ON THE DYNAMICS OF A GENERALIZED LOGISTIC FUNCTION

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In this paper, we generalize the chaotic characteristic of the logistic function F_μ for $\mu \geq 4$ on its invariant subset to an arbitrary real function. In fact, we describe sufficient conditions for a real arbitrary function to be chaotic on an invariant subset. Then we present some functions that satisfy these sufficient conditions, although these functions have some differences with the logistic function.

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1. INTRODUCTION

The logistic family $F_\mu(x) = \mu x(1-x)$ is a well-known family of polynomials in one dimensional discrete dynamical systems. Although F_μ is expressed with a simple formula, however many different dynamical behaviors are observed in this family as the parameter μ increases. Specially, when $\mu \geq 4$, F_μ is chaotic on the invariant set $\Lambda_\mu = \cap_{n=1}^{\infty} F_\mu^{-n}([0,1])$. The study of this chaotic behavior is done in several ways. A well known theorem from complex dynamics shows that when the orbit of each critical point of a polynomial tends to infinity, then the Julia set of the polynomial is totally disconnected. Thus Λ_μ that is the Julia set of F_μ for $\mu > 4$ is totally disconnected (see [4, Theorem 9.8.1]). Henry in [11] proves that for $\mu > 4$, the Lebesgue measure of Λ_μ is zero, therefore Λ_μ does not contain any interval. The totally disconnectedness of Λ_μ for $\mu > 4$ can be proved by using [10, Proposition 2.8] and negativeness of the Schwarzian derivative of Λ_μ. In [13] Robinson shows, by using Schwarz Lemma from complex analysis, that for $\mu > 4$, Λ_μ is a Cantor set. In [12] it is shown that when $\mu > 4$, Λ_μ is a Cantor set and $F_\mu|_{\Lambda_\mu}$ is chaotic by employing the negative Schwarzian derivative and the Return Lemma. Glendinning [9], by conjugating F_μ, gives an estimate of the expansion rate of the function on the invariant set by employing mathematical techniques of elementary calculus. In [3] Aulbach and Kieninger give an elementary and self-contained proof

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of the hyperbolicity of Λ_μ for $\mu > 4$. They also prove that F_μ is chaotic on Λ_μ. In [7] it is proved, when $\mu = 4$, for each interval $J \subseteq [0, 1]$ there is some $n \in \mathbb{N}$ such that $F_4^n(J) \supseteq [0, 1]$ and it is deduced that F_4 is chaotic on $[0, 1]$. In this method the first return map and the negative Schwarzian derivative play essential roles.

In this paper we recognize the main features that cause the logistic function F_μ to be chaotic on Λ_μ, and based on them we introduce some sufficient conditions for a real function f in order to be chaotic on an invariant subset of its domain. We call this function *a generalized logistic function* since the dynamics of f restricted to this invariant subset is similar to $F_\mu|_{\Lambda_\mu}$ when $\mu \geq 4$. Although, for $\mu > 1$, the logistic function F_μ is concave downward and 0 is a repelling fixed point, we show that for a generalized logistic function f, $f'(0) = 1$ is possible and there is no such restriction on the concavity of f (see Section 4, Example 2). Also, we do not consider the behavior of f on the complement of the invariant subset (see Section 4, Examples 2 and 3). We introduce these conditions and their conclusions in Section 2. We show in Section 3, Theorem 1, that for a generalized logistic function there is some invariant subset on which it is chaotic. The method that is used has been adapted from [7] where it is employed for F_4. In Section 4 some examples of the generalized logistic functions are presented.

We next describe our terminology and notations. Let I be an interval and $f : I \to I$ be a C^1 function. By f^n we mean $f \circ f^{n-1}$, where f^0 is the identity function. A point $x_0 \in I$ is called a *fixed point* of f if $f(x_0) = x_0$ and it is called a *periodic point* of f of *period* n, if n is the least natural number that $f^n(x_0) = x_0$. In this case, the set $\{x_0,\ f(x_0),\ f^2(x_0),\ \cdots,\ f^{n-1}(x_0)\}$ is called a *cycle of period* n. The *basin of* this cycle is $\cup_{i=0}^{n-1}\{x :\ \lim_{k\to\infty} f^{kn}(x) = f^i(x_0)\}$. The *immediate basin* of this cycle is the union of the connected components of its basin which contain a point of the cycle. A cycle $\{f^i(x_0) : 0 \leq i \leq n-1\}$ is called an *attracting cycle, repelling cycle*, or *neutral cycle* if $|(f^n)'(x_0)| < 1$, $|(f^n)'(x_0)| > 1$, or $|(f^n)'(x_0)| = 1$, respectively.

Let f be a C^3 function such that $f^{'}(x) \neq 0$. The *Schwarzian derivative* of f at x is defined by:

$$Sf(x) = \frac{f^{'''}(x)}{f^{'}(x)} - \frac{3}{2}\left(\frac{f^{''}(x)}{f^{'}(x)}\right)^2.$$

Every function $f : I \to I$ with negative Schwarzian derivative has the following properties (see [7, 6] for details):

1. The Schwarzian derivative of f^n is negative.
2. $f^{'}$ does not have a positive local minimum or a negative local maximum.

3. The immediate basin of a neutral cycle of f contains some intervals.
4. The immediate basin of any attracting (neutral) cycle contains either a critical point of f or a boundary point of the interval I.
5. If f has finitely many critical points then it has finitely many periodic points of period m for each $m \in \mathbb{N}$.

For $\Lambda \subseteq I$, a function $f : \Lambda \to \Lambda$ is called *chaotic on* Λ if the following three conditions hold (see [7] for more details):

1. The set of the *periodic points* of f is dense in Λ.
2. f is topologically *transitive* (i.e., for every pair of open subsets U and V of Λ, there exists some $k > 0$ such that $f^k(U) \cap V \neq \emptyset$).
3. f has *sensitive dependence on initial conditions* (i.e., there exists $\delta > 0$ such that for any $x \in \Lambda$ and any open subset U of Λ containing x, there exist $y \in U$ and $n \geq 0$ such that $|f^n(x) - f^n(y)| > \delta$).

2. PROPERTIES OF A GENERALIZED LOGISTIC FUNCTION

In this section we introduce a real function f on an interval I that satisfies some conditions which make f to be chaotic on an invariant subset of I. We introduce these conditions in several steps and in each step we derive the properties of f.

Let I be an interval, $f : I \to I$ be a continuous function, and $[0, 1] \subseteq I$. In the first step we suppose that f satisfies the following conditions.

(c1) $f|_{[0,1]}$ is a C^3 function.

(c2) $f(0) = f(1) = 0$.

(c3) f has just one critical point c in the interval $(0, 1)$ and $f(c) \geq 1$.

Note that under these conditions, $f(x) > 0$, for $x \in (0, 1)$ and the critical point c is the maximum point of f on the interval $[0, 1]$, consequently f is strictly increasing on $[0, c]$ and strictly decreasing on $[c, 1]$, as well. Therefore f has only one non zero fixed point p in the interval $(c, 1)$, since $f(1) - 1 < 0$ and $f(c) - c \geq 1 - c > 0$.

The following conditions guarantee that f has no fixed point in interval $(0, c)$.

(c4) $f(x) > x$ for every positive x that is close enough to 0.

(c5) $f|_{(0,1)}$ has negative Schwarzian derivative.

Suppose that now f satisfies conditions (c1)-(c5). Since $f|_{(0,1)}$ has negative Schwarzian derivative, the number of fixed points of f is finite in $[0,1]$. If f has some fixed points in interval $(0,c)$, then we call the least fixed point in interval $(0,c)$ by x_0. Now, condition (c4) guarantees $f(x) > x$, for $x \in (0,x_0)$. Thus x_0 is attracting from the left and there must be a critical point in the immediate basin of x_0, that is a contradiction. Therefore, f has no fixed point on the interval $(0,c)$.

Also, under the conditions (c1)-(c3) the equation $f(x) = a$ with $0 \le a < f(c)$ has exactly two solutions in the interval $[0,1]$. Let $\hat{q} \le q$ be the solutions of $f(x) = 1$ in the interval $[0,1]$, and $0 < \hat{p} < p$ be such that $f(\hat{p}) = f(p) = p$. Note that if $f(c) = 1$, then $\hat{q} = q = c$, also note that $\hat{p} < \hat{q} \le q < p$. Let

$$\Lambda = \{x \in [0,1] : f^n(x) \in [0,1] \text{ for all } n \in \mathbb{N}\}. \tag{1}$$

It can be shown that $f(\Lambda) = \Lambda$. Thus Λ is an invariant subset of $[0,1]$ under f. In order to prove that f is chaotic on Λ, we need further assumptions to guarantee that for each $x \in (\hat{p},\hat{q}) \cup (q,p)$ there is some n such that $|(f^n)'(x)| > 1$. Hence we suppose f satisfies the following conditions as well.

(c6) $f'(\hat{p}) \ge 1$, where $0 < \hat{p} < 1$ is the preimage of the non-zero fixed point p.

(c7) $\hat{p} \ge \max\{p - q, \hat{q} - \hat{p}\}$, where $0 < \hat{q} \le q < 1$ are the preimages of 1.

We call a function f that satisfies conditions (c1)-(c7) a *generalized logistic function.* In the next section, we study the dynamics of $f|_\Lambda$.

Remark 1. Note that in this section by some modifications in the conditions, we can choose any other interval $[x_0, x_1]$ instead of interval $[0,1]$. More precisely, in conditions (c1)-(c5) we should replace 0 with x_0 and 1 with x_1, in (c6) we should replace $0 < \hat{p} < 1$ with $x_0 < \hat{p} < x_1$, and also in condition (c7) we should replace $\hat{p} \ge \max\{p - q, \hat{q} - \hat{p}\}$ with $\hat{p} - x_0 \ge \max\{p - q, \hat{q} - \hat{p}\}$.

Remark 2. Note that under the conditions (c2) and (c3) when $f(c) > 1$, there exists a compact set $X \subseteq I$ such that $f(X) = X$ and $f : X \to X$ is semi-conjugate to $\sigma : \Sigma_2 \to \Sigma_2$ (see [5, Proposition 15. page 35] for details), and when $f(c) > 1$ and $f'(0) > 1$, for each neighborhood U of 0, there is an integer $n > 0$ such that f^n has a hyperbolic invariant subset in U on which f^n is topologically conjugate to $\sigma : \Sigma_2 \to \Sigma_2$ (see [7, Theorem 16.5] for details).

3. CHAOS IN A GENERALIZED LOGISTIC FUNCTION

Our aim in this section is to prove the following theorem.

Theorem 1. *Suppose that f is a generalized logistic function. Let Λ be as defined in (1). Then f is chaotic on Λ.*

We use the following lemmas to prove Theorem 1. In these lemmas we suppose that f is a generalized logistic function and Λ is as defined in (1).

Lemma 2. *Suppose U is an open interval in $[0,1]$ and $0 \in U$. Then there are some closed interval $U' \subseteq U$ and some $n \geq 1$ such that $0 \in U'$, $f^n(U') = [0,1]$ and $f^k(U') \subseteq [0,\hat{q}]$ for $k = 0, 1, \cdots, n-1$.*

Proof. Note that $f : [0,\hat{q}] \to [0,1]$ is increasing and by (c4) the fixed point 0 is an attracting fixed point from the right of $(f|_{[0,\hat{q}]})^{-1}$. Thus there is the decreasing sequence $(\hat{q}_{-n})_{n\in\mathbb{N}}$ such that $f(\hat{q}_{-n}) = \hat{q}_{-n+1}$ and $\lim_{n\to\infty} \hat{q}_{-n} = 0$ where $\hat{q}_0 = \hat{q}$. Therefore there are some closed interval $U' \subseteq U$ and some $n \geq 1$ such that $0 \in U'$, $f^n(U') = [0,1]$ and $f^k(U') \subseteq [0,\hat{q}]$ for $k = 0, 1, \cdots, n-1$. □

Remark 3. Note that by (c5) and this fact that f is strictly increasing on $[0,c]$ and strictly decreasing on $[c,1]$, we can conclude that $f'(p) < -1$.

Lemma 3. *Suppose $U \subseteq [0,1]$ is an open interval that contains p. Then there are some closed interval $U' \subseteq U$ and some $n \geq 1$ such that $p \in U'$, $f^n(U') \supseteq [0,1]$ and $f^k(U') \subseteq [0,1]$ for $k = 0, 1, \cdots, n-1$*

Proof. Let $x \in U \cap (p,1)$ such that for all $t \in (f(x), x)$, $f'(t) < -1$. Since f is strictly decreasing on $[c,1]$ and $Sf < 0$, there is the least n_0 such that $f^{2n_0+1}(x) \leq q$, otherwise f has an attracting (neutral) 2-cycle in $(q,1)$ which is a contradiction.

Next note that $q < f^k(x) < 1$, for $k = 0, 1, \cdots, 2n_0$. Therefore, there is $z \in (p,x)$ such that $f^{2n_0+1}(z) = q$. Let $U' = [p,z]$. Then $f^{2n_0+4}(U') \supseteq [0,1]$ and $f^k(U') \subseteq [0,1]$, for $k = 0, 1, \cdots, 2n_0+3$. Therefore there are some closed interval $U' \subseteq U$ and some $n \geq 1$ such that $p \in U'$, $f^n(U') \supseteq [0,1]$ and $f^k(U') \subseteq [0,1]$, for $k = 0, 1, \cdots, n-1$. □

Here we use the method of [7] in partitioning the set $(\hat{p},\hat{q}) \cup (q,p)$. Note that $f(q,p) = (p,1)$ and $f(p,1) = (0,p)$, thus there is an interval $A_2 = [a_2,\ b_2) \subset (q,p)$ such that $f^2(A_2) = [\hat{p},p)$. Let the subset whose image under f^2 is $(0,\hat{p})$ be $W_2 = (q,a_2)$. Moreover, $f(0,\hat{p}) = (0,p)$, therefore there is an interval $A_3 \subseteq W_2$ such that $f^3(A_3) = [\hat{p},p)$. By continuing this process we construct subsets A_n and W_n for $n \geq 2$ such that

$$q < \cdots < b_n < \cdots < b_3 < b_2 = p,\ b_{n+1} = a_n,$$

$$W_n = (q,a_n),\ A_n = [a_n,b_n),$$
$$f^n(A_n) = [\hat{p},p),\ f^n(W_n) = (0,\hat{p})$$

and

$$\cup A_n = (q, p).$$

Note that $b_n - a_n \to 0$ as $n \to \infty$.

Since $f^n([a_n, b_n)) = [\hat{p}, p)$, we have

$$\frac{| f^n(a_n) - f^n(b_n) |}{| a_n - b_n |} > \frac{p - \hat{p}}{p - q} > 1,$$

also, $f^n((q, a_n)) = (0, \hat{p})$ and by condition (c7),

$$\frac{| f^n(a_n) - f^n(q) |}{| a_n - q |} > \frac{\hat{p}}{p - q} \geq 1.$$

By the Mean Value Theorem we have $| (f^n)'(c_n) |> 1$ and $| (f^n)'(d_n) |> 1$ for some $c_n \in (a_n, b_n)$ and $d_n \in (q, a_n)$. Since the Schwarzian derivative of $f|_{(0,1)}$ is negative, we conclude that $| (f^n)'(a_n) |> 1$. Since $b_n = a_{n-1}$, for $n \geq 3$, we obtain

$$\begin{aligned} |(f^n)'(b_n)| &= |f'(f^{n-1}(b_n))||(f^{n-1})'(b_n)| \\ &= |f'(f^{n-1}(a_{n-1}))||(f^{n-1})'(a_{n-1})| \\ &= |f'(\hat{p})||(f^{n-1})'(a_{n-1})| > 1. \end{aligned}$$

Also note that $|(f^2)'(b_2)| = |(f^2)'(p)| > 1$. Therefore if $x \in [a_n, b_n)$, then $| (f^n)'(x) |> 1$.

The subintervals $\hat{W}_n$ and $\hat{A}_n$ in $(\hat{p},\ \hat{q})$ are constructed in a similar way. Note that condition (c6) guarantees $|(f^2)'(\hat{b_2})| > 1$, as well. See [7] for more details.

Now suppose that the open interval U is a subset of $(\hat{p}, \hat{q}) \cup (q, p)$ and $U \cap \Lambda \neq \emptyset$. In this case we are going to show that there is some $n \geq 2$ such that$f^n(U) \supseteq [0, 1]$. In the proof of the following lemma,we use the fact that f^n is expanding on $[a_n, b_n)$.

Lemma 4. *Let $U \subseteq (\hat{p}, \hat{q}) \cup (q, p)$ be an open interval such that $U \cap \Lambda \neq \emptyset$. Then there are a closed interval $U' \subseteq U$ and an $n \geq 1$ such that $f^n(U') \supseteq [0, 1]$ and $f^k(U') \subseteq [0, 1]$ for $k = 0, 1, \cdots, n - 1$.*

Proof. Let $U \subseteq (q, p)$. If there exists some m such that $f^m(U)$ contains $\hat{b}_k$ or b_k for some $k \geq 2$, then $p \in f^{k+m}(U)$ and the claim holds by Lemma 3. Otherwise let $U_0 = U \subseteq (a_{n_0}, b_{n_0})$ and $U_1 = f^{n_0}(U_0) \subseteq (\hat{b}_{n_1},\ \hat{a}_{n_1}) \cup (a_{n_1}, b_{n_1})$. Note that $U \cap \Lambda \neq \emptyset$, therefore $f^n(U) \nsubseteq (\hat{q}, q)$ for all n. By induction on k, the sequences (U_k) of open intervals and (n_k) of integers are constructed such that $U_{k+1} = f^{n_k}(U_k) \subseteq (\hat{b}_{n_{k+1}},\ \hat{a}_{n_{k+1}}) \cup (a_{n_{k+1}},\ b_{n_{k+1}})$ and $b_n,\ \hat{b}_n \notin U_k$ for all n and all k.

This is impossible, since f^n is expanding on $(\hat{b}_n, \hat{a}_n) \cup (a_n, b_n)$ and the length of $(\hat{b}_n, \hat{a}_n) \cup (a_n, b_n)$ tends to zero as $n \to \infty$. □

LEMMA 5. *If U is an open interval in $[0,1]$ such that $U \cap \Lambda \neq \emptyset$. Then there are a closed interval $U' \subseteq U$ and an $n \geq 1$ such that $f^n(U') \supseteq [0,1]$ and $f^k(U') \subseteq [0,1]$ for $k = 0, 1, \cdots, n-1$.*

Proof. Note that $f(\hat{p}) = f(p) = p$ and $f(1) = 0$. Thus, by Lemmas 2-4, it is enough that we consider the case $U \subseteq (0, \hat{p}) \cup (p, 1)$. Now let $x \in U \cap \Lambda \subseteq (0, \hat{p})$. The sequence $\left(f^n(x)\right)_{n \geq 0}$ is increasing, so $f^n(x) > \hat{p}$, for some $n \geq 1$. Thus $f^n(x) \in (\hat{p}, \hat{q}) \cup (q, p)$, for some $n \geq 1$. Choose an open interval $V \subseteq U$ containing x such that $f^n(V) \subseteq (\hat{p}, \hat{q}) \cup (q, p)$ is an open interval and then use lemma 4 for $f^n(V)$. The case $U \subseteq (p, 1)$ is clear, since $f(p, 1) = (0, p)$. □

The following lemma is useful in proving the existence of a fixed point in a closed interval.

LEMMA 6 ([2, Theorem 3.17]). *Suppose $g : \mathbb{R} \to \mathbb{R}$ is a continuous function and I is a closed interval such that $g(I) \supseteq I$. Then g has a fixed point in I.*

We use the following lemma to prove that f is transitive and has sensitive dependence on initial conditions.

LEMMA 7. *Let A be a bounded closed subset of $\mathbb{R}$ and $g : A \to A$ be a continuous function. Suppose for every open subset $U \subseteq A$, there is some n such that $g^n(U) = A$. Then g is transitive and has sensitive dependence on initial conditions.*

Proof. Let $a = \inf A, b = \sup A$ and $\delta = (b-a)/4$. Suppose $x \in A$ and the open subset U contains x. Then $g^n(U) = A$ for some n. We choose $z \in A$ and $y \in U$ such that $|g^n(x) - z| > \delta$ and $z = g^n(y)$. Thus g has sensitive dependence on initial conditions. Also for every open set V, we have $g^n(U) \cap V \neq \emptyset$. Hence g is transitive. □

We are ready to prove our main result.

Proof of Theorem 1. Let U_1 be an open subset in Λ, i.e., $U_1 = U \cap \Lambda$ for some open interval in $[0,1]$. By Lemma 5, there are a closed interval $U' \subseteq U$ and an $n \geq 1$ such that $f^n(U') \supseteq [0,1]$ and $f^k(U') \subseteq [0,1]$ for $k = 0, 1, \cdots, n-1$. Thus, by Lemma 6, U' and hence U_1 contains a periodic point of $f|_\Lambda$. Also, since $f(\Lambda) = \Lambda$, we have $f^n(U_1) \subseteq \Lambda$. For every $y \in \Lambda$ there is some $x \in U'$ such that $f^n(x) = y$ and $f^i(x) \in [0,1]$ for $i = 0, 1, \cdots, n-1$. Therefore $x \in \Lambda$ and consequently $x \in U_1$. Hence $f^n(U_1) = \Lambda$. Therefore by Lemma 7, f is transitive and has sensitive dependence on initial conditions on Λ.

Thus f is chaotic on Λ. □

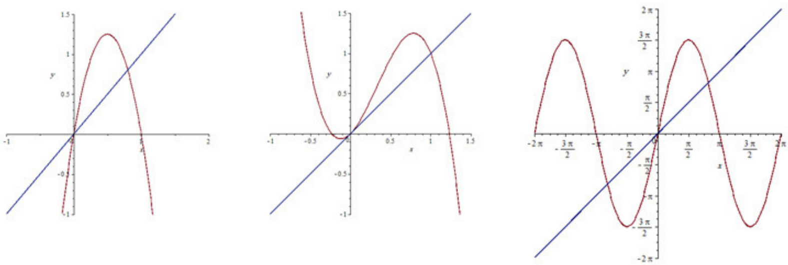

Fig. 1 – The graphs of F_5, $f_{-7/2}$ and $S_{3\pi/2}$, respectively from left to right

4. **EXAMPLES**

In this section we present three examples of the generalized logistic function. Therefore they are chaotic on Λ as defined in (1). The graphs and graphical analysis of these examples are shown in Fig. 1 and Fig. 2.

Example 1. Let $F_\mu(x) = \mu x(1-x)$, $\mu \geq 4$.

Then, $F_\mu(x)$ satisfies Conditions (c1)-(c4). Also, we have

1. $SF_\mu(x) = -(3/2)(2/(1-2x))^2 < 0$.
2. The non-zero fixed point is $p = (\mu - 1)/\mu$.
3. $\hat{p} = 1/\mu$, $\hat{q} = (\mu - \sqrt{\mu^2 - 4\mu})/(2\mu)$ and $q = (\mu + \sqrt{\mu^2 - 4\mu})/(2\mu)$.
4. $F'_\mu(\hat{p}) = \mu - 2 > 1$.

Finally, standard calculation shows that $2\hat{p} \geq \hat{q}$, for $\mu \geq 4$. Hence (c7) is satisfied since $p - q = \hat{q} - \hat{p}$. Therefore F_μ is chaotic on Λ_μ, where $\Lambda_\mu = \{x \in [0,1] : F_\mu^n(x) \in [0,1] \text{ for all } n \in \mathbb{N}\}$.

Note that $F'_\mu(0) = \mu > 1$. Thus, 0 is a repelling fixed point. Also, F_μ is concave downward and if for some n, $F_\mu^n(x) \notin [0,1]$, then $F_\mu^m(x) \notin [0,1]$ for all $m \geq n$, and $\lim_{m\to\infty} F_\mu^m(x) = -\infty$.

In the next two examples $[0,1]$ is replaced with another suitable interval.

Example 2. Let $f_a(x) = ax^2(x-1) + x$, $-3.59 \leq a \leq -3.42$.

The dynamics of the family $f_a(x)$, $a \neq 0$ has been studied in [1]. It has been shown that f_a is chaotic on Λ_a, for $-3.59 \leq a \leq -3.42$, where $\Lambda_a = \{x \in [0, x_1] : f_a^n(x) \in [0, x_1]; \forall n \geq 1\}$, $x_1 > 0$ and $f_a(x_1) = f_a(0) = 0$.

In this example 0 is a neutral fixed point of f_a and it is repelling from the right and attracting from the left, thus (c4) is satisfied. The function f_a has only one critical point c_1 in $[0, x_1]$ and $f_a(c_1) \geq x_1$, hence (c3) is satisfied.

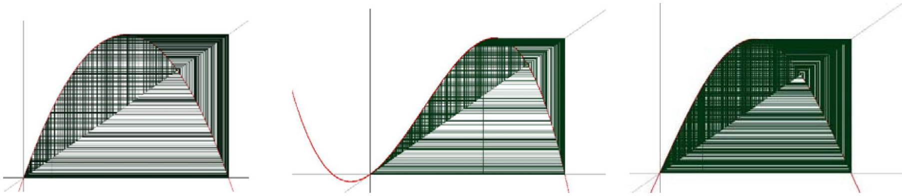

Fig. 2 – Graphical analysis of $F_4, f_{-3.42}$ and S_π, respectively from left to right.

Also, by [1, Lemma 2.3], condition (c7) is satisfied. The investigation of the other conditions is straightforward.

In contrary to the logistic function, the concavity of f_a changes in $[0, x_1]$, $f'_a(0) = 1$ and when $-3.59 \leq a \leq -3.42$, the points of interval $[0, x_1]$ whose orbits leave $[0, x_1]$ are attracted to 0.

Example 3. Let $S_\lambda(x) = \lambda \sin x,\ \pi \leq \lambda \leq \sqrt{1+\pi^2}$.

One can show that

1. $S_\lambda|_{[0,\ \pi]}$ is a C^3 function.
2. $S_\lambda(0) = S_\lambda(\pi) = 0$.
3. S_λ has just one critical point $c = \pi/2$ in interval $(0, \pi)$, and $S_\lambda(c) \geq \pi$.
4. 0 is a repelling fixed point.
5. $S_\lambda|_{(0,\pi)}$ has negative Schwarzian derivative.

Let p, $\hat{p}$, q and $\hat{q}$ have the similar roles that defined in Section 2. Thus $\sin \hat{p} = p/\lambda$. Then for $0 < x < p$, we have $S_\lambda(x) > x$ and for $p < x < \pi$, we have $S_\lambda(x) < x$. Now, to verify (c6) and (c7), note that for $\pi \leq \lambda \leq \sqrt{1+\pi^2}$, we have $S_\lambda(3\pi/4) < 3\pi/4$, and thus

$$p < \frac{3\pi}{4}. \tag{2}$$

From (2) we conclude $p^2 < \pi^2 - 1 \leq \lambda^2 - 1$. Hence $S'_\lambda(\hat{p}) = \lambda \cos \hat{p} \geq 1$ Thus (c6) is satisfied. Also, from (2) we conclude $\pi - p > p - \frac{\pi}{2}$, thus $\hat{p} = \pi - p > p - (\pi/2) > p - q = \hat{q} - \hat{p}$. Therefore (c7) is satisfied too.

Note that in contrary to the other two examples there are some points in $[0, \pi]$ whose orbits leave $[0, \pi]$ and return to it frequently.

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