RINGS SATISFYING THE STRONGLY HOPFIAN AND S-STRONGLY HOPFIAN PROPERTIES

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A commutative ring R is said to be strongly Hopfian if the chain of annihilators $ann(a) \subseteq ann(a^2) \subseteq \cdots$ is stationary for each $a \in R$ ([10]). In this paper we give a necessary and sufficient condition for the amalgamated duplication of R along an ideal I, $R \bowtie I$, to be strongly Hopfian. In the second part of this paper we introduce the notion of S-strongly Hopfian rings. Let R be a commutative ring and S a multiplicative subset of R. An increasing sequence of ideals $(I_k)_{k\in\mathbb{N}}$ of R is called S-stationary if there exist a positive integer n and an $s \in S$ such that for each $k \geq n$, $sI_k \subseteq I_n$ ([7]). We say that R is an S-strongly Hopfian ring if the chain of annihilators $ann(a) \subseteq ann(a^2) \subseteq \cdots$ is S-stationary for each $a \in R$. We investigate the class of such rings R and we give an S-version of classical results regarding the strongly Hopfian property.

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1. INTRODUCTION

A commutative ring R is said to be strongly Hopfian if the chain of annihilators $ann(a) \subseteq ann(a^2) \subseteq \cdots$ is stationary for each $a \in R$ [10]. The class of strongly Hopfian rings contains Noetherian rings, Laskerian rings, rings satisfying acc on d-annihilators and those satisfying acc on d-colons, rings satisfying accr [14], rings which are embeddable in a zero dimensional ring [11] (Gilmer called strongly Hopfian rings (Qp) rings), in particular zero dimensional rings are strongly Hopfian. In the first part of this paper we study several rings that satisfy the strongly Hopfian property. First let us recall the following notions. Let $\mathcal{R} = (R_n)_{n\in\mathbb{N}}$ be an increasing sequence of unitary commutative rings and $R = \cup_{n\in\mathbb{N}} R_n$, their union. Let $\mathcal{R}[X]$ be the ring of polynomials with coefficients of degree i in R_i. Then $\mathcal{R}[X]$ is a subring of the ring of polynomials $R[X]$ ([8], [6]). Also the amalgamated duplication of a ring R along an ideal I is a ring that is defined as the following subring with unit element $(1, 1)$ of $R \times R$:

$$R \bowtie I = \{(r, r + i) | r \in R, i \in I\}.$$

This construction has been studied, in the general case, and from the different points of view of pullbacks, by D'Anna and Fontana [3].

Let $\mathcal{R} = (R_n)_{n\in\mathbb{N}}$ be an increasing sequence of unitary commutative rings, $R = \cup_{n\in\mathbb{N}} R_n$ their union. We show that R is strongly Hopfian if and only if $\mathcal{R}[X]$ is strongly Hopfian. Also we give a necessary and sufficient condition for the amalgamated duplication of R along an ideal I, $R \bowtie I$ to be strongly Hopfian. We prove that R is strongly Hopfian if and only if $R \bowtie I$ is strongly Hopfian.

On the other hand, let R be a commutative ring and S a multiplicative subset of R. In [7], the authors introduced the concept of "S-stationary sequence" as follows, let $(I_k)_{k\in\mathbb{N}}$ be an increasing sequence of ideals of R. Then $(I_k)_{k\in\mathbb{N}}$ is called S-stationary if there exist a positive integer n and an $s \in S$ such that for each $k \geq n$, $sI_k \subseteq I_n$. Note that if S consists of units of R, then the two notions, S-stationary and stationary, coincide.

The main purpose of the second part of this paper is to introduce and to investigate the notion of rings satisfying the S-strongly Hopfian property. Let R be a commutative ring and S a multiplicative subset of R. We say that R is an S-strongly Hopfian ring if the chain of annihilators $ann(a) \subseteq ann(a^2) \subseteq \cdots$ is S-stationary for each $a \in R$. So every S-Noetherian ring (for all ideals I of R, $sI \subseteq J \subseteq I$ for some finitely generated ideal J and some $s \in S$) is an S-strongly Hopfian ring. We show that the S-strongly Hopfian and the strongly Hopfian properties are the same when S consists of nonzero divisors. Also, we prove that R is an S-strongly Hopfian ring if and only if for each finitely generated ideal I of R, the sequence $(ann(I^k))_k$ is S-stationary. Let R be a commutative ring and $S \subseteq R$ a multiplicative subset of R. Recall from [7] that R is said to satisfy the S-accr condition if the ascending chain of colon ideals of the form $(I : a) \subseteq (I : a^2) \subseteq (I : a^3) \subseteq \cdots$ is S-stationary for each $a \in R$ and every ideal I of R. We show that R satisfies the S-accr condition if and only if R/I is an $\overline{S}$-strongly Hopfian ring for each ideal I of R disjoint from S, where $\overline{S} = \{\overline{s} \mid s \in S\}$ (multiplicative subset of R/I). Recall that a multiplicative set S of R is anti-Archimedean if for each $s \in S$, $S \bigcap (\bigcap_{n\geq 1} s^n R) \neq \emptyset$, see [1]. We study the Hilbert basis theorem for an S-strongly Hopfian (S-SH) ring. When S is anti-Archimedean we prove that R is an S-strongly hopfian ring if and only if $R[X]$ is an S-strongly Hopfian ring. Finally, we study the Nagata's idealization (as a particular case of the amalgamation) $R(+)M$ related to those of R and M to be an S-strongly Hopfian ring. Let first recall some facts of the idealization. Let R be a commutative ring with identity and M a unitary $R-$module. The idealization of M in R (or trivial extension of R by M) is a commutative ring $R(+)M = \{(r, m) \mid r \in R \text{ and } m \in M\}$ under the usual addition and the multiplication defined as $(r_1, m_1)(r_2, m_2) = (r_1 r_1, r_1 m_2 + r_2 m_1)$ for all

$(r_1, m_1), (r_2, m_2) \in R(+)M$. It is easy to show that if S is a multiplicative subset of R, then, $S(+)M$ is a multiplicative subset of $R(+)M$. We show that $R(+)M$ is an $S(+)M$-strongly Hopfian ring if and only if R is an S-strongly Hopfian ring and for each $a \in R$, the sequence $(ann_M(a^k))_k$ is S-stationary.

2. STRONGLY HOPFIAN RING

Recall from [10] that a commutative ring R is said to be strongly Hopfian if the chain of annihilators $ann(a) \subseteq ann(a^2) \subseteq \cdots$ is stationary for each $a \in R$. In this section we study when the polynomial ring of the form $\mathcal{R}[X]$ and the amalgamated ring $R \bowtie I$ satisfy the strongly Hopfian property. First let us recall the following notion: let $\mathcal{R} = (R_n)_{n\in\mathbb{N}}$ be an increasing sequence of unitary commutative rings and $R = \cup_{n\in\mathbb{N}} R_n$, their union. Let $\mathcal{R}[X]$ (resp. $\mathcal{R}[[X]]$) be the ring of polynomials (resp. power series) with coefficients of degree i in R_i. Then $\mathcal{R}[X]$ (resp. $\mathcal{R}[[X]]$) is a subring of the ring of polynomials $R[X]$ (resp. formal power series $R[[X]]$) ([8], [6]). Our next Theorem gives a necessary and sufficent condition for the ring $\mathcal{R}[X]$ to be strongly Hopfian.

THEOREM 2.1. *Let $\mathcal{R} = (R_n)_{n\in\mathbb{N}}$ be an increasing sequence of unitary commutative rings, $R = \cup_{n\in\mathbb{N}} R_n$ their union. Then R is strongly Hopfian if and only if $\mathcal{R}[X]$ is strongly Hopfian.*

Proof. "⇒" If R is strongly Hopfian, then by [10, Theorem 5.1], $R[X]$ is strongly Hopfian. So by [9, Proposition 1.2], $\mathcal{R}[X]$ is strongly Hopfian.

"⇐" Assume that $\mathcal{R}[X]$ is strongly Hopfian and we show that R is strongly Hopfian. Let $a \in R$. There exist $p \in \mathbb{N}$ such that $a \in R_p$. Since $aX^p \in \mathcal{R}[X]$, then by hypothesis there exist a positive integer n such that for each $k \geq n$, $ann_{\mathcal{R}[X]}(aX^p)^k \subseteq ann_{\mathcal{R}[X]}(aX^p)^n$. We show that for each $k \geq n$, $ann_R(a^k) \subseteq ann_R(a^n)$. Let $k \geq n$ and $b \in R$ such that $ba^k = 0$. There exist $m \in \mathbb{N}$ such that $b \in R_m$. Since $bX^m \in \mathcal{R}[X]$ and $bX^m(aX^p)^k = 0$, then $bX^m(aX^p)^n = 0$ and hence $ba^n = 0$. □

Remark 2.1. The strongly Hopfian property does not carry over to the power series ring. In fact, there is an example of a commutative ring R satisfing the strongly Hopfian property such that $R[[X]]$ does not have the strongly Hopfian property[9].

PROPOSITION 2.1. *Let $A \subseteq B$ be an extension of commutative rings. Then $A+XB[[X]]$ is strongly Hopfian if and only if $B[[X]]$ is strongly Hopfian.*

Proof. "⇐" $A + XB[[X]]$ is a subring of $B[[X]]$. So we can conclude by [9, Proposition 1.2].

"$\Rightarrow$" Assume that $A + XB[[X]]$ is strongly Hopfian and we show that $B[[X]]$ is strongly Hopfian. Let $f \in B[[X]]$. Then $Xf \in A + XB[[X]]$. So by hypothesis there exist a positive integer n such that for each $k \geq n$, $ann_{A+XB[[X]]}(Xf)^k \subseteq ann_{A+XB[[X]]}(Xf)^n$. We show that for each $k \geq n$, $ann_{XB[[X]]}(f)^k \subseteq ann_{B[[X]]}(f)^n$. Let $k \geq n$ and $g \in B[[X]]$ such that $gf^k = 0$. Since $Xg \in A + XB[[X]]$ and $Xg(Xf)^k = 0$. Then $Xg(Xf)^n = 0$ and hence $gf^n = 0$. □

Recall that a ring R is chained if its lattice of ideals is linearly ordered under inclusion ([13]).

Corollary 2.1. *Let $A \subseteq B$ be an extension of commutative rings such that B is a chained ring. Then $A + XB[[X]]$ is strongly Hopfian if and only if B is strongly Hopfian.*

Proof. By [13, Theorem 4.2], B is strongly Hopfian if and only if $B[[X]]$ is strongly Hopfian. So by Proposition 2.1, $A + XB[[X]]$ is strongly Hopfian if and only if B is strongly Hopfian. □

Recall that R is said power serieswise Armendariz if for each $f = \sum_{i\geq 0} a_i X^i$ and $g = \sum_{i\geq 0} b_i X^i \in R[[X]]$ such that $fg = 0$, then $a_i b_j = 0$ for each i and j ([12]).

Corollary 2.2. *Let $A \subseteq B$ be an extension of commutative rings such that B is a power serieswise Armendariz ring (then so is A). Then $A+XB[[X]]$ is strongly Hopfian if and only if B is strongly Hopfian.*

Proof. Follows from [9, Theorem 2.28] and Proposition 2.1. □

Remark 2.2. Using the same idea as in the proof of Proposition 2.1, we can show that if the sequence $\mathcal{R} = (R_n)_{n\in\mathbb{N}}$ is stationary at rank n, then the power series ring $\mathcal{R}[[X]]$ is strongly Hopfian if and only if $R_n[[X]]$ is strongly Hopfian. In particuler if R_n is a chained ring, then, $\mathcal{R}[[X]]$ is strongly Hopfian if and only if R_n is strongly Hopfian. Note that, if the power series ring $\mathcal{R}[[X]]$ is strongly Hopfian, then the sequence $\mathcal{R} = (R_n)_{n\in\mathbb{N}^*}$ is not stationary in general. Indeed, let $(Y_i)_{i\in\mathbb{N}^*}$ be a family of indeterminates over $R_0 = \mathbb{Z}$ and put $R_n = \mathbb{Z}[Y_1, ..., Y_n]$ for each $n \geq 1$. Then $R = \mathbb{Z}[Y_1, ...,]$ is a domain. So $R[[X]]$ is a domain. Therefore $R[[X]]$ is strongly Hopfian. Hence $\mathcal{R}[[X]]$ is strongly Hopfian but the sequence $\mathcal{R} = (R_n)_{n\in\mathbb{N}}$ is not stationary.

Let R be a commutative ring and I be a non-zero ideal of R. Let $R \bowtie I$ be the subring of $R \times R$ consisting of the elements $(r, r + i)$ for $r \in R$ and $i \in I$. We next give a necessary and sufficient condition for the ring $R \bowtie I$ to be strongly Hopfian.

THEOREM 2.2. *Let R be a commutative ring and I be a non-zero ideal of R. Then R is strongly Hopfian if and only if $R \bowtie I$ is strongly Hopfian.*

Proof. " $\Leftarrow$" Let $r \in R$. Since $(r,r) \in R \bowtie I$, there exist a positive integer n such that $ann_{R \bowtie I}((r,r)^k) \subseteq ann_{R \bowtie I}((r,r)^n)$. We show that $ann_R(r^k) \subseteq ann_R(r^n)$. Let $b \in ann_R(r^k)$. Since $(b,b) \in ann_{R \bowtie I}((r,r)^k)$, then $(b,b)(r,r)^n = (0,0)$. This implies that $br^n = 0$ and hence $b \in ann_R(r^n)$.

"$\Rightarrow$" Let $(r, r+i) \in R \bowtie I$. There exist positive integers n_1, n_2 such that for each $k \geq n_1$, $ann_R(a^k) \subseteq ann_R(a^n)$ and for each $k \geq n_2$, $ann_R((a+i)^k) \subseteq ann_R((a+i)^n)$. Let $n = \max\{n_1, n_2\}$. Then, for each $k \geq n$, $ann_{R \bowtie I}((r, r+i)^k) \subseteq ann_{R \bowtie I}((r, r+i)^n$. □

Definition 2.1 ([4]). Let A and B be commutative rings with unit, let J be an ideal of B, and let $f : A \longrightarrow B$ be a ring homomorphism. We define the subring of $A \times B$ as follows:

$$A \bowtie^f J = \{(a, f(a) + j) \mid a \in A \text{ and } j \in J\}.$$

We call the ring $A \bowtie^f J$ the amalgamation of A with B along J with respect to f.

PROPOSITION 2.2. *Let A and B be a commutative rings with unit, let J be an ideal of B, and let $f : A \longrightarrow B$ be a ring homomorphism. If A and $f(A) + J$ are strongly Hopfian, then $A \bowtie^f J$ is strongly Hopfian.*

Proof. Let $(a, f(a) + i) \in A \bowtie^f J$. Since A and $f(A) + J$ are strongly Hopfian, there exist a positive integer n such that for each $k \geq n$, $ann_A(a^k) \subseteq ann_A(a^n)$ and $ann_{f(A)+J}((f(a) + i)^k) \subseteq ann_{f(A)+J}((f(a) + i)^n)$. It is easy to show that $ann_{A \bowtie^f J}((a, f(a) + i)^k))) \subseteq ann_{A \bowtie^f J}((a, f(a) + i)^n))$. □

PROPOSITION 2.3. *Let A and B be a commutative rings with unit, let J be an ideal of B and let $f : A \longrightarrow B$ be a ring homomorphism such that there exists an ideal I of A with $f^{-1}(J) = ann(I)$. Then $A \bowtie^f J$ is strongly Hopfian if and only if A and $f(A) + J$ are strongly Hopfian.*

Proof. "$\Leftarrow$" Follows from Proposition 2.2.

"$\Rightarrow$" It is easy to show that A is strongly Hopfian. On the other hand by [5, Proposition 5.1], $A \bowtie^f J/(f^{-1}(J) \times \{0\}) \simeq f(A) + J$ and $f^{-1}(J) \times \{0\}$ $= ann(I \times B)$. Then by [9, Remark 1.14], $f(A) + J$ is a strongly Hopfian ring. □

Remark 2.3. Let A and B be a commutative rings with unit, let J be an ideal of B, and let $f : A \longrightarrow B$ be a ring homomorphism.

1. If $A \bowtie^f J$ is strongly Hopfian, it is not necessary for $f(A) + J$ to be strongly Hopfian. Indeed, let K be a field, $(X_n)_n$ be a set of indeterminates over K and $A = K[X_i, i \in \mathbb{N}]$. Let I be the ideal of A generated by the elements X_0X_1, $X_0^2X_2$, ...$X_0^nX_n$, ..., $B = A/I$, $f : A \to B$ the canonical mapping and $J = 0$. By [5, Proposition 5.1], we have $A \bowtie^f J \simeq A$. Then $A \bowtie^f J$ is a strongly Hopfian ring but $f(A) + J = B$ is not strongly Hopfian.

2. If A is strongly Hopfian, $A \bowtie^f J$ is not in general strongly Hopfian. Indeed, let $A \subseteq B$ be an extension of rings such that A is strongly Hopfian, B is not strongly Hopfian, $J = XB[X]$ and $f : A \longrightarrow B[X]$ the natural embedding. We have $A \bowtie^f J \simeq f(A) + J = A + XB[X]$ ([5, Proposition 5.1]) and A is strongly Hopfian but $A \bowtie^f J$ is not strongly Hopfian because $A + XB[X]$ is strongly Hopfian if and only if B is strongly Hopfian (Theorem 2.1).

3. S-STRONGLY HOPFIAN

Let R be a commutative ring and S be a multiplicative subset of R. Recall from [7] that an increasing sequence $(I_k)_{k\in\mathbb{N}}$ of ideals of R is called S-stationary if there exist a positive integer n and an $s \in S$ such that for each $k \geq n$, $sI_k \subseteq I_n$. We start this section by introducing the following definition in order to generalize some known results about Strongly Hopfian rings.

Definition 3.1. Let R be a commutative ring and S a multiplicative subset of R. We say that R is an S-strongly Hopfian ring (S.SH) if the chain of annihilators $ann(a) \subseteq ann(a^2) \subseteq \cdots$ is S-stationary for each $a \in R$.

Let R be a commutative ring and S be a multiplicative subset of R. Recall from [1] that, an ideal I of R is called S-finite, if $sI \subseteq J \subseteq I$ for some finitely generated ideal J of R and some $s \in S$. Also, R is called S-Noetherian if each ideal of R is S-finite.

Examples 3.1. Let R be a commutative ring and S a multiplicative subset of R.

1. If R is strongly Hopfian, then R is an S-strongly Hopfian ring.

2. If S consists of units of R, then R is an S-strongly Hopfian ring if and only if R is strongly Hopfian.

3. If R is an S-Noetherian ring, then R is an S-strongly Hopfian ring. Indeed by [7, Remark 2.3], every increasing sequence of ideals of R is S-stationary, in particular for each $a \in R$ the sequence $(ann(a^k))_k$ is S-stationary. So R is S-strongly Hopfian.

Proposition 3.1. *Let R be a commutative ring and S be a multiplicative subset of R consisting of nonzero divisors. Then R is an S-strongly Hopfian ring if and only if R is strongly Hopfian.*

Proof. "$\Leftarrow$" Clear.

"$\Rightarrow$" Let $a \in R$. Since R is S-strongly Hopfian, then there exist $s \in S$ and a positive integer n such that $s(ann(a^k)) \subseteq ann(a^n)$ for each $k \geq n$. We show that $ann(a^{n+1}) = ann(a^n)$. Let $b \in ann(a^{n+1})$, then $sba^n = 0$. As S is consisting of nonzero divisors, then $ba^n = 0$. Hence R is strongly Hopfian. □

Proposition 3.2. *Let $A \subseteq B$ be an extension of commutative rings and S a multiplicative subset of A. If B is an S-strongly Hopfian ring, then A is also an S-strongly Hopfian ring.*

Proof. Let $a \in A$. Since B is an S-strongly Hopfian ring, then the sequence $(ann_B(a^k))_k$ is S-stationary. So there exist an $s \in S$ and a positive integer n such that for each $k \geq n$, $s(ann_B(a^k)) \subseteq ann_B(a^n)$. This implies that for each $k \geq n$, $s(ann_B(a^k) \bigcap A) \subseteq ann_B(a^n) \bigcap A$. Thus for each $k \geq n$, $s(ann_A(a^k)) \subseteq ann_A(a^n)$, and hence A is an S-strongly Hopfian ring. □

Let R be a commutative ring and $S \subseteq R$ a multiplicative subset of R. Recall from [7] that R is said to satisfy the S-accr condition if the ascending chain of residuals of the form $(I : a) \subseteq (I : a^2) \subseteq (I : a^3) \subseteq \cdots$ is S-stationary for each $a \in R$ and every ideal I of R. Let I be an ideal of R and $\overline{S} = \{\overline{s} \mid s \in S\}$. Then it is easy to see that $\overline{S}$ is a multiplicative subset of R/I.

Proposition 3.3. *Let R be a commutative ring and S a multiplicative subset of R. Then R satisfies the S-accr condition if and only if for each ideal I of R such that $S \cap I = \emptyset$, R/I is an $\overline{S}$-strongly Hopfian ring.*

Proof. "$\Rightarrow$" Let I be an ideal of R such that $S \cap I = \emptyset$ and $\overline{a} \in R/I$. Since R satisfies the S-accr condition, then there exist an $s \in S$ and a positive integer n such that for each $k \geq n$, $s(I :_R a^k) \subseteq (I :_R a^n)$. We show that $\overline{s}(ann(\overline{a}^k)) \subseteq ann(\overline{a}^n)$ for each $k \geq n$. Let $\overline{\alpha} \in ann_{R/I}\overline{a}^k$. Then $\overline{\alpha a^k} = \overline{0}$ which implies that $s\alpha \in (I : a^n)$. This implies that $\overline{s\alpha a^n} = \overline{0}$ in R/I. So $\overline{s\alpha} \in ann_{R/I}(\overline{a}^n)$, and hence R/I is an $\overline{S}$-SH.

"$\Leftarrow$" Let I be an ideal of R and $a \in R$.
First case: $S \cap I = \emptyset$. As R/I is an $\overline{S}$-strongly Hopfian ring, then there exist an $s \in S$ and a positive integer n such that for each $k \geq n$, $\overline{s}(ann_{R/I}(\overline{a}^k)) \subseteq ann(\overline{a}^n)$. Let $\alpha \in (I :_R a^k)$. Then $\alpha a^k \in I$ which implies that $\overline{s\alpha a}^n = \overline{0}$ in R/I. Therefore $s\alpha a^n \in I$, so $s\alpha \in (I :_R a^n)$.

Second case: $S \cap I \neq \emptyset$. Let $s \in S \cap I$. Then $s(I : a^{k+1}) \subseteq (I : a^k)$ for each integer k, and hence R satisfies the S-accr condition. □

In the particular case when S consists of units of R we find the following Corollary.

COROLLARY 3.1. *The following assertions are equivalent for a commutative ring R.*

1. *R satisfies the accr condition.*

2. *R/I is strongly Hopfian for each ideal I of R.*

Remark 3.1. Recall that if $f \in Hom(A, B)$ and A is strongly Hopfian, then it is not necessary for B to be strongly Hopfian. Indeed let K be a field, $(X_n)_n$ be a set of indeterminates over K and $A = K[X_i, i \in \mathbb{N}]$. Let I be the ideal of A generated by the elements $X_0X_1,\ X_0^2X_2,\ ..,X_0^nX_n, ...,$. $B = A/I$ and f the canonical mapping. A is strongly Hopfian but $B = f(A)$ is not strongly Hopfian(see [9, Remark 1.14]).

PROPOSITION 3.4. *Let R be an S-strongly Hopfian ring and I an ideal of R such that there exists an ideal J of R with $I = ann(J)$ and $S \cap I = \emptyset$. Then R/I is $\overline{S}$-strongly Hopfian, where $\overline{S} = \{\overline{s} \mid s \in S\}$ (multiplicative subset of R/I).*

Proof. Let $\overline{a} \in R/I$. Since R is S-strongly Hopfian, there exist an $s \in S$ and a positive integer n such that for each $k \geq n$, $s(ann(a^k)) \subseteq ann(a^n)$. We show that $\overline{s}(ann_{R/I}(\overline{a}^k)) \subseteq ann_{R/I}(\overline{a}^n)$ for each $k \geq n$. Let $\overline{b} \in ann_{R/I}(\overline{a}^k)$. Then $ba^k \in I$ which implies that for each $j \in J$, $ba^kj = 0$. Therefore $bj \in ann(a^k)$ for each $j \in J$. So for each $j \in J$, $sba^nj = 0$. Thus $sba^n \in I$ and hence $\overline{s}\overline{b}\overline{a}^n = \overline{0}$. □

PROPOSITION 3.5. *Let R be a commutative ring and S a multiplicative subset of R. If R is an S-strongly Hopfian ring, then $S^{-1}R$ is strongly Hopfian.*

Proof. Let $\frac{a}{s} \in S^{-1}R$. Since the ring R is S-strongly Hopfian, then there exist an $s' \in S$ and a positive integer n such that for each $k \geq n, s'(ann(a^k)) \subseteq ann(a^n)$. We show that $ann(\frac{a}{s})^{n+1} \subseteq ann(\frac{a}{s})^n$. Let $\frac{b}{t} \in ann(\frac{a}{s})^{n+1}$. Then $\frac{ba^{n+1}}{ts^{n+1}} = 0$. So there exists $r \in S$ such that $rba^{n+1} = 0$. This implies that $rb \in ann(a^{n+1})$. So $s'rb \in ann(a^n)$ and hence $s'rba^n = 0$. Then $ann(\frac{a}{s})^{n+1} \subseteq ann(\frac{a}{s})^n$ and we can conclude that $S^{-1}R$ satisfies the strongly Hopfian property. □

Let $I =< b_1, ..., b_t >$ be a finitely generated ideal of a ring R. According to [14], if k and n are any two positive integers such that $k > nt$, then $I^k =< b_1^n, ..., b_t^n > I^{k-n}$ ([14, Lemma 2]).

PROPOSITION 3.6. *Let R be a commutative ring and S a multiplicative subset of R. Then R is an S-strongly hopfian ring if and only if for each finitely generated ideal I of R, the sequence $(ann(I^k))_k$ is S-stationary.*

Proof. "$\Leftarrow$" Let $a \in R$ and put $I =< a >$. Since $I^k =< a^k >$ and $ann_R(I^k) = ann_R(a^k)$, then by hypothesis the sequence $(ann(a^k))_k$ is S-stationary. So R is S-strongly Hopfian.

"$\Rightarrow$"Let $I =< b_1, ..., b_t >$. For each $i \in \{1, .., t\}$ there exist an $s_i \in S$ and a positive integer n_i such that for each $k \geq n_i$, $s_i(ann(b_i^k)) \subseteq ann(b_i^{n_i})$. Let $n = max\{n_i, 1 \leq i \leq t\}$ and $s = s_1 \cdots s_t$. It is easy to prove that, for each $i \in \{1, .., t\}$ and $k \geq n$, $s(ann(b_i^k)) \subseteq ann(b_i^n)$. If we take $n_0 = nt + 1$, then $n_0 > nt$; so by [14, Lemma 2], $I^{n_0} \subseteq< b_1^n, ..., b_t^n >$. We show that $s(ann(I^k)) \subseteq ann(I^{n_0})$ for each $k \geq n_0$. Since $I^{n_0} \subseteq< b_1^n, ..., b_t^n >$, then $ann(< b_1^n, ..., b_t^n >) \subseteq ann(I^{n_0})$. On the other hand as $b_i^k \in I^k$ for each $i \in \{1, .., t\}$, thus $ann(I^k) \subseteq ann(b_i^k)$, for each $i \in \{1, .., t\}$. Therefore, for each $k \geq n_0$ and $i \in \{1, .., t\}$, $s(ann(I^k)) \subseteq s(ann(b_i^k)) \subseteq ann(b_i^n)$. So for each $k \geq n_0$, $s(ann(I^k)) \subseteq \bigcap_{1 \leq i \leq t} ann(b_i^n) \subseteq ann(< b_1^n, ..., b_t^n >)$. Then for each $k \geq n_0$, $s(ann(I^k)) \subseteq ann(I^{n_0})$. $\square$

Recall that a multiplicative set S of R is anti-Archimedean if for each $s \in S$, $S \bigcap (\bigcap_{n \geq 1} s^n R) \neq \emptyset$, see [1]. Note that, in [7], the authors showed that, if S is a finite multiplicative set of a commutative ring R, then S is an anti-Archimedean set. For example, let $R = \mathbb{Z}/12\mathbb{Z}$ and $S = \{\overline{1}, \overline{2}, \overline{4}, \overline{8}\} \nsubseteq U(R)$ is an anti-Archimedean multiplicative set of R.

Our next result is to study the Hilbert basis theorem for an S-strongly Hopfian ring when S has a special property. To prove this, we need the following lemma.

LEMMA 3.1. *Let S be an anti-Archimedean multiplicative subset of a commutative ring R and $p(X) = a_0 + a_1X + \cdots + a_rX^r$ in $R[X]$. If there exist a $t \in S$ and a positive integer n such that for all $k \geq n$, $t(ann(a_j^k)) \subseteq ann(a_j^n)$ for each $0 \leq j \leq r$, then there exist an $\alpha \in S$ such that for all $k \in \mathbb{N}^*$, $\alpha(ann(p(X)^{(r+1)n+k})) \subseteq ann(p(X)^{(r+1)n})$.*

Proof. By induction on $r = \deg(p(X))$. For $r = 0$ we take $\alpha = t$.

We suppose that is true for all polynomials of degree$\leq r - 1$.

Let $p(X)=a_0+a_1X+\cdots+a_rX^r$ of degree r such that there exist a $t\in S$ and a positive integer n satisfy for each $k\geq n$, $t(ann(a_j^k))\subseteq ann(a_j^n)$ for all $j\in\{0,..,r\}$. Let $h(X)=b_0+b_1X+\cdots+b_sX^s\in ann((p(X))^{(r+1)n+k}$. We have for all $j\in\{0,..,s\}, t^{j+1}b_{s-j}\in ann(a_r^n)$. Indeed, $h(X)p(X)^{(r+1)n+k}=0$ implies that $(b_0+b_1X+\cdots+b_sX^s)(a_rX^r+\cdots+a_0)^{(r+1)n+k}=0$. Then $b_sa_r^{(r+1)n+k}=0$. Therefore $b_s\in ann(a_r^{(r+1)n+k})$. So $tb_s\in ann(a_r^n)$.

Since $ta_r^nh(X)(p(X))^{(r+1)n+k}=0$, then $tb_{s-1}a_r^na_r^{(r+1)n+k}=0$. Thus, $tb_{s-1}\in ann(a_r^{(r+1)n+k+n})$ and hence $t^2b_{s-1}\in ann(a_r^n)$. It is easy to show that $t^{j+1}b_{s-j}\in ann(a_r^n)$ for each $j\in\{0,..,s\}$. S is anti-Archimedean. Then $S\bigcap(\bigcap_{n\in\mathbb{N}}t^nR)\neq\emptyset$. Let $t'\in S\bigcap(\bigcap_{n\in\mathbb{N}}t^nR)$. We have $t'b_j\in ann(a_r^n)$ for all $j\in\{0..,s\}$.

On the other hand, write $p(X)=a_rX^r+p_{r-1}(X)$. Then,

$$h(X)p(X)^{(r+1)n+k}=0.$$

So $t'h(X)a_r^{n-1}(a_rX^r+p_{r-1}(X))^{(r+1)n+k}=0$ and hence

$$t'h(X)a_r^{n-1}(p_{r-1}(X))^{(r+1)n+k}=0.$$

Since $degp_{r-1}(X)\leq r-1$, then by recurrence hypothesis there exist an $s'\in S$ such that $s'(ann(p_{r-1}(X)^{rn+k}))\subseteq ann(p_{r-1}(X)^{rn})$, for all $k\in\mathbb{N}^*$. So $s't'h(X)a_r^{n-1}(p_{r-1}(X))^{rn}=0$. We have

$$s't'h(X)a_r^{n-2}(a_rX^r+p_{r-1}(X))^{(r+1)n+k}=0.$$

Since $t'h(X)a_r^n=0$ and $s't'h(X)a_r^{n-1}(p_{r-1}(x))^{rn}=0$. Then,

$$s't'h(X)a_r^{n-2}((p_{r-1}(X))^{(r+1)n+k}=0.$$

We deduce that $s't'h(X)a_r^{n-2}\in ann(p_{r-1}(X)^{(r+1)n+k})$. Then $s'^2t'h(X)a_r^{n-2}\in ann(p_{r-1}(X)^{rn})$. So by induction we can show that $(s')^nt'h(X)(p_{r-1}(X))^{rn}=0$. Then

$$\begin{aligned}(s')^nt'h(X)(p(X))^{(r+1)n+k}&=(s')^nt'h(X)(a_rX^r+p_{r-1}(X))^{(r+1)n}\\&=(s')^nt'h(X)\left[\sum_{i=0}^{n-1}\binom{(r+1)n}{i}(a_rX^r)^i(p_{r-1}(x))^{(r+1)n-i}\right.\\&\left.+\sum_{i=n}^{(r+1)n}\binom{(r+1)n}{i}(a_rX^r)^i(p_{r-1}(X))^{(r+1)n-i}\right]=0.\end{aligned}$$

Indeed in the first sum $i\leq n-1$ so $(r+1)n-i\geq rn$ and in the second sum $s't'h(X)a_r^i=0$ since $i\geq n$. Then $(s')^nt'ann(p(X))^{(r+1)n+k}\subseteq ann(p(X))^{(r+1)n}$. We have $\alpha=(s')^nt'\in S$ and for all $k\in\mathbb{N}^*$,

$$\alpha(ann(p(X))^{(r+1)n+k})\subseteq ann(p(X))^{(r+1)n}.$$

□

Theorem 3.1. *Let R be a commutative ring and let $S \subseteq R$ be an anti-Archimedean multiplicative set. Then R is an S-strongly Hopfian ring if and only if $R[X]$ is an S-strongly Hopfian ring.*

Proof. "$\Leftarrow$" Clear.

"$\Rightarrow$" Let $p(X) = a_rX^r + \cdots\cdot + a_0$ in $R[X]$. Since R satisfies the S-strongly Hopfian property, then for each $0 \leq j \leq r$ there exist an $s_j \in S$ and a positive integer n_j such that for all $k \geq n_j$, $s_j(ann(a_j^k)) \subseteq ann(a_j^n)$. Let $s = s_1 \cdots s_r$ and $n = max\{n_j, 0 \leq j \leq r\}$. Then for each $k \geq n$ and $0 \leq j \leq r$, $s(ann(a_j^k)) \subseteq ann(a_j^n)$. By lemma 3.1 there exist an $s' \in S$ such that $s'(ann(p(X))^{(r+1)n+k}) \subseteq ann(p(X))^{(r+1)n}$ for all $k \in \mathbb{N}^*$. This implies that $R[X]$ is S-strongly Hopfian. $\square$

In the particular case when S consists of units of R we find the following Corollary. So we give an S-version of the result of Hmaimou, Kaidi and Sánchez Campos [10].

Corollary 3.2. *Let R be a commutative strongly Hopfian ring. Then the polynomial ring $R[X]$ is strongly Hopfian.*

Corollary 3.3. *Let R be a commutative ring and $S \subseteq R$ an anti-Archimedean multiplicative set. If R is S-strongly Hopfian, then $R[X, X^{-1}]$ is S-strongly Hopfian.*

Proof. Let $p(X) = a_nX^{-n} + \cdots + a_1X + a_0 + b_1X + \cdots + b_pX^p$ in $R[X, X^{-1}]$. Since $X^np(X) \in R[X]$ and $R[X]$ is $S-$strongly Hopfian, there exist an $s \in S$ and a positive integer r such that for all $k \geq r$, $s(ann(X^np(X))^k) \subseteq ann(X^np(X))^r$. We show that $s(ann(p(X))^k) \subseteq ann(p(X))^r$ for all $k \geq r$. Let $k \geq r$ and $f(X) \in ann(p(X)^k$. There exist $m \in \mathbb{N}$ such that $X^mf(X) \in R[X]$. Then $X^{m+kn}f(X)(p(X))^k = 0$ which implies that $X^mf(X)(X^np(X))^k = 0$. So $X^mf(X) \in ann(X^np(X))^k)$. Therefore $sX^mf(X)(X^np(X))^r) = 0$. Then $sf(X)(p(X))^r = 0$ and hence $sf(X) \in ann((p(X))^r$. $\square$

Proposition 3.7. *Let I be a nonempty set, R a commutative ring and $S \subseteq R$ an anti-Archimedean multiplicative set . Then R is an S-strongly Hopfian ring if and only if $R[X_i, i \in I]$ is an S-strongly Hopfian ring.*

Proof. "$\Leftarrow$" Since R is a subring of $R[X_i, i \in I]$, then we can conclude with Proposition 3.2.

"$\Rightarrow$"Let $f \in R[X_i, i \in I]$. Then there exist $p \in \mathbb{N}$ such that $f \in R[X_{i_1}, .., X_{i_p}]$. Since R is S-strongly Hopfian so by Theorem 3.1, $R[X_{i_1}, .., X_{i_p}]$ is S-strongly Hopfian. Therefore there exist an $s \in S$ and a positive integer n such that for each $k \geq n$, $s(ann_{R[X_{i_1},..,X_{i_p}]}(f^k)) \subseteq ann_{R[X_{i_1},..,X_{i_p}]}(f^n)$. We show that $s(ann_A(f^k)) \subseteq ann_A(f^n)$ where $A = R[X_i, i \in I]$. Let $k \geq n$ and $g \in A$ such that $gf^k = 0$.

If $g \in R[X_{i_1}, .., X_{i_p}]$, then $sg \in ann_{R[X_{i_1},..,X_{i_p}]}(f^n) \subseteq ann_A(f^n)$.
If not, i.e., $g \notin R[X_{i_1}, .., X_{i_p}]$ there exist $q > p$ such that $g \in R[X_{i_1}, .., X_{i_q}]$. Then there exist $g_i \in R[X_{i_1}, .., X_{i_p}]$ such that $g = \sum_{i=p+1}^{q} g_i X_{i_{p+1}}^{\alpha_{i,p+1}} .. X_{i_q}^{\alpha_{i,q}}$. Therefore for each $i \in \{p+1, .., q\}$, $g_i f^k = 0$. So $g_i \in ann_{R[X_{i_1},..,X_{i_p}]}(f^k)$. Then $sg_i \in ann(f^n)$ for each $p+1 \leq i \leq q$. Thus $sg_i f^n = 0$ for all $i \in \{p+1, .., q\}$ and hence $sgf^n = 0$. □

Finally, we study how the S-strongly Hopfian property of Nagata's idealization (as a particular case of the amalgamation) $R(+)M$ is related to those of R and M.To do this,we recall some facts on the idealization. Let R be a commutative ring with identity and M a unitary $R-$module. The idealization of M in R (or trivial extension of R by M) is a commutative ring $R(+)M = \{(r, m) \mid r \in R \text{ and } m \in M\}$ under the usual addition and the multiplication defined as $(r_1, m_1)(r_2, m_2) = (r_1 r_1, r_1 m_2 + r_2 m_1)$ for all $(r_1, m_1), (r_2, m_2) \in R(+)M$. It is easy to show that if S is a multiplicative subset of R, then, $S(+)M$ is a multiplicative subset of $R(+)M$.

Definition 3.2. Let R be a commutative ring, $S \subseteq R$ be a multiplicative set, and M an R-module. An increasing sequence $(N_k)_{k \in \mathbb{N}}$ of submodules of M is called S-stationary if there exist a positive integer k and an $s \in S$ such that for each $k \geq n$, $sN_k \subseteq N_n$.

THEOREM 3.2. *Let R be a commutative ring, M a unitary R-module and S a multiplicative subset of R. Then, $R(+)M$ is an $S(+)M$-strongly Hopfian ring if and only if R is an S-strongly Hopfian ring and for each $a \in R$, the sequence $(ann_M(a^k))_k$ is S-stationary.*

Proof. "⇒" Let $a \in R$. Since $(a, 0) \in R(+)M$, then there exist a positive integer n and an $(s, m) \in S(+)M$ such that for each $k \geq n$,

$$(s, m)(ann_{R(+)M}((a, 0)^k) \subseteq ann_{R(+)M}((a, 0)^n.$$

It is easy to show that for each $k \geq n$, $s(ann_R(a^k)) \subseteq ann_R(a^n)$ and $s(ann_M(a^k)) \subseteq ann_M(a^n)$.

"⇐"Let $(a, m) \in R(+)M$. There exist an $s \in S$ and a positive integer n such that $s(ann_R(a^k)) \subseteq ann_R(a^n)$ and $s(ann_M(a^k)) \subseteq ann_M(a^n)$. We show that for each $k \geq n+1$, $(s, 0)(ann_{R(+)M}((a, m)^k) \subseteq ann_{R(+)M}((a, m)^{n+1})$. Let $k \geq n+1$ and $(b, m') \in ann_{R(+)M}((a, m)^k)$. Then $ba^k = 0$ and $kba^{k-1}m +$

$a^k m' = 0$. this implies that $sba^n = 0$. So $sba^{n+1} = (n+1)sba^n m + sa^{n+1}m' = 0$. Hence $(s,0)(b,m')(a,m)^{n+1} = (0,0)$. □

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REFERENCES

[1] D. D. Anderson and T. Dumitrescu, *S-Noetherian rings.* Comm. Algebra **30** (2002), 4407–4416.

[2] M. D'Anna, *A construction of Gorenstein rings.* J. Algebra **306** (2006), 507–519.

[3] M. D'Anna and M. Fontana, *An amalgamated duplication of a ring along a multiplicative-canonical ideal.* Arkiv Mat. **6** (2007), 241–252.

[4] M. D'Anna and M. Fontana, *An amalgamated duplication of a ring along an ideal: the basic properties.* J. Algebra Appl **6** (2007), 443–459.

[5] M. D'Anna, C.A. Finocchiaro, and M. Fontana, *Amalgamated algebras along an ideal.* In: Com. Alg. Appl. Proc. of the fifth international Fez conference on commutative algebra and applications, Fez, Morocco, June 23–28. (2009). Berlin: Walter de Gruyter, pp. 155–172.

[6] A. Hamed and S. Hizem, *S-Noetherian rings of the forms $\mathcal{A}[X]$ and $\mathcal{A}[[X]]$.* Comm. Algebra **43** (2015), 3848–3856.

[7] A. Hamed and S. Hizem, *Modules satisfying the S-Noetherian property and S-ACCR.* Comm. Algebra **44**(2016), 1941-1951.

[8] Y. Haouat, PhD Thesis, University of Elmanar, Tunisia, 1988.

[9] S. Hizem, *Formal power series over strongly hopfian rings.* Comm. Algebra **39** (2011), 279–291.

[10] A. Hmaimou, A. Kaidi, and E. Sànchez Campos, *Fitting modules and rings.* J. Algebra **308** (2007), 199–214.

[11] R. Gilmer, *A New Criterion for Embeddability in a Zero Dimensional Commutative Ring.* Lecture Notes in Pure and Applied Mathematics, Marcel Dekker **220** (2001), 223–229.

[12] N. K. Kim, K. H. Lee, and Y. Lee, *Power series rings satisfying a zero divisor property.* Comm. Algebra **34** (2006), 2205–2218.

[13] M. Khalifa, *Power series over strongly Hopfian bounded rings.* Comm.Algebra **45** (2017), 3587–3593.

[14] C. P. LU, *Modules satisfying ACC on a certain type of colons.* Pacific J. of Math **131** (1988), 303–318.

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