ANALYTIC EXTENSION OF TOTALLY POLYNOMIALLY POSINORMAL OPERATORS

SALAH MECHERI and T. PRASAD

Communicated by Mihai Putinar

In this paper, we show that every analytic extension of totally polynomially posinormal operator has a scalar extension. As a consequence, we obtain that analytic extension of totally polynomially posinormal operator with thick spectra has nontrivial invariant subspace. We show that if T is an analytic extension of totally polynomially posinormal operator, then $f(T)$ satisfies generalized Weyl's theorem for every analytic functions f which are defined on an open neighborhood of the spectrum of T.

AMS 2010 Subject Classification: Primary 47A11, Secondary 47A15, 47B20.

Key words: Totally polynomially posinormal operators, subscalar operator, invariant subspaces, Bishop's property (β), Weyl's theorem.

1. INTRODUCTION

Let $B(\mathcal{H})$ be the algebra of all bounded linear operators acting on infinite dimensional separable complex Hilbert space $\mathcal{H}$. Let $\mathbb{C}$ denote the set of complex numbers. Throughout this paper $\mathcal{R}(T)$, For $T \in B(\mathcal{H})$, let $\mathcal{R}(T)$, $\mathcal{N}(T)$, $\sigma(T)$, $\sigma_p(T)$ and iso $\sigma(T)$ mean the range, null space, spectrum, the point spectrum and the set of isolated points of $\sigma(T)$ of T, respectively. Recall that an operator $T \in B(\mathcal{H})$ is said to be *hyponormal* if $T^*T \geq TT^*$. Investigating new generalizations of hyponormal operators is one of recent interest in operator theory. An operator is said to be *M-hyponormal*$(M \geq 1)$ if $||(T-z)^*x|| \leq M||(T-z)x||$ for each $x \in \mathcal{H}$ and $z \in \mathbb{C}$, *dominant* if to each $z \in \mathbb{C}$ there corresponds a real number $M(z)$ such that $||(T-z)^*x|| \leq M(z)||(T-z)x||$ for each $x \in \mathcal{H}$ [25], and *posinormal* if $TT^* \leq \lambda^2 T^*T$ for some $\lambda \geq 0$[24]. Kostov and Todorov [23] introduced and studied totally polynomially posinormal operators. Let $P(z) = z^n + \sum_{m=1}^{n-1} a_m z^m$ be a polynomial, $a_m \in \mathbb{C}$. As usual $\overline{P}(\overline{z}) = \overline{z}^n + \sum_{m=1}^{n-1} \overline{a}_m \overline{z}^m$. An operator $T \in B(\mathcal{H})$ is said to be *polynomially posinormal* if there exist a constant $M > 0$ such that $||\overline{P}(T^*)x|| \leq M||Tx||$ for each $x \in \mathcal{H}$ [23]. Note that every posinormal operator is *polynomially posinormal* with $P(z) = z$. An operator $T \in B(H)$ is said to be *totally polynomially posinormal* if

$$||\overline{P}(T-z)^*x|| \leq M(z)||(T-z)x||$$

for each $x \in \mathcal{H}$, where $M(z)$ is bounded on the compacts of $\mathbb{C}$ [23].
In general, the following inclusion relations hold:

hyponormal $\subset$ M-hyponormal$\subset$ dominant
M-hyponormal $\subset$ polynomially posinormal
M-hyponormal $\subset$ totally polynomially posinormal

As it is shown in [23, Corollary 5.4], the class of totally polynomially-posinormal operators includes all finite dimensional and nilpotent operators. Thus it is much larger than the class of M-hyponormal operators, since for such operators these properties yield normality (see [26]). For another example of totally polynomially posinormal operator (see [23, Example 5.3]).

We say that an operator $T \in B(\mathcal{H})$ is *analytic* if there exists a nonconstant analytic function F on a neighborhood of $\sigma(T)$ such that $F(T) = 0$. If there is a nonconstant polynomial p such that $p(T) = 0$, we call T is *algebraic.* If an operator $T \in B(\mathcal{H})$ is analytic, then $F(T) = 0$ for some nonconstant analytic function F on a neighborhood U of $\sigma(T)$. Since F cannot have infinitely many zeros in U, we write $F(z) = G(z)p(z)$ where the function G is analytic and does not vanish on U and p is a nonconstant polynomial with zeros in U. By Riesz-Dunford calculus, $G(T)$ is invertible and the invertibility of $G(T)$ induces that $p(T) = 0$, which means that T is algebraic (See [7]). Throughout this paper, we say that T *is analytic with order k when p has degree k*. In order to generalize totally polynomially posinormal and totally k-quasi-polynomially posinormal operators we introduce the class of analytic extension of totally polynomially posinormal operator as follows:

Definition 1.1. An operator $T \in B(\mathcal{H}_1 \oplus \mathcal{H}_2)$ is said to be an analytic extension of totally polynomially posinormal operator if $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \in B(\mathcal{H}_1 \oplus \mathcal{H}_2)$, where T_1 is totally polynomially posinormal and T_3 is analytic of order k, where k is a positive integer.

Let $0 \le m \le \infty$ and let $C_0^m(\mathbb{C})$ be the space of all compactly supported functions on complex plane $\mathbb{C}$ with continuous derivative of order m. Recall that an operator $T \in B(\mathcal{H})$ is said to be *scalar* operator of order m if it possess a spectral distribution of order m, i.e., there exist a continuous unital morphism of topological algebras

$$\Phi : C_0^m(\mathbb{C}) \to B(\mathcal{H})$$

such that $\Phi(z) = T$, where z stands for the identity function on $\mathbb{C}$. An operator T is said to be *subscalar* of order m if T is similar to the restriction of a scalar operator of order m to an invariant subspace (See [16]).

Recall that operator $T \in B(\mathcal{H})$ has the *single valued extension property*(SVEP) at $\lambda_0 \in \mathbb{C}$, if for every open disk D_{λ_0} centered at λ_0 the only analytic function $f : D_{\lambda_0} \to \mathcal{H}$ which satisfies $(T-\lambda)f(\lambda) = 0$ for all $\lambda \in D_{\lambda_0}$ is the function $f \equiv 0$. We say that T has SVEP if it has SVEP at every $\lambda \in \mathbb{C}$. The *local resolvent set* of $T \in B(\mathcal{H})$ at $x \in \mathcal{H}$, denoted by $\rho_T(x)$, is the set of elements $\lambda_0 \in \mathbb{C}$ such that there exists an analytic function $f(\lambda)$ defined in a neighborhood of λ_0, with values in $\mathcal{H}$, which verifies $(T-\lambda)f(\lambda) \equiv x$. The set $\sigma_T(x)$, the compliment of $\rho_T(x)$ is called the *local spectrum* of T at x. For each subset U of $\mathbb{C}$, the *local spectral subspace* of T denoted by $\mathcal{H}_T(U)$ is the set $\mathcal{H}_T(U) = \{x \in \mathcal{H} : \sigma_T(x) \subset U\}$. An operator $T \in B(\mathcal{H})$ is said to have *Bishop's property* (β) if, for every open subset U of $\mathbb{C}$ and every sequence $f_n : U \longrightarrow \mathcal{H}$ of analytic functions such that $(T-\lambda)f_n(\lambda)$ converges uniformly to 0 in norm on compact subsets of U, it follows that $f_n(\lambda)$ converges uniformly to 0 in norm on compact subsets of U(See [1, 15]). It is well known from [1, 15] that every subscalar operators has Bishop's property (β) and

$$\text{Bishop's property } (\beta) \Rightarrow \text{SVEP}.$$

There are many outstanding problems which are still open for hyponormal operators, for example, the invariant subspace problem. The *invariant subspace problem* poses the question: does every operator have a nontrivial invariant subspace? In [22], M. Putinar proved subscalarity for hyponormal opeators. S. Brown [6] proved if T is hyponormal operator with thick spectra then T has non trivial invariant subspace. In [9], Eschmeier proved that a Banach space operator T has a nontrivial invariant subspace if T has the property (β) with thick spectra. The study of subscalarity for non hyponormal operators have been attracted a lot of attention of researchers; see for instance [13, 14, 18, 20, 22] and the references therein.

The aim of this paper is to study subscalarity and Weyl type theorems for analytic extension of totally polynomially posinormal operators. In section three, we prove analytic extension of totally polynomially posinormal operators are subscalar. As a corollary of this result we obtain that such class of operators with thick spectra has a nontrivial invariant subspace. In section four, we show that $f(T)$ satisfies generalized Weyl's theorem for every analytic functions f which are defined on an open neighborhood of the spectrum of an analytic extension of totally polynomially posinormal operator T.

2. PRELIMINARIES

Let D be a bounded open disk in $\mathbb{C}$. The space $L^2(D, \mathcal{H})$ defined as follows is a Hilbert space

$$L^2(D,\mathcal{H}) = \left\{ f : D \to \mathcal{H} : f \text{ is measurable}, \|f\|_{2,D} = \left(\int_D \|f(z)\|^2 \mathrm{d}\mu(z) \right)^{\frac{1}{2}} < \infty \right\},$$

where $d\mu(z)$ be the planar Lebesgue measure. The Bergman space for D, denoted by $A^2(D,\mathcal{H})$, is a subspace of $L^2(D,\mathcal{H})$ in which each function is analytic in D (ie., $\frac{\partial f}{\partial \bar{z}} = 0$). Let $\mathcal{O}(D,\mathcal{H})$ be the Fréchet space of $\mathcal{H}$-valued analytic functions on D with respect to the uniform topology. Note that

$$A^2(D,\mathcal{H}) = L^2(D,\mathcal{H}) \cap \mathcal{O}(D,\mathcal{H})$$

is a Hilbert space. The following functional space $W^m(D,\mathcal{H})$ is a Sobolev type space with respect to $\bar{\partial}$ and of order m

$$W^m(D,\mathcal{H}) = \{f \in L^2(D,\mathcal{H}) : \bar{\partial}^i f \in L^2(D,\mathcal{H}), \text{for } i = 1, 2, ..., m\}.$$

Note that $W^2(D,\mathcal{H})$ is a Hilbert space with respect to the norm

$$||f||^2_{W^m} = \sum_{i=0}^{m} ||\overline{\partial}^i f||^2_{2,U},$$

$W^m(U,H)$ becomes a Hilbert space contained continuously in $L^2(U,H)$. A bounded linear operator S on H is called scalar of order m if it possesses a spectral distribution of order m, *i.e.*, if there is a continuous unital morphism of topological algebras

$$\Phi : C_0^m(\mathbb{C}) \to B(H)$$

such that $\Phi(z) = S$, where z is the identity function on $\mathbb{C}$, and $C_0^m(\mathbb{C})$ is the space of compactly supported functions on $\mathbb{C}$, continuously differentiable of order m, where $0 \leq m \leq \infty$. An operator is subscalar if it is similar to the restriction of a scalar operator to an invariant subspace. Let U be a (connected) bounded open subset of $\mathbb{C}$, and let m be a nonnegative integer. The linear operator M_f of multiplication by f on $W^m(U,H)$ is continuous, has a spectral distribution of order m, and is defined by the functional calculus

$$\Phi_M : C_0^m(\mathbb{C}) \to B(W^m(U,H)),\ \Phi_M(f) = M_f.$$

Therefore, M is a scalar operator of order m. Let

$$V : W^m(U,H) \to \oplus_0^\infty L^2(U,H)$$

be the operator $V(f) = (f, \overline{\partial} f, ..., \overline{\partial}^m f)$. Then V is an isometry such that $VM_z = (\oplus_0^m M_z)V$. Therefore, M_z is a subnormal operator.

Let $\alpha(T)$ and $\beta(T)$ denote the *nullity* and the *deficiency* of $T \in B(\mathcal{H})$, defined by $\alpha(T)= dim(\mathcal{N}(T)$ and $\beta(T)=dim(\mathcal{N}(T^*)$. An operator T is said to be *upper semi-Fredholm* (resp., *lower semi- Fredholm*) if $\mathcal{R}(T)$ of $T \in B(\mathcal{H})$ is closed and $\alpha(T) < \infty$ (resp., $\beta(T) < \infty$). Let $SF_+(\mathcal{H})$ (resp., $SF_-(\mathcal{H})$) denote the semigroup of upper *semi-Fredholm* (resp., *lower semi-Fredholm*) operators on $\mathcal{H}$. An operator $T \in B(\mathcal{H})$ is said to be *semi-Fredhom*, $T \in SF(\mathcal{H})$, if $T \in SF_+(\mathcal{H}) \cup SF_-(\mathcal{H})$ and *Fredholm*, $T \in F(\mathcal{H})$, if $T \in SF_+(\mathcal{H}) \cap SF_-(\mathcal{H})$. The *index* of semi-Fredholm operator T is defined by ind $(T) = \alpha(T) - \beta(T)$.

Recall[11], the *ascent* of an operator $T \in B(\mathcal{H})$, $a(T)$, is the smallest non negative integer $\mathtt{p}:=\mathtt{p}(T)$ such that $\mathcal{N}(T^{\mathtt{p}}) = \mathcal{N}(T^{(\mathtt{p}+1)})$. Such $\mathtt{p}$ does not exist, then $\mathtt{p}(T) = \infty$. The *descent* of $T \in B(\mathcal{H})$, $d(T)$, is defined as the smallest non negative integer $\mathtt{q} := \mathtt{q}(T)$ such that $\mathcal{R}(T^{\mathtt{q}}) = \mathcal{R}(T^{(\mathtt{q}+1)})$. An operator $T \in B(\mathcal{H})$ is *Weyl*, $T \in W(\mathcal{H})$ it is Fredholm of index zero and *Browder* if T is Fredholm of finite ascent and descent. The *Weyl spectrum* of T, denoted by $\sigma_W(T)$, is given by

$$\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin W(\mathcal{H})\}.$$

We say that $T \in B(\mathcal{H})$ satisfies Weyl's theorem if

$$\sigma(T) \setminus \sigma_W(T) = E_0(T).$$

where $E_0(T)$ denote the set of eigenvalues of T of finite geometric multiplicity. Let $SF_+^-(\mathcal{H}) = \{T \in SF_+(\mathcal{H}) : \text{ind(T)} \leq 0\}$. The *essential approximate point spectrum* $\sigma_{SF_+^-}(T)$ of T is defined by

$$\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin SF_+^-(\mathcal{H})\}.$$

Let $\sigma_a(T)$ denote the approximate point spectrum of $T \in B(\mathcal{H})$. An operator $T \in B(\mathcal{H})$ holds *a-Weyl's theorem* if,

$$\sigma_{SF_+^-}(T) = \sigma_a(T) \setminus E_0^a(T),$$

where $E_0^a(T) = \{\lambda \in \mathbb{C} : \lambda \in \text{iso } \sigma_a(T) \text{ and } 0 < \alpha(T - \lambda) < \infty\}$. We say that an operator $T \in B(\mathcal{H})$ satisfies *a-Browder's* theorem if

$$\sigma_{SF_+^-}(T) = \sigma_a(T) \setminus \Pi_0^a(T),$$

where $\Pi_0^a(T)$ denote the set the left poles of T of finite rank. An operator $T \in B(\mathcal{H})$ is called *B-Fredholm*, $T \in BF(\mathcal{H})$ if there exist a non negative integer n for which the induced operator

$$T_{[n]} : \mathcal{R}(T_{[n]}) \to \mathcal{R}(T_{[n]})(\text{ in particular } T_{[0]} = T).$$

is Fredholm in the usual sense (see [4]). An operator $T \in B(\mathcal{H})$ is called *B-Weyl*, $T \in BW(\mathcal{H})$, if it is B-Fredholm with $\text{ind}(T_{[n]}) = 0$. The *B-Weyl spectrum* $\sigma_{BW}(T)$ is defined by

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin BW(\mathcal{H})\}$$

(see [4]). Let $E(T)$ is the set of all eigenvalues of T which are isolated in $\sigma(T)$. We say that T satisfies *generalized Weyl's theorem* if

$$\sigma_{BW}(T) = \sigma(T) \setminus E(T).$$

A bounded operator $T \in B(\mathcal{H})$ is said to satisfy *generalized Browders's theorem* if

$$\sigma(T) \setminus \sigma_{BW}(T) = \Pi(T),$$

where $\Pi(T)$ is the set of poles of T. See [5] for more information. From [5], it is known that an operator $T \in B(\mathcal{H})$ satisfying a-Weyl's theorem theorem satisfies generalized a-Browders's theorem. It is known from [5] that an operator $T \in B(\mathcal{H})$ satisfying generalized Weyl's theorem satisfies generalized Browders's theorem . It is well known that that if $T \in B(\mathcal{H})$ obeys generalized Weyl's theorem, then it is also obeys Weyl's theorem and if T obeys generalized Browders's theorem, then it obeys Browders's theorem (see [4]).

3. SUBSCALARITY

In this section we shall prove that every analytic extension of totally polynomially posinormal operator is subscalar. We start with the following Lemmas.

Lemma 3.1 (See [22, Proposition 2.1]). *For a bounded open disk D in the complex plane $\mathbb{C}$, there is a constant C_D such that for an arbitrary operator $T \in B(\mathcal{H})$ and $f \in W^2(D, \mathcal{H})$ we have*

$$||(I-P)f||_{2,D} \leq C_D(||(T-z)\bar{\partial} f||_{2,D} + ||(T-z)\bar{\partial}^2 f||_{2,D})$$

where P denote the orthogonal projection of $L^2(D, \mathcal{H})$ on to the Bergman space $A^2(D, \mathcal{H})$

Lemma 3.2. *Let $T \in B(\mathcal{H})$ be a totally polynomially posinormal operator and let $\{f_j\}$ is a sequence in $W^m(D, \mathcal{H})$ $(m \geq 2n)$ such that*

$$\lim_{j\to\infty} ||(T-z)\bar{\partial}^i f_j||_{2,D} = 0$$

for $i = 1, 2, ..., m$, where D be bounded disc in $\mathbb{C}$ and n is the degree of polynomial. Then,

$$\lim_{j\to\infty} ||(I-Q)\bar{\partial}^i f_j||_{2,D_1} = 0$$

for $i = 1, 2, ..., m-2n$, where $D_1 \subsetneqq D$ and $Q : L^2(D, \mathcal{H}) \to A^2(D, \mathcal{H})$ is orthogonal projection. Furthermore, if $m > 2n$ then we have

$$\lim_{j\to\infty} ||\bar{\partial}^i f_j||_{2,D_2} = 0$$

for $i = 1, 2, ..., m-2n$, where $D_2 \subsetneqq D_1 \subsetneqq D$.

Proof. Suppose $T \in B(\mathcal{H})$ be a totally polynomially posinormal. From [20, Corollary 1], there exist a constant C_D such that

$$||(I-Q)\bar{\partial}^i f_j||_{2,D} \leq C_D \sum_{k=0}^{n} ||(T-z)\bar{\partial}^{i+n+k} f_j||_{2,D} \tag{3.1}$$

for $i = 1, 2, ..., m-2n$. From (3.1), we have

$$\lim_{j\to\infty} ||(I-Q)\bar{\partial}^i f_j||_{2,D_1} = 0 \tag{3.2}$$

for $i = 1, 2, ..., m-2n$, where $D_1 \subsetneqq D$. Thus we have,

$$\lim_{j\to\infty} ||(T-z)Q\bar{\partial}^i f_j||_{2,D_1} = 0 \tag{3.3}$$

for $i = 1, 2, ..., m-2n$. From [20, Proposition 1], totally polynomially posinormal operators satisfies Bishop's property (β). Then from (3.3), we have

$$\lim_{j\to\infty} ||Q\bar{\partial}^i f_j||_{2,D_2} = 0 \tag{3.4}$$

for $i = 1, 2, ..., m-2n$, where $D_2 \subsetneqq D_1 \subsetneqq D$. From (3.2) and (3.4), we get

$$\lim_{j\to\infty} ||\bar{\partial}^i f_j||_{2,D_2} = 0$$

for $i = 1, 2, ..., m-2n$, where $D_2 \subsetneqq D_1 \subsetneqq D$. $\square$

LEMMA 3.3. *Let $T \in B(\mathcal{H}_1 \oplus \mathcal{H}_2)$ be an analytic extension of a totally polynomially posinormal operator, ie., $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \in B(\mathcal{H}_1 \oplus \mathcal{H}_2)$, where T_1 is totally polynomially posinormal and T_3 is analytic with order k. For any bounded disk D which contains $\sigma(T)$, define the map $A : \mathcal{H}_1 \oplus \mathcal{H}_2 \to K(D)$ by*

$$Ax = 1 \otimes x + \overline{(T-z)W^{2k+4n}(D, \mathcal{H}_1) \oplus W^{2k+4n}(D, \mathcal{H}_2)} (= \widetilde{1 \otimes x}),$$

where

$$K(D) = \frac{W^{2k+4n}(D, \mathcal{H}_1) \oplus W^{2k+4n}(D, \mathcal{H}_2)}{\overline{(T-z)W^{2k+4n}(D, \mathcal{H}_1) \oplus W^{2k+4n}(D, \mathcal{H}_2)}}$$

and $1 \otimes x$ denotes the constant function sending any $z \in D$ to $x \in \mathcal{H}_1 \oplus \mathcal{H}_2$. Then, A is injective with closed range.

Proof. Let $f_j = f_{j,1} \oplus f_{j,2} \in W^{2k+4n}(D, \mathcal{H}_1) \oplus W^{2k+4n}(D, \mathcal{H}_2)$ and $x_j = x_{j,1} \oplus x_{j,2} \in \mathcal{H}_1 \oplus \mathcal{H}_2$ be sequences such that

$$\lim_{j\to\infty} ||(T-z)f_j + 1 \otimes x_j||_{W^{2k+4n}(D,\mathcal{H}_1)\oplus W^{2k+4n})(D,\mathcal{H}_2)} = 0 \tag{3.5}$$

From (3.5), we write

$$\lim_{j\to\infty} ||(T_1-z)f_{j,1} + T_2 f_{j,2} + 1 \otimes x_{j,1}||_{W^{2k+4n}} = 0, \tag{3.6}$$

$$\lim_{j\to\infty}||(T_3-z)f_{j,2}+1\otimes x_{j,2}||_{W^{2k+4n}}=0 \tag{3.7}$$

Then from the definition of the norm of Sobolev space, (3.6) and (3.7) yields

$$\lim_{j\to\infty}||(T_1-z)\bar{\partial}^i f_{j,1}+T_2\bar{\partial}^i f_{j,2}||_{2,D}=0 \tag{3.8}$$

and

$$\lim_{j\to\infty}||(T_3-z)\bar{\partial}^i f_{j,2}||_{2,D}=0 \tag{3.9}$$

for $i=1,2,....,2k+4n$.

Since T_3 is analytic of order k, there exists a nonconstant analytic function F on a neighborhood of $\sigma(T_3)$ such that $F(T_3)=0$. We write $F(z)=G(z)p(z)$, where G is non vanishing analytic function on a neighborhood of $\sigma(T)$ and nonconstant polynomial $p(z)=(z-z_1)(z-z_2)......(z-z_k)$. Set $q_j=(z-z_{j+1})....(z-z_k)$ for $j=0,1,2,..,k-1$ and $q_k(z)=1$.

Now we need to provethat for all $s=0,1,2,....,k$ the following equation

$$\lim_{j\to\infty}||q_s(T_3)\bar{\partial}^i f_{j,2}||_{2,D_s}=0 \tag{3.10}$$

holds for $i=1,2,....,2k+4n-2s$, where $\sigma(T)\subsetneqq D_k\subsetneqq D_{k-1}\subsetneqq ....\subsetneqq D_1\subset D$. We use induction on s for the proof (3.10). Since T_3 is analytic, $0=F(T_3)=G(T_3)p(T_3)$. Since $G(T_3)$ is invertible, we have $q_0(T_3)=p(T_3)=0$. That is, (3.10) is true for $s=0$. Suppose that

$$\lim_{j\to\infty}||q_s(T_3)\bar{\partial}^i f_{j,2}||_{2,D_s}=0$$

holds for $0<s<k$ and holds for $i=1,2,....,2k+4n-2s$. From (3.9) and (3.10), we obtain that

$$0=\lim_{j\to\infty}\|q_{s+1}(T_3-z)\bar{\partial}^i f_{j,2}\|_{2,D_s}=\lim_{j\to\infty}\|(z_{s+1}-z)q_{s+1}(T_3)\bar{\partial}^i f_{j,2}\|_{2,D_s} \tag{3.11}$$

holds for $i=1,2,....,2k+4n-2s$. Then by applying [13, lemma 3.2], it follows that

$$\lim_{j\to\infty}||q_{s+1}(T_3)\bar{\partial}^i f_{j,2}||_{2,D_{s+1}}=0 \tag{3.12}$$

holds for $i=1,2,....,2(k-s-1)+4n$, where $\sigma(T)\subsetneqq D_{s+1}\subsetneqq D_s$. Which completes the proof of (3.10). Now consider $s=k$ in (3.10), we get

$$\lim_{j\to\infty}||\bar{\partial}^i f_{j,2}||_{2,D_k}=0 \tag{3.13}$$

for $i=1,2,3....,4n$. then by (3.8) and (3.9), we have

$$\lim_{j\to\infty}||(T_1-z)\bar{\partial}^i f_{j,1}||_{2,D_k}=0$$

for $i = 1, 2, 3..., 4n$. Since T_1 is totally polynomially posinormal operator, from Lemma 3.2, we have

$$\lim_{j\to\infty} ||(I - Q_{\mathcal{H}_1})f_{j,1}||_{2,D_k} = 0, \tag{3.14}$$

where $Q_{\mathcal{H}_1}$ denotes the orthogonal projection of $L^2(D_k, \mathcal{H}_1)$ onto $A^2(D_k, \mathcal{H}_1)$. From (3.13) and Lemma 3.1, it follows that

$$\lim_{j\to\infty} ||(I - Q_{\mathcal{H}_2})f_{j,2}||_{2,D_k} = 0, \tag{3.15}$$

where $Q_{\mathcal{H}_2}$ denotes the orthogonal projection of $L^2(D_k, \mathcal{H}_2)$ onto $A^2(D_k, \mathcal{H}_2)$. Set $Qf_j := Q_{\mathcal{H}_1}f_{j,1} \oplus Q_{\mathcal{H}_2}f_{j,2}$. Then from (3.5), (3.14) and (3.15), we have

$$\lim_{j\to\infty} ||(T - z)Qf_j + 1 \otimes x_j||_{2,D_k} = 0.$$

Let γ be a closed curve in D_k surrounding $\sigma(T)$. Then,

$$\lim_{j\to\infty} ||Qf_j + (T - z)^{-1}(1 \otimes x_j)(z)|| = 0$$

uniformly for all $z \in \gamma$. Then by Riesz-Dunford functional calculus, we get

$$\lim_{j\to\infty} ||\frac{1}{2\pi i}\int_\gamma Qf_j(z)\mathrm{d}z + x_j|| = 0.$$

Then by Cauchy's theorem, we have $\frac{1}{2\pi i}\int_\gamma Qf_j(z)\mathrm{d}z = 0$. Thus we have

$$\lim_{j\to\infty} ||x_j|| = 0.$$

This completes the proof. $\square$

Now we are ready to show that every analytic extension of totally polynomially posinormal operator has scalar extension of order $2k + 4n$.

THEOREM 3.4. *Let $T \in B(\mathcal{H}_1 \oplus \mathcal{H}_2)$ be an analytic extension of a totally polynomially posinormal operator, ie., $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \in B(\mathcal{H}_1 \oplus \mathcal{H}_2)$, where T_1 is totally polynomially posinormal and T_3 is analytic with order k. Then T is subscalar of order $2k + 4n$.*

Proof. Suppose that $T \in B(\mathcal{H}_1 \oplus \mathcal{H}_2)$ is an analytic extension of a totally polynomially posinormal operator. For any bounded disk D which contains $\sigma(T)$, the map

$$A : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \frac{W^{2k+4n}(D, \mathcal{H}_1) \oplus W^{2k+4n}(D, \mathcal{H}_2)}{\overline{(T - z)W^{2k+4n}(D, \mathcal{H}_1) \oplus W^{2k+4n}(D, \mathcal{H}_2)}}$$

by

$$Ax = 1 \otimes x + \overline{(T - z)W^{2k+4n}(D, \mathcal{H}_1) \oplus W^{2k+4n}(D, \mathcal{H}_2)} (= \widetilde{1 \otimes x}),$$

where $1 \otimes x$ denotes the constant function sending any $z \in D$ to $x \in \mathcal{H}_1 \oplus \mathcal{H}_2$ is injective with closed range by Lemma 3.3. Let x and $\tilde{U}$ denotes the class of vector and operator on $\frac{W^{2k+4n}(D,\mathcal{H}_1) \oplus W^{2k+4n}(D,\mathcal{H}_2))}{\overline{(T-z)W^{2k+4n}(D,\mathcal{H}_1) \oplus W^{2k+4n}(D,\mathcal{H}_2))}}$ respectively. Consider M, which is the operator of multiplication by z on $W^{2k+4n}(D,\mathcal{H}_1) \oplus W^{2k+4n}(D,\mathcal{H}_2)$. Then M is scalar operator of order $2k+4n$ and has spectral distribution

$$\Phi : C_0^{2k+4n}(\mathbb{C}) \to B(W^{2k+4n}(D,\mathcal{H}_1) \oplus W^{2k+4n}(D,\mathcal{H}_2))$$

defined by $\Phi(\nu)x = \nu x$ for $x \in W^{2k+4n}(D,\mathcal{H}_1) \oplus W^{2k+4n}(D,\mathcal{H}_2)$ and $\nu \in C_0^{2k+4n}(\mathbb{C})$. Since $\overline{(T-z)W^{2k+4n}(D,\mathcal{H}_1) \oplus W^{2k+4n}(D,\mathcal{H}_2)}$ is invariant under M, $\tilde{M}$ is scalar operator of order $2k+4n$ with $\tilde{\Phi}$ as a spectral distribution. From the definition of map A, we have $AT = \tilde{M}A$. In particular $\mathcal{R}(A)$ is an invariant subspace for $\tilde{M}$. By Lemma 3.3, A is one-to-one and has closed range. Since T is similar to the restriction $\tilde{M}\mid_{\mathcal{R}(A)}$ and $\tilde{M}$ is scalar of order $2k+4n$, T is subscalar of order $2k+4n$. $\square$

We give some important facts which follow from subscalarity of analytic extension of totally polynomially posinormal operators.

Corollary 3.5. *Let T be an analytic extension of totally polynomially posinormal operator. Then T satisfies Bishop's property(β).*

Corollary 3.6. *Let T be an analytic extension of totally polynomially posinormal operator. Then T satisfies single valued extension property ($SVEP$)*

Let $\mathcal{H}^\infty(U)$ denote the space of all bounded analytic functions on bounded open set U in $\mathbb{C}$. A subset σ of $\mathbb{C}$ is dominating in U if $\|f\| = \sup_{x\in\sigma\cap U}|f(x)|$ holds for each function $f \in \mathcal{H}^\infty(U)$. Recall [6], a subset σ is thick if there is a bounded open set U in $\mathbb{C}$ such that σ is dominating in U.

Corollary 3.7. *Let T be an analytic extension of totally polynomially posinormal operator with thick spectra. Then T has a nontrivial invariant subspace.*

Proof. Suppose T is an analytic extension of totally polynomially posinormal operator. Then by corollary 3.5, T satisfies Bishop's property (β). Hence the required result follows from [9]. $\square$

Recall that a closed subspace $\mathcal{M}$ of $\mathcal{H}$ is said to be hyperinvariant for T if $\mathcal{M}$ is invariant under every operator in the commutant $\{T\}'$ of T.

Corollary 3.8. *Let T be an analytic extension of totally polynomially posinormal. If there exists a nonzero $x \in \mathcal{H}_1 \oplus \mathcal{H}_2$ such that $\sigma_T(x) \subsetneq \sigma(T)$, then T has a nontrivial hyperinvariant subspace.*

Proof. Suppose T is an analytic extension of totally polynomially posinormal operator such that there exists a nonzero $x \in \mathcal{H}_1 \oplus \mathcal{H}_2$ such that $\sigma_T(x) \subsetneqq \sigma(T)$. From corollary 3.5, T satisfies Bishop's property (β). Applying [18, Theorem 5.1], we obtain that T has a nontrivial hyperinvariant subspace. $\square$

THEOREM 3.9. *Let $T \in B(\mathcal{H}_1 \oplus \mathcal{H}_2)$ be an analytic extension of totally polynomially posinormal operator, i.e.,*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix},$$

is an operator matrix on $\mathcal{H}_1 \oplus \mathcal{H}_2$, where T_1 is totally polynomially posinormal operator and $F(T_3) = 0$ for a nonconstant analytic function F on a neighborhood D of $\sigma(T_3)$. Then the following statements hold

(i) $\mathcal{H}_T(E) \subseteq \mathcal{H}_{T_1}(E) \oplus \{0\}$ *for every subset E of $\mathbb{C}$.*

(ii) $\sigma_{T_1}(x_1) = \sigma_T(x_1 \oplus 0)$ *and* $\sigma_{T_3}(x_2) \subset \sigma_T(x_1 \oplus x_2)$ *where* $x_1 \oplus x_2 \in \mathcal{H}_1 \oplus \mathcal{H}_2$.

(iii) $\mathcal{R}_{T_1}(F) \oplus 0 \subset \mathcal{H}_T(F)$ *where* $\mathcal{R}_{T_1}(F) := \{y \in \mathcal{H}_1 : \sigma_{T_1}(y) \subset F\}$ *for any subset* $F \in \mathbb{C}$.

Proof. (i) Let $x_1 \in \mathcal{H}_{T_1}(E)$, where E be any subset of $\mathbb{C}$. Since T is analytic extension of totally polynomially posinormal operator, T has single valued extension property by Corollary 3.6. Then there exists an $\mathcal{H}$-valued analytic function f_1 on $\mathbb{C} \setminus E$ such that $(T_1 - z)f_1(z) \equiv x_1$ on $\mathbb{C} \setminus E$. Hence

$$(T - z)(f_1(z) \oplus 0) \equiv x_1 \oplus 0 \text{ on } \mathbb{C} \setminus E.$$

Thus, $x_1 \oplus 0 \in \mathcal{H}_T(E)$.

(ii) Let $x_1 \oplus x_2 \in \mathcal{H}_1 \oplus \mathcal{H}_2$. Suppose that $z_0 \in \rho_T(x_1 \oplus 0)$. Then there exists an $\mathcal{H}$- valued analytic function f defined on a neighborhood U of z_0 such that $(T - \lambda)f(\lambda) = x_1 \oplus 0$ for all $\lambda \in U$. We can write $f = f_1 \oplus f_2$ where $f_1 \in O(U; \mathcal{H}_1)$ and $f_2 \in O(U; \mathcal{H}_2)$. Then we get

$$T = \begin{pmatrix} T_1 - \lambda & T_2 \\ 0 & T_3 - \lambda \end{pmatrix} \begin{pmatrix} f_1(\lambda) \\ f_2(\lambda) \end{pmatrix} \equiv \begin{pmatrix} x_1 \\ 0 \end{pmatrix}.$$

Thus,

$$(T_1 - \lambda)f_1(\lambda) + T_2 f_2(\lambda) \equiv x_1 \text{ and } (T_3 - \lambda)f_2(\lambda) \equiv 0.$$

Since T_3 is analytic of order k, T_3 has single valued extension property. Hence we have $f_2(\lambda) \equiv 0$. Thus $(T_1 - \lambda)f_1(\lambda) \equiv x_1$, and so $z_0 \in \rho_{T_1}(x_1)$. Conversely suppose $z_0 \in \rho_{T_1}(x_1)$. Then there exists a function $f_1 \in O(U; \mathcal{H}_1)$ for some neighborhood U of λ_0 such that $(T_1 - \lambda)f_1(\lambda) \equiv x_1$. Then

$$(T - \lambda)f_1(\lambda \oplus 0) \equiv x_1 \oplus 0.$$

Hence $z_0 \in \rho_T(x_1 \oplus 0)$. On the other hand, If $z_0 \in \rho_T(x_1 \oplus x_2)$, then there exists an $\mathcal{H}$-valued analytic function g defined on a neighborhood U of z_0 such that $(T-\lambda)g(\lambda) = x_1 \oplus x_2$ for all $\lambda \in U$. We can write $g = g_1 \oplus g_2$ where $g_1 \in O(U; \mathcal{H}_1)$ and $g_2 \in O(U; \mathcal{H}_2)$. Then we get

$$T = \begin{pmatrix} T_1 - \lambda & T_2 \\ 0 & T_3 - \lambda \end{pmatrix} \begin{pmatrix} g_1(\lambda) \\ g_2(\lambda) \end{pmatrix} \equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Thus $(T_3 - \lambda)g_2(\lambda) = x_2$. Hence $z_0 \in \rho_{T_3}(x_2)$.
(iii) Let $x_1 \in \mathcal{R}_{T_1}(F)$. Then $\sigma_{T_1}(x_1) \subset F$. From (ii), we have the equality $\sigma_{T_1}(x_1) = \sigma_{T_1}(x_1 \oplus 0)$. Therefor, $\sigma_{T_1}(x_1 \oplus 0) \in F$. Thus $x_1 \oplus 0 \in \mathcal{H}_T(F)$, and hence $\mathcal{R}_{T_1}(F) \oplus 0 \subset \mathcal{H}_T(F)$. □

4. WEYL TYPE THEOREMS

Recall that an operator $T \in B(\mathcal{H})$ is called *isoloid* if every isolated point of spectrum of T is an eigenvalue and $T \in B(H)$ is said to be *polaroid* if every isolated point of spectrum of T is a pole of resolvent of T. Note that if T is polaroid then T is isoloid. Moreover, T is polaroid if and only if T^* is polaroid.

Since T is subscalar, it follows that T is polaroid [18] and hence isoloid.

Theorem 4.1. *Let T be an analytic extension of totally polynomially posinormal operator. Then generalized Weyl's theorem holds for both T and T^*.*

Proof. Suppose T is an analytic extension of totally polynomially posinormal operator. From Theorem 3.4, T is subscalar. Then by [18, Corollary 2.2], T is polaroid. From Corollary 3.5, T has single valued extension property (SVEP). Then it follows from [2, Theorem 4.1] that generalized Weyl's theorem holds for both T and T^*. □

According to Berkani and Koliha [5], an operator $T \in B(\mathcal{H})$ is said to be Drazin invertible if T has finite ascent and descent. The *Drazin spectrum* of $T \in B(\mathcal{H})$, denoted by $\sigma_D(T)$, is defined $\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Drazin invertible}\}$ (See, [4]). Let $H(\sigma(T))$ denote the set of analytic functions which are defined on an open neighborhood of $\sigma(T)$.

Theorem 4.2. *Let T be an analytic extension of totally polynomially posinormal. Then $f(T)$ satisfies generalized Weyl's theorem for every $f \in H(\sigma(T))$.*

Proof. Suppose T is an analytic extension of totally polynomially posinormal operator. To prove $f(T)$ satisfies generalized Weyl's theorem for every

$f \in H(\sigma(T))$, it is enough to prove equality $\sigma_{BW}(f(T)) = f(\sigma_{BW}(T))$ holds for every $f \in H(\sigma(T))$. Since T is analytic extension of totally polynomially posinormal, T has SVEP corollary 3.6. Thus, $f(T)$ satisfies generalized Browder's theorem. Then by [8, Theorem 2.1] we have

$$\sigma_{BW}(f(T)) = \sigma_D(f(T)).$$

Since $\sigma_D(f(T)) = f(\sigma_D(T))$ (See [8, Theorem 2.7]),

$$\sigma_{BW}(f(T)) = f(\sigma_D(T)).$$

Since T satisfies generalized Weyl's theorem (see Theorem 4.1), T satisfies generalized Browder's theorem. Hence the following equality holds

$$f(\sigma_D(T)) = f(\sigma_{BW}(T)).$$

Hence, $\sigma_{BW}(f(T)) = f(\sigma_{BW}(T))$. This completes the proof. □

THEOREM 4.3. *If T^* is an analytic extension of totally polynomially posinormal operator, then a-Weyl's theorem holds for T.*

Proof. Suppose T^* is an analytic extension of totally polynomially posinormal operator. Then T^* has SVEP by Corollary 3.6. From Theorem 3.4, T^* is subscalar. Then by [18, Corollary 2.2], T^* is polaroid. Since T^* is polaroid, T is polaroid. By applying [3, theorem 3.19], it follows that a-Weyl's theorem holds for T. □

Acknowledgments. We wish to thank the referee for careful reading of the paper and valuable comments for the origin draft.

REFERENCES

[1] P. Aiena, *Fredhlom and local specral theory with application to multipliers*. Kluwer Acad. Publishers,Dordrecht, 2004.

[2] P. Aiena, J. R. Guillen, and P. Peña, *A unifying approch to Weyl type theorems for Banach space operators*. Integral Equations Operator Theory. **77** (2013), 371–384.

[3] P. Aiena, E. Aponte, and E. Balzan, *Weyl Type theorems for left and right polaroid operators*. Integral Equations Operator Theory. **66** (2010), 1–20.

[4] M. Berkani, *B-Weyl spectrum and poles of the resolvant*. J. Math. Anal. Appl. **272** (2002), 596–603

[5] M. Berkani and J. Koliha, *Weyl Type theorems for bounded linear operators*. Acta. Sci Math(Szeged). **69** (2003), 359–376.

[6] S. Brown, *Hyponormal operator with thick spectrum have invariant subspaces*. Ann. Math. **125** (1987), 93–103.

[7] X. H. Cao, *Analytically class A operators and Weyls theorem*, J. Math. Anal. Appl. **320** (2006), 795–803.

[8] R. E. Curto and Y. M. Han, *Generalized Browders and Weyls theorems for Banach space operators*. J. Math. Anal. Appl. **336** (2007), 1424–1442.

[9] J. Eschmeier, *Invariant subspaces for subscalar operators.* Arch. Math. **52** (1989), 562–570.

[10] J. K. Han, H. Y. Lee, and W. Y. Lee, *Invertible completions of* 2×2 *upper triangular operator matrices*. Proc. Amer. Math. Soc. **128** (1999), 119–123.

[11] H. Heuser, *Functional Analysis*. Dekker, New York, 1982.

[12] J. C. Hou and X. L. Zhang, *On the Weyl spectrum: spectral mapping theorem and Weyl's theorem*. J. Math. Anal. Appl. **220** (1998), 760–768.

[13] S. Jung, E. Ko, and M. Lee, *Subscalarity of* (p, k) *quasihyponormal operators*. J. Math Anal. Appl. **380** (2011), 76–86.

[14] S. Jung and E. Ko, *Class A operators and their extensions*. Operator and Matrices **7**(1)(2013), 167–182.

[15] K. B. Laursen and M. M. Neumann, *Introduction to local spectral theorey*. Clarendon Press, Oxford, 2000.

[16] M. Martin and M. Putinar, *Lectures on Hyponormal operators*. Opertor Theorey Adv. Appl. 39, Birkhauser, Boston, 1989.

[17] S. Mecheri, *Bishop's property* (β)*, hypercyclicity and hyperinvariant subspaces*. Operators and Matrices **8** (2014) 555–562.

[18] S. Mecheri, *Subscalarity, invariant, and hyperinvariant subspaces for upper triangular operator matrices*. Bull. Malays. Math. Sci. Soc. **41** (2018), 1085–1104.

[19] S. Mecheri and F. Zou, *Analytic extension of M-hyponormal operators*. J. Korean Math. Soc. **53** (2016), 233–246.

[20] R. Nickolov and Zh. Zhelev, *Totally P-posinormal operators are subscalar*. Integral Equations Operator Theory **43** (2002), 346–355.

[21] M. Oudghiri, *a-Weyl's Theorem and the single valued extension property*. Extracta Mathematicae **21** (2006), 41–50.

[22] M. Putinar, *Hyponormal operators are subscalar*. J. Operator Theory **12** (1984), 385–395.

[23] S. Kostov and I. Todorov, *A functional model for polynomially posinormal operators*. Integral Equations Operator Theory **40** (2001), 61–79.

[24] H. C. Rhaly, Jr, textit Posinormal operators. J. Math. Soc. Japan **46** (1994), 587–605.

[25] J. Stampfli and B. L. Wadhwa, *On dominant operators*. Monatsh. Math. **84** (1978), 143–153.

[26] B. L. Wadhwa, *M-hyponormal operators*. Duke Math. J. **41** (1974), 655–660.

Received December 23, 2017

Mohamed El Bachir El Ibrahimi University
Faculty of Mathematics and Informatics
Department of Mathematics
Bordj Bou Arreridj, Algeria
`mecherisalah@hotmail.com`

University of Calicut
Department of Mathematics
Kerala-673635, India
`prasadvalapil@gmail.com`