1.9966	5.8957e-6
2^{-19}	6.9210e-3 2.0365	1.6870e-3 1.9149	4.4737e-4 2.0013	1.1174e-4 1.9854	2.8219e-5 1.9976	7.0664e-6
2^{-23}	7.7973e-3 2.0450	1.8895e-3 2.0507	4.5605e-4 1.9718	1.1626e-4 2.0144	2.8776e-5 1.9876	7.2563e-6
2^{-27}	7.8616e-3 1.9929	1.9751e-3 2.0101	4.9034e-4 2.0270	1.2031e-4 2.0222	2.9619e-5 2.0019	7.3950e-6
2^{-31}	8.4431e-3 2.0891	1.9843e-3 2.0190	4.8957e-4 2.0329	1.1963e-4 2.0009	2.9888e-5 1.9976	7.4846e-6
2^{-35}	8.3261e-3 2.1621	1.8603e-3 1.9156	4.9311e-4 2.0410	1.1982e-4 2.0036	2.9880e-5 2.0036	7.4306e-6
2^{-39}	8.5866e-3 2.0853	2.0234e-3 2.0431	4.9095e-4 2.0213	1.2094e-4 2.0153	2.9917e-5 2.0192	7.3802e-6

Table 8 – $E^N_{\varepsilon_1,u_2}$ *and* $r^N_{\varepsilon_1,u_2}$ *for Example 6.3 on the adaptive grid*

ε_1	Number of intervals N					
	32	64	128	256	512	1024
2^{-15}	8.3425e-3 1.9289	2.1910e-3 1.2378	9.2903e-4 1.8558	2.5668e-4 2.7357	3.8537e-5 1.1288	1.7623e-5
2^{-19}	1.0756e-2 2.119	2.4750e-3 1.8300	6.9615e-4 2.2778	1.4355e-4 2.0036	3.5799e-5 1.7813	1.0415e-5
2^{-23}	1.0801e-2 2.1486	2.4359e-3 1.9656	6.2368e-4 2.0486	1.5075e-4 2.0373	3.6726e-5 1.7611	1.0835e-5
2^{-27}	1.3771e-2 2.7083	2.1071e-3 1.8561	5.8202e-4 1.6072	1.9104e-4 1.9646	4.8948e-5 2.2076	1.0597e-5
2^{-31}	9.6603e-3 2.0496	2.3335e-3 2.1389	5.2983e-4 1.9068	1.4130e-4 1.9533	3.6486e-5 1.2376	1.5473e-5
2^{-35}	1.1367e-2 1.8835	3.0807e-3 2.2317	6.5591e-4 2.1559	1.4718e-4 1.7972	4.2349e-5 2.1704	9.4081e-6
2^{-39}	1.1189e-2 2.0490	2.7038e-3 2.3023	5.4815e-4 1.4159	2.0544e-4 1.6391	6.5958e-5 1.0021	3.2931e-5

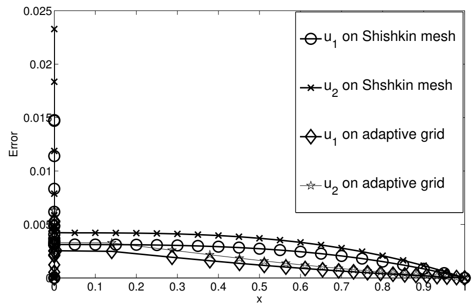


Fig. 5 – Error in Example 6.3 for $\varepsilon_1 = 2^{-26}$ and $\varepsilon_2 = 10^{-6}$.

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REFERENCES

[1] V. B. Andreev and N. Kopteva, *On the convergence, uniform with respect to a small parameter, of monotone three - point finite difference approximation.* Differ. Equ. **34** (1998), 921–929.

[2] M.G. Beckett and J.A. Mackenzie, *Convergence analysis of finite difference approximations on equidistributed grids to a singularly perturbed boundary value problem.* Appl. Numer. Math. **35** (2000), 87–109.

[3] M.G. Beckett, J. A. Mackenzie, A. Ramage and D. M. Sloan, *On the numerical solution of one-dimensional PDEs using adaptive methods based on equidistribution* J. Comput. Phys. **167** (2001), 372–392.

[4] S. Bellew and E. ORiordan, *A parameter robust numerical method for a system of two singularly perturbed convection-diffusion equations.* Appl. Numer. Math. **51** (2004), 171–186.

[5] A. Cangiani, E. H. Georgoulis, and S. Metcalfe, *Adaptive discontinuous Galerkin methods for nonstationary convection-diffusion problems.* IMA J. Numer Anal. **34** (2014), *4*, 1578–1597.

[6] P. Das and S. Natesan, *Optimal error estimate using mesh equidistribution technique for singularly perturbed system of reaction-diffusion boundary-value problems.* Appl. Math. Comput. **249** (2014), 265–277.

[7] V. Erivin and W. Layton, *On the approximation of derivatives of singulalrly perturbed boundary value problem.* SIAM J. Sci. Comput. **8** (1987), 256–277.

[8] P. A. Farrell, A. F. Hegarty, J. J. H. Miller, E. O'Riordan, and G. I. Shishkin, *Robust Computational Techniques for Boundary Layers.* Chapman & Hall/CRC Press, Boca Raton, FL, 2000.

[9] J. Giesselmann, C. Makridakis, and T. Pryer, *A posteriori analysis of discontinuous Galerkin schemes for systems of hyperbolic conservation laws.* SIAM J. Numer. Anal. **53** (2015), 1280–1303.

[10] S. Gowrisankar and S. Natesan, *The parameter uniform numerical method for singularly perturbed parabolic reaction–diffusion problems on equidistributed grids*. Appl. Math. Lett. **26** (2013), 1053–1060.

[11] S. Gowrisankar and S. Natesan, *Robust numerical scheme for singularly perturbed convection diffusion parabolic initial boundary value problems on equidistributed grids.* Comput. Phys. Commun. **185** (2014), 2008–2019.

[12] V. John, *A numerical study of a posteriori error estimators for convection-diffusion equations.* Comput. Methods Appl. Mech. Engrg. **190** (2000), 757–781.

[13] R.B. Kellogg and A. Tsan, *Analysis of some differences approximations for a singular perturbation problem without turning point.* Math. Comp. **32** (1978), 1025–1039.

[14] N. Kopteva and M. Stynes, *A robust adaptive grid for a quasi-linear one dimensional convection-diffusion problem.* SIAM J. Numer. Anal. **39** (2001), 1446–1467.

[15] N. Kopteva and M. Stynes, *Approximation of derivatives in a convection diffusion two point boundary value problem.* Appl. Numer. Math. **39** (2001), 47–60.

[16] T. Linß *Sufficient conditions for uniform convergence on layer-adapted grids.* Appl. Numer. Math. **37** (2001), 241–255.

[17] T. Linß, *On a set of singularly perturbed convectiondiffusion equations.* J. Comput. Appl. Math. **180** (2005), 173–179.

[18] T. Linß and M. Stynes, *Numerical solution of systems of singularly perturbed differential equations.* Comput. Methods Appl. Math. **9** (2009), 165–191.

[19] J. Mackenzie, *Uniform convergence analysis of an upwind finite differences approximation of a convection-diffusion boundary value problem on an adaptive grid.* IMA J. Numer Anal. **19** (1999), 233–249.

[20] J. J. H. Miller, E. O'Riordan, and G.I. Shishkin, *Fitted Numerical Methods for Singular Perturbation Problems (revised edition).* World Scientific, Singapore, 2012.

[21] J. Mohapatra and S. Natesan, *Parameter-uniform numerical method for global solution and global normalized flux of singularly perturbed boundary value problems using grid equidistribution.* Comput. Math. Appl. **60** (2010), 1924–1939.

[22] S. Natesan, R.K. Bawa, and C. Clavero, *Uniformly convergent compact numerical scheme for the normalized flux of singularly perturbed reaction diffusion problems.* Int. J. Inf. Syst. Sci. **3** (2007), 207–221.

[23] Y. Qiu, D. M. Sloan, and T. Tang, *Numerical solution of a singularly perturbed two-point boundary value problem using equidistribution analysis of convergence.* J. Comput. Appl. Math. **116** (2000), 121–143.

[24] H. G. Roos, M. Stynes, and L. Tobiska, *Numerical Methods for Singularly Perturbed Differential Equations.* Springer, Berlin, 2008.

[25] D. Shakti and J. Mohapatra, *A second order numerical method for a class of parameterized singular perturbation problems on adaptive grid.* Nonlinear Engineering **6** (2017), 221–228.

[26] D. Shakti and J. Mohapatra. *Numerical simulation and convergence analysis for a system of nonlinear singularly perturbed differential equations arising in population dynamics.* J. Differ. Equations Appl. **24** (2018), 1185–1196.

[27] G. I. Shishkin and L. P. Shishkina, *Differences Methods For Singular Perturbation Problems.* CRC Press, Boca Raton, 2009.

[28] M. Stynes and H. G. Roos, *The midpoint upwind scheme.* Appl. Numer. Math. **23** (1997), 361–374.

[29] R. Verfürth, *A posteriori error estimators for convection–diffusion equations.* Numer. Math. **80** (1998), 641–663.

[30] Q. Zheng, X. Li, and Y. Gao, *Uniformly convergent hybrid schemes for solutions and derivatives in quasilinear singularly perturbed BVPs.* Appl. Numer. Math. **91** (2015), 46–59.

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