MULTIPLICATIVE LIE TRIPLE HIGHER DERIVATIONS ON STANDARD OPERATOR ALGEBRAS

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Let $\mathcal{X}$ be a Banach space of dimension $n > 1$ and $\mathfrak{A} \subset \mathcal{B}(\mathcal{X})$ be a standard operator algebra. In the present paper it is shown that if a mapping $d : \mathfrak{A} \to \mathfrak{A}$ (not necessarily linear) satisfies $d([[U, V], W]) = [[d(U), V], W] + [[U, d(V)], W] + [[U, V], d(W)]$ for all $U, V, W \in \mathfrak{A}$, then $d = \psi + \tau$, where ψ is an additive derivation of $\mathfrak{A}$ and $\tau : \mathfrak{A} \to \mathbb{F}I$ vanishes at second commutator $[[U, V], W]$ for all $U, V, W \in \mathfrak{A}$. Moreover, if d is linear and satisfies the above relation, then there exists an operator $S \in \mathfrak{A}$ and a linear mapping τ from $\mathfrak{A}$ into $\mathbb{F}I$ satisfying $\tau([[U, V], W]) = 0$ for all $U, V, W \in \mathfrak{A}$, such that $d(U) = SU - US + \tau(U)$ for all $U \in \mathfrak{A}$. Further, this result is extended to the multiplicative Lie triple higher derivations on $\mathfrak{A}$.

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1. INTRODUCTION

Let $\mathfrak{A}$ be an associative algebra over a field $\mathbb{F}$. Recall that a linear mapping $d : \mathfrak{A} \to \mathfrak{A}$ is said to be a derivation if $d(UV) = d(U)V + Ud(V)$ holds for all $U, V \in \mathfrak{A}$. If the condition of linearity is replaced by additivity in the above definition, then d is said to be an additive derivation. In particular, derivation d is called an inner derivation if there exists some $X \in \mathfrak{A}$ such that $d(U) = UX - XU$ for all $U \in \mathfrak{A}$. A linear mapping $d : \mathfrak{A} \to \mathfrak{A}$ is called a Lie (resp. Lie triple) derivation if $d([U, V]) = [d(U), V] + [U, d(V)]$ (resp. $d([[U, V], W]) = [[d(U), V], W] + [[U, d(V)], W] + [[U, V], d(W)]$) holds for all $U, V, W \in \mathfrak{A}$, where $[U, V] = UV - VU$ is the usual Lie product. If the condition of linearity is dropped from the above definition, then the corresponding Lie derivation and Lie triple derivation are called multiplicative Lie derivation and multiplicative Lie triple derivation respectively. Obviously, every derivation is a Lie derivation and every Lie derivation is a Lie triple derivation. However, the converse statements are not true in general.

There has been a great interest in the study of characterization of Lie derivations and Lie triple derivations for many years. The first quite surprising result is due to Martindale who proved that every multiplicative bijective mapping from a prime ring containing a nontrivial idempotent onto an arbitrary ring is additive (see [16]). Miers [17] initially established that every Lie derivation d on a von Neumann algebra $\mathfrak{A}$ can be uniquely written as the sum $d = \delta + \tau$ where δ is an inner derivation of $\mathfrak{A}$ and τ is a linear mapping from $\mathfrak{A}$ into its center $Z(\mathfrak{A})$ vanishing on each commutator. Furthermore, Miers [18] obtained an analogous decomposition for Lie triple derivations of von Neumann algebras with no abelian summands. Yu and Zhang [25] proved that every nonlinear Lie derivation of triangular algebras is the sum of an additive derivation and a map from triangular algebra into its center sending commutators to zero. Ji, Liu and Zhao [9] proved the similar result for nonlinear Lie triple derivation of triangular algebras. Zhang, Wu and Cao [26] studied Lie triple derivation on nest algebras. Mathieu and Villena [15] gave the characterizations of Lie derivations on C^*-algebras. In addition, the characterization of Lie derivations and Lie triple derivations on various algebras are considered in [1–3, 8–10, 12, 13, 21, 22, 27].

Let us recall some basic facts related to Lie derivations, Lie higher derivations and Lie triple higher derivation of an associative algebra. Many different kinds of higher derivations, which consist of a family of some additive mappings, have been widely studied in commutative and noncommutative rings. Let $\mathbb{N}$ be the set of non-negative integers and $\mathcal{D} = \{d_n\}_{n\in\mathbb{N}}$ be a family of linear mappings $d_n : \mathfrak{A} \to \mathfrak{A}$ such that $d_0 = id_{\mathfrak{A}}$, the identity map on $\mathfrak{A}$. Then $\mathcal{D}$ is called

(i) a *higher derivation* on $\mathfrak{A}$ if for every $n \in \mathbb{N}$, $d_n(UV) = \sum\limits_{i+j=n} d_i(U)d_j(V)$ for all $U, V \in \mathfrak{A}$.

(ii) a *Lie higher derivation* on $\mathfrak{A}$ if for every $n \in \mathbb{N}$, $d_n([U, V]) = \sum\limits_{i+j=n} [d_i(U), d_j(V)]$ for all $U, V \in \mathfrak{A}$.

(iii) a *Lie triple higher derivation* on $\mathfrak{A}$ if for every $n \in \mathbb{N}$, $d_n([[U, V], W]) = \sum\limits_{i+j+k=n} [[d_i(U), d_j(V)], d_k(W)]$ for all $U, V, W \in \mathfrak{A}$.

(iv) an *inner higher derivation* on $\mathfrak{A}$ if there exist two sequences $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ in $\mathfrak{A}$ satisfying the conditions $X_0 = Y_0 = 1$ and $\sum_{i=0}^{n} X_iY_{n-i} = \delta_{n0} = \sum_{i=0}^{n} Y_iX_{n-i}$ such that $d_n(U) = \sum_{i=0}^{n} X_iUY_{n-i}$, for all $U \in \mathfrak{A}$ and for every $n \in \mathbb{N}$, where δ_{n0} is the Kronecker sign.

If we assume $\mathcal{D} = \{d_n\}_{n\in\mathbb{N}}$ be a family of mappings $d_n : \mathfrak{A} \to \mathfrak{A}$ (not necessarily linear) in the above definitions then the corresponding higher derivation, Lie higher derivation and Lie triple higher derivation is said to be multiplicative higher derivation, multiplicative Lie higher derivation and multiplicative Lie triple higher derivation respectively. Moreover, if $\mathcal{D} = \{d_n\}_{n\in\mathbb{N}}$ is assumed to be the family of additive mappings, then in the above definition higher derivation, Lie higher derivation and Lie triple higher derivation is said to be additive higher derivation, additive Lie higher derivation and additive Lie triple higher derivation respectively. Note that d_1 is always a derivation, Lie derivation and Lie triple derivation if $\mathcal{D} = \{d_n\}_{n\in\mathbb{N}}$ is a higher derivation, Lie higher derivation and Lie triple higher derivation respectively.

It is the objective of this article is to investigate multiplicative Lie triple derivations and multiplicative Lie triple higher derivation on Banach space standard operator algebras. Many researchers have made important contributions to the related topics (see [5], [7], [23]). Xiao [24] proved that every multiplicative Lie higher derivation of triangular algebras is the sum of an additive higher derivation and a multiplicative functional vanishing on all commutators. Qi and Hou [20] gave a characterization of Lie higher derivations on nest algebras. Motivated by the work of F. Lu and B. Liu [14], in Section 2, we study the characterization of multiplicative Lie triple derivations on standard operator algebras and in the subsequent Section 3, we extend the result to the multiplicative Lie triple higher derivations on standard operator algebras.

2. MULTIPLICATIVE LIE TRIPLE DERIVATIONS

Throughout this paper, $\mathcal{X}$ represents a Banach space over $\mathbb{F}$, where $\mathbb{F}$ is the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. By $\mathcal{X}^*$ and $\mathcal{B}(\mathcal{X})$ we denote the topological dual space of $\mathcal{X}$ and the algebra of all linear bounded operators on $\mathcal{X}$, respectively. If $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$, then rank one operator is $x \otimes f$ is defined by $y \mapsto f(y)x$ for $y \in \mathcal{X}$. A subalgebra $\mathfrak{A} \subset \mathcal{B}(\mathcal{X})$ is called a *standard operator algebra* if all the bounded finite rank operators are contained in $\mathfrak{A}$. An algebra $\mathfrak{A}$ is said to be prime if $A\mathfrak{A}B = 0$ implies either $A = 0$ or $B = 0$. It is to be noted that every standard operator algebra is prime. Motivated by the work of Jing [14], we have obtained the following main result.

THEOREM 2.1. *Let $\mathcal{X}$ be a Banach space of dimension $n > 1$ and $\mathfrak{A} \subset \mathcal{B}(\mathcal{X})$ be a standard operator algebra. Suppose that a map $d : \mathfrak{A} \to \mathfrak{A}$ satisfies*

$$d([[U,V],W]) = [[d(U),V],W] + [[U,d(V)],W] + [[U,V],d(W)], \tag{2.1}$$

for all $U,V,W \in \mathfrak{A}$. Then $d = \psi + \tau$, where ψ is an additive derivation and τ is a mapping from $\mathfrak{A}$ into $\mathbb{F}I$ satisfying $\tau([[U,V],W]) = 0$ for all $U,V,W \in \mathfrak{A}$.

In particular, if d is linear and satisfies equation (2.1), then there exist an operator $S \in \mathfrak{A}$ and a linear mapping τ from $\mathfrak{A}$ into $\mathbb{F}I$ that vanishes at second commutators $[[U,V],W]$, such that $d(U) = SU - US + \tau(U)$ for all $U \in \mathfrak{A}$.

For the convenience, in the sequel, take $x_0 \in \mathcal{X}$, $f_0 \in \mathcal{X}^*$ satisfying $f_0(x_0) = 1$. Let $P = x_0 \otimes f_0$ and $Q = I - P$ be idempotent of $\mathfrak{A}$, it is obvious that $PQ = QP = 0$. Then $\mathfrak{A} = \mathfrak{A}_{11} + \mathfrak{A}_{12} + \mathfrak{A}_{21} + \mathfrak{A}_{22}$, where $\mathfrak{A}_{11} = P\mathfrak{A}P$, $\mathfrak{A}_{12} = P\mathfrak{A}Q$, $\mathfrak{A}_{21} = Q\mathfrak{A}P$ and $\mathfrak{A}_{22} = Q\mathfrak{A}Q$. We facilitate our discussion with the following known results.

LEMMA 2.1 ([6, Problem 230]). *Suppose $\mathcal{A}$ is a Banach algebra with the identity I. If $A, B \in \mathcal{A}$ and $\lambda \in \mathbb{F}$ are such that $[A,B] = \lambda I$, then $\lambda = 0$.*

LEMMA 2.2 ([11, Lemma 2 (ii)]). *For $U = U_{11} + U_{12} + U_{21} + U_{22} \in \mathfrak{A}$. If $U_{ij}V_{jk} = 0$ for every $U_{ij} \in \mathfrak{A}_{ij}$, $1 \le i,j,k \le 2$, then $V_{jk} = 0$. Dually, if $V_{ki}U_{ij} = 0$ for every $U_{ij} \in \mathfrak{A}_{ij}$, $1 \le i,j,k \le 2$, then $V_{ki} = 0$.*

Now we shall use the hypothesis of Theorem 2.1 freely without any specific mention in proving the following lemmas.

LEMMA 2.3. *Let $U_{ii} \in \mathfrak{A}_{ii}$, $i = 1,2$. If $U_{11}V_{12} = V_{12}U_{22}$ for all $V_{12} \in \mathfrak{A}_{12}$, then $U_{11} + U_{22} \in \mathbb{F}I$.*

Proof. For any $V_{11} \in \mathfrak{A}_{11}$ and $V_{12} \in \mathfrak{A}_{12}$, we get $U_{11}V_{11}V_{12} = V_{11}V_{12}U_{22} = V_{11}U_{11}V_{12}$ for all $V_{12} \in \mathfrak{A}_{12}$. As $\mathfrak{A}$ is prime, we have $U_{11}V_{11} = V_{11}U_{11}$.

For any $V_{12} \in \mathfrak{A}_{12}$ and $V_{22} \in \mathfrak{A}_{22}$, we get $V_{12}V_{22}U_{22} = U_{11}V_{12}V_{22} = V_{12}U_{22}V_{22}$ for all $V_{12} \in \mathfrak{A}_{12}$. It follows by the primeness of $\mathfrak{A}$ that $V_{22}U_{22} = U_{22}V_{22}$.

For any $V_{12} \in \mathfrak{A}_{12}$ and $V_{21} \in \mathfrak{A}_{21}$, we get $U_{22}V_{21}V_{12} = V_{21}V_{12}U_{22} = V_{21}U_{11}V_{12}$ for all $V_{12} \in \mathfrak{A}_{12}$. It follows that $U_{22}V_{21} = V_{21}U_{22}$.

For any $V \in \mathfrak{A}$, we have

$$\begin{aligned}(U_{11} + U_{22})V &= (U_{11} + U_{22})(V_{11} + V_{12} + V_{21} + V_{22}) \\ &= U_{11}V_{11} + U_{11}V_{12} + U_{22}V_{21} + U_{22}V_{22} \\ &= V_{11}U_{11} + V_{12}U_{11} + V_{21}U_{22} + V_{22}U_{22} \\ &= (V_{11} + V_{12} + V_{21} + V_{22})(U_{11} + U_{22}) \\ &= V(U_{11} + U_{22}).\end{aligned}$$

Hence it follows that $U_{11} + U_{22} \in \mathbb{F}I$. □

LEMMA 2.4. *$d(0) = 0$.*

Proof. $d(0) = d([[0,0],0]) = [[d(0),0],0] + [[0,d(0)],0] + [[0,0],d(0)] = 0$. □

LEMMA 2.5. *$Pd(P)P + Qd(P)Q \in \mathbb{F}I$.*

Proof. Let $x \in \mathcal{X}$, $f \in \mathcal{X}^*$. Then

$$\begin{aligned} d(Px \otimes Q^* f) &= d([[Px \otimes Q^* f, P], P]) = [[d(Px \otimes Q^* f), P], P] \\ &\quad + [[Px \otimes Q^* f, d(P)], P] + [[Px \otimes Q^* f, P], d(P)] \\ &= Qd(Px \otimes Q^* f)P + Pd(Px \otimes Q^* f)Q - Px \otimes Q^* f d(P)Q \\ &\quad + Pd(P)Px \otimes Q^* f - Px \otimes Q^* f d(P) + d(P)Px \otimes Q^* f. \end{aligned}$$

Multiplying the above identity from the left by P and from the right by Q, we arrive at

$$Px \otimes Q^* f d(P)Q = Pd(P)Px \otimes Q^* f.$$

Equivalently,

$$Px \otimes fQd(P)Q = Pd(P)Px \otimes fQ.$$

It follows that $Pd(P)P = \lambda P$ and $Qd(P)Q = \lambda Q$ for some $\lambda \in \mathbb{C}$. Hence $Pd(P)P + Qd(P)Q = \lambda I$. □

In the sequel, we define $\phi : \mathfrak{A} \to \mathfrak{A}$ by

$$\phi(U) = d(U) + d_{Pd(P)Q - Qd(P)P}(U) \text{ for all } U \in \mathfrak{A}$$

where $d_{Pd(P)Q-Qd(P)P}$ is the inner derivation determined by $Pd(P)Q - Qd(P)P$. It is easy to verify that

$$\phi([[U, V], W]) = [[\phi(U), V], W] + [[U, \phi(V)], W] + [[U, V], \phi(W)]$$

holds for all $U, V, W \in \mathfrak{A}$. Moreover, by Lemma 2.5, we have

$$\begin{aligned} \phi(P) &= d(P) - Pd(P)Q - Qd(P)P \\ &= d(P)P + d(P)Q - Pd(P)Q - Qd(P)P \\ &= Pd(P)P + Qd(P)Q \\ &= \lambda I. \end{aligned}$$

Thus $\phi(P) \in \mathbb{F}I$.

Lemma 2.6. *$\phi(PUQ + QUP) = P\phi(U)Q + Q\phi(U)P$ for all $U \in \mathfrak{A}$.*

Proof. Since $[[U, P], P] = PU - 2PUP + UP = PUQ + QUP$, it follows that

$$\begin{aligned} \phi(PUQ + QUP) &= \phi([[U, P], P]) = [[\phi(U), P], P] \\ &= P\phi(U)Q + Q\phi(U)P. \end{aligned}$$

□

Lemma 2.7. *$\phi(Q) \in \mathbb{F}I$.*

Proof. Using arguments similar to those used in the proof of Lemma 2.5, we get

$$P\phi(Q)P + Q\phi(Q)Q \in \mathbb{F}I.$$

Since $\phi(Q) = P\phi(Q)P + P\phi(Q)Q + Q\phi(Q)P + Q\phi(Q)Q$, by Lemma 2.6, we have

$$P\phi(Q)Q + Q\phi(Q)P \quad = \quad \phi(PQQ + QQP) = 0.$$

Consequently, we get $\phi(Q) = P\phi(Q)P + Q\phi(Q)Q \in \mathbb{F}I.$ □

LEMMA 2.8. *If $[U, V] \in \mathbb{F}I$ for any $U, V \in \mathfrak{A}$, then $[\phi(U), V] + [U, \phi(V)] \in \mathbb{F}I$.*

Proof. For $[U, V] \in \mathbb{F}I$, we have $[[U, V], W] = 0$ for all $W \in \mathfrak{A}$.

$$\begin{aligned} 0 &= \phi(0) = \phi[[U, V], W] = [[\phi(U), V], W] + [[U, \phi(V)], W] \\ &= [[\phi(U), V] + [U, \phi(V)], W] \end{aligned}$$

for all $W \in \mathfrak{A}$. Thus $[\phi(U), V] + [U, \phi(V)] \in \mathbb{F}I.$ □

LEMMA 2.9. $\phi(U_{ij}) \subseteq \mathfrak{A}_{ij},\ 1 \leq i \neq j \leq 2.$

Proof. For $U_{12} \in \mathfrak{A}_{12}$, we have $U_{12} = [[U_{12}, P], P]$. Thus

$$\phi(U_{12}) \quad = \quad \phi([[U_{12}, P], P]) = [[\phi(U_{12}), P], P] = P\phi(U_{12})Q + Q\phi(U_{12})P,$$

and hence we see that $P\phi(U_{12})P = Q\phi(U_{12})Q = 0$. Now for $U_{12}, V_{12} \in \mathfrak{A}_{12}$, by Lemma 2.8, we have

$$[\phi(U_{12}), V_{12}] + [U_{12}, \phi(V_{12})] \quad = \quad \lambda I \in \mathbb{F}I. \tag{2.2}$$

Since $U_{12} = [P, U_{12}]$, by using (2.2), we find that

$$\begin{aligned} [\phi(U_{12}), V_{12}] &= [\phi([P, U_{12}]), V_{12}] = \lambda I - [[P, U_{12}], \phi(V_{12})] \\ &= \lambda I - \phi([[P, U_{12}], V_{12}]) + [[\phi(P), U_{12}], V_{12}] + [[P, \phi(U_{12})], V_{12}] \\ &= \lambda I + [[P, \phi(U_{12})], V_{12}]. \end{aligned}$$

This implies that

$$\begin{aligned} [P\phi(U_{12})Q + Q\phi(U_{12})P, V_{12}] &= \lambda I + [[P, P\phi(U_{12})Q + Q\phi(U_{12})P], V_{12}] \\ &= \lambda I + [P\phi(U_{12})Q - Q\phi(U_{12})P, V_{12}]. \end{aligned}$$

Hence $[Q\phi(U_{12})P, V_{12}] = \frac{1}{2}\lambda I \in \mathbb{F}I$. It follows from Lemma 2.1 that $[Q\phi(U_{12})P, V_{12}] = 0$. Thus $Q\phi(U_{12})V_{12} = 0$ and hence by Lemma 2.2, we have $Q\phi(U_{12})P = 0$. So $\phi(U_{12}) = P\phi(U_{12})Q \in \mathfrak{A}_{12}$ for each $U_{12} \in \mathfrak{A}_{12}$. This implies that $\phi(\mathfrak{A}_{12}) \subseteq \mathfrak{A}_{12}$.

Similarly, $\phi(U_{21}) = Q\phi(U_{21})P \in \mathfrak{A}_{21}$ for each $U_{21} \in \mathfrak{A}_{21}$ and therefore $\phi(\mathfrak{A}_{21}) \subseteq \mathfrak{A}_{21}$. □

LEMMA 2.10. *There exists a functional* $f_i : \mathfrak{A}_{ii} \to \mathbb{F}I$ *such that* $\phi(U_{ii}) - f_i(U_{ii})I \in \mathfrak{A}_{ii}$ *for all* $U_{ii} \in \mathfrak{A}_{\rm ii}$, $i = 1, 2$.

Proof. For $U_{11} \in \mathfrak{A}_{11}$, by Lemma 2.6, we have

$$P\phi(U_{11})Q + Q\phi(U_{11})P \;=\; \phi(PU_{11}Q + QU_{11}P) = 0.$$

Thus, it can be assumed that $\phi(U_{11}) = A_{11} + A_{22}$ and $\phi(U_{22}) = B_{11} + B_{22}$, here $A_{ii}, B_{ii} \in \mathfrak{A}_{\rm ii}, i = 1, 2$. Since $[U_{11}, U_{22}] = 0$, then by Lemma 2.1, we have $[\phi(U_{11}), U_{22}] + [U_{11}, \phi(U_{22})] = \lambda I \in \mathbb{F}I$. Multiplying both sides by Q, we arrive at $[Q\phi(U_{11})Q, U_{22}] = \lambda Q$. Consequently, by Lemma 2.1, $[Q\phi(U_{11})Q, U_{22}] = 0$ for all $U_{22} \in \mathfrak{A}_{22}$. Similarly $[U_{11}, P\phi(U_{22})P] = 0$ for all $U_{11} \in \mathfrak{A}_{11}$.

Equivalently, $[A_{22}, U_{22}] = 0$ for all $U_{22} \in \mathfrak{A}_{22}$ and $[U_{11}, B_{11}] = 0$ for all $U_{11} \in \mathfrak{A}_{11}$. Therefore, there exist scalars $f_1(U_{11})$ and $f_2(U_{22})$ such that $A_{22} = f_1(U_{11})Q$ and $B_{11} = f_2(U_{22})P$. Hence $\phi(U_{11}) - f_1(U_{11})I \in \mathfrak{A}_{11}$ and $\phi(U_{22}) - f_2(U_{22})I \in \mathfrak{A}_{22}$. □

Our next aim is to show that ϕ is additive on $\mathfrak{A}_{12}$ and $\mathfrak{A}_{21}$.

LEMMA 2.11. *Let* $U_{ii} \in \mathfrak{A}_{\rm ii}$ *and* $U_{ij} \in \mathfrak{A}_{\rm ij}$, $1 \le i \ne j \le 2$. *Then* $\phi(U_{ii} + U_{ij}) - \phi(U_{ii}) - \phi(U_{ij}) \in \mathbb{F}I$.

Proof. Let $U_{11} \in \mathfrak{A}_{11}$, $U_{12} \in \mathfrak{A}_{12}$. We have

$$\begin{aligned}\phi([[U_{11} + U_{12}, P], P]) &= [[\phi(U_{11} + U_{12}), P], P] + [[U_{11} + U_{12}, \phi(P)], P] \\ &\quad + [[U_{11} + U_{12}, \phi(P)], \phi(P)] \\ &= [[\phi(U_{11} + U_{12}), P], P].\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\phi([[U_{11} + U_{12}, P], P]) &= \phi([[U_{11}, P], P]) + \phi([[U_{12}, P], P]) \\ &= [[\phi(U_{12}), P], P] + [[\phi(U_{12}), P], P].\end{aligned}$$

Combining the above two identities, we get
$[[\phi(U_{11} + U_{12}) - \phi(U_{12}) - \phi(U_{12}), P], P] = 0$, that is

$$\begin{aligned}0 &= P(\phi(U_{11} + U_{12}) - \phi(U_{12}) - \phi(U_{12}))Q \\ &\quad + Q(\phi(U_{11} + U_{12}) - \phi(U_{12}) - \phi(U_{12}))P.\end{aligned} \tag{2.3}$$

Now, for any $V_{12} \in \mathfrak{A}_{12}$ and by Lemma 2.5, we have

$$\phi([[U_{11} + U_{12}, V_{12}], P]) \;=\; [[\phi(U_{11} + U_{12}), V_{12}], P] + [[U_{11} + U_{12}, \phi(V_{12})], P].$$

On the other hand, we have

$$\begin{aligned}\phi([[U_{11} + U_{12}, V_{12}], P]) &= \phi([[U_{11}, V_{12}], P]) + \phi([[U_{12}, V_{12}], P]) \\ &= [[\phi(U_{11}), V_{12}], P] + [[U_{11}, \phi(V_{12})], P]\end{aligned}$$

$$+[[\phi(U_{12}), V_{12}], P] + [[U_{12}, \phi(V_{12})], P].$$

Combining the above two identities, we arrive at

$$[[\phi(U_{11} + U_{12}) - \phi(U_{11}) - \phi(U_{12}), V_{12}], P] \quad = \quad 0.$$

In other words,

$$\begin{aligned} (2.4) \qquad & P\phi(U_{11} + U_{12}) - \phi(U_{11}) - \phi(U_{12})PV_{12} \\ & = \quad V_{12}Q\phi(U_{11} + U_{12}) - \phi(U_{11}) - \phi(U_{12})Q. \end{aligned}$$

Equations (2.3) and (2.4), together with Lemma 2.3, gives that $\phi(U_{11} + U_{12}) - \phi(U_{11}) - \phi(U_{12}) \in \mathbb{F}I$. Similarly, one can easily prove the other part. □

Lemma 2.12. *ϕ is additive on $\mathfrak{A}_{12}$ and $\mathfrak{A}_{21}$.*

Proof. Let $U_{12}, V_{12} \in \mathfrak{A}_{12}$. By Lemmas 2.5, 2.7 and 2.11, we see that

$$\begin{aligned} \phi(U_{12} + V_{12}) &= \phi([[P + U_{12}, Q + V_{12}], Q]) \\ &= [[\phi(P + U_{12}), Q + V_{12}], Q] + [[P + U_{12}, \phi(Q + V_{12})], Q] \\ &\quad +[[P + U_{12}, Q + V_{12}], \phi(Q)] \\ &= [[\phi(P) + \phi(U_{12}), Q + V_{12}], Q] + [[P + U_{12}, \phi(Q) + \phi(V_{12})], Q] \\ &= \phi(U_{12}) + \phi(V_{12}). \end{aligned}$$

Hence ϕ is additive on $\mathfrak{A}_{12}$. Similarly ϕ is additive on $\mathfrak{A}_{21}$. □

Now for any $U \in \mathfrak{A}$, define $\Delta(U) = \phi(PUP) + \phi(PUQ) + \phi(QUP) + \phi(QUQ) - (f_1(PUP) + f_2(QUQ))I$. By Lemmas 2.9 and 2.10, we have

Lemma 2.13. *Let $U_{ij} \in \mathfrak{A}_{ij}$, $1 \leq i \neq j \leq 2$. Then*

(i) $\Delta(U_{ij}) \in \mathfrak{A}_{ij}$, $1 \leq i \neq j \leq 2$,

(ii) $\Delta(U_{12}) = \phi(U_{12})$ *and* $\Delta(U_{21}) = \phi(U_{21})$,

(iii) $\Delta(U_{11} + U_{12} + U_{21} + U_{22}) = \Delta(U_{11}) + \Delta(U_{12}) + \Delta(U_{21}) + \Delta(U_{22})$.

Now, we shall show that Δ is an additive derivation. First, we shall prove the additivity of Δ.

By Lemma 2.12 and (ii) part of Lemma 2.13, we immediately get the following result.

Lemma 2.14. *Δ is additive on $\mathfrak{A}_{12}$ and $\mathfrak{A}_{21}$.*

Lemma 2.15. *Let $U_{ii} \in \mathfrak{A}_{ii}$, $U_{ij} \in \mathfrak{A}_{ij}$, $1 \leq i \neq j \leq 2$. Then*

(i) $\Delta(U_{ii}V_{ij}) = \Delta(U_{ii})V_{ij} + U_{ii}\Delta(V_{ij})$,

(ii) $\Delta(V_{ij}U_{jj}) = \Delta(V_{ij})U_{jj} + V_{ij}\Delta(U_{jj})$.

Proof. Since $U_{11}V_{12} = [[U_{11}, V_{12}], Q]$, by Lemmas 2.7 & 2.13, we have

$$\begin{aligned}\Delta(U_{11}V_{12}) &= \phi(U_{11}V_{12}) = \phi([[U_{11}, V_{12}], Q]) \\ &= [[\phi(U_{11}), V_{12}], Q] + [[U_{11}, \phi(V_{12})], Q] + [[U_{11}, V_{12}], \phi(Q)] \\ &= [[\Delta(U_{11}), V_{12}], Q] + [[U_{11}, \Delta(V_{12})], Q] \\ &= \Delta(U_{11})V_{12} + U_{11}\Delta(V_{12}).\end{aligned}$$

Similarly, it is easy to prove the other identities. □

LEMMA 2.16. *Δ is additive on $\mathfrak{A}_{11}$ and $\mathfrak{A}_{22}$.*

Proof. Let $U_{11}, V_{11} \in \mathfrak{A}_{11}$. For any $W_{12} \in \mathfrak{A}_{12}$, by Lemma 2.15, we have

$$\Delta((U_{11} + V_{11})W_{12}) = \Delta(U_{11} + V_{11})W_{12} + (U_{11} + V_{11})\Delta(W_{12}).$$

On the other hand, by Lemmas 2.14 & 2.15, we have

$$\begin{aligned}\Delta((U_{11} + V_{11})W_{12}) &= \Delta(U_{11}W_{12} + V_{11}W_{12}) = \Delta(U_{11}W_{12}) + \Delta(V_{11}W_{12}) \\ &= \Delta(U_{11})W_{12} + U_{11}\Delta(W_{12}) + \Delta(V_{11})W_{12} + V_{11}\Delta(W_{12}).\end{aligned}$$

Comparing the above two identities, we get $(\Delta(U_{11} + V_{11}) - \Delta(U_{11}) - \Delta(V_{11}))W_{12} = 0$. In other words $(\Delta(U_{11} + V_{11}) - \Delta(U_{11}) - \Delta(V_{11}))P\mathfrak{A}Q = 0$. Since $\mathfrak{A}$ is prime, it follows that $(\Delta(U_{11}+V_{11})-\Delta(U_{11})-\Delta(V_{11}))P = 0$. Hence, $\Delta(U_{11} + V_{11}) = \Delta(U_{11}) + \Delta(V_{11})$ as $\Delta(\mathfrak{A}_{11}) \subseteq \mathfrak{A}_{11}$. Similarly, Δ is additive on $\mathfrak{A}_{22}$. □

LEMMA 2.17. *Δ is additive.*

Proof. Let $U = \sum_{i,j=1}^{2} U_{ij}$, $V = \sum_{i,j=1}^{2} V_{ij}$ be in $\mathfrak{A}$. By Lemmas, 2.13, 2.14 & 2.16, we have

$$\begin{aligned}\Delta(U + V) &= \Delta\Big\{\sum_{i,j=1}^{2}(U_{ij} + V_{ij})\Big\} \\ &= \sum_{i,j=1}^{2}\Delta(U_{ij} + V_{ij}) = \sum_{i,j=1}^{2}(\Delta(U_{ij}) + \Delta(V_{ij})) \\ &= \Delta(\sum_{i,j=1}^{2}U_{ij}) + \Delta(\sum_{i,j=1}^{2}V_{ij}) = \Delta(U) + \Delta(V).\end{aligned}$$

□

In the sequel, we shall prove that Δ is a derivation.

LEMMA 2.18. *Let $U_{ii}, V_{ii} \in \mathfrak{A}_{ii}, i = 1, 2$. Then $\Delta(U_{ii}V_{ii}) = \Delta(U_{ii})V_{ii} + U_{ii}\Delta(V_{ii})$*

Proof. For any $U_{11}, V_{11} \in \mathfrak{A}_{11}$ and $W_{12} \in \mathfrak{A}_{12}$, we have by Lemma 2.15 that

$$\Delta(U_{11}V_{11}W_{12}) = \Delta(U_{11}V_{11})W_{12} + U_{11}V_{11}\Delta(W_{12}).$$

On the other hand we have,

$$\begin{aligned}\Delta(U_{11}V_{11}W_{12}) &= \Delta(U_{11})V_{11}W_{12} + U_{11}\Delta(V_{11}W_{12}) \\ &= \Delta(U_{11})V_{11}W_{12} + U_{11}\Delta(V_{11})W_{12} + U_{11}V_{11}\Delta(W_{12}).\end{aligned}$$

Comparing the above two identities, we get $\big(\Delta(U_{11}V_{11}) - \Delta(U_{11})V_{11} - U_{11}\Delta(V_{11})\big)W_{12} = 0$. In other words, we have $\big(\Delta(U_{11}V_{11}) - \Delta(U_{11})V_{11} - U_{11}\Delta(V_{11})\big)P\mathfrak{A}Q = 0$. Since $\mathfrak{A}$ is prime, we get that $\big(\Delta(U_{11}V_{11}) - \Delta(U_{11})V_{11} - U_{11}\Delta(V_{11})\big)P = 0$. Hence, $\Delta(U_{11}V_{11}) = \Delta(U_{11})V_{11} + U_{11}\Delta(V_{11})$ as $\Delta(\mathfrak{A}_{11}) \subseteq \mathfrak{A}_{11}$. Similarly, $\Delta(U_{22}V_{22}) = \Delta(U_{22})V_{22} + U_{22}\Delta(V_{22})$. □

LEMMA 2.19. *Let $U_{11} \in \mathfrak{A}_{11}$ and $V_{22} \in \mathfrak{A}_{22}$. Then $\phi(U_{11}+V_{22}) - \Delta(U_{11}) - \Delta(V_{22}) \in \mathbb{F}I$.*

Proof. For any $U_{11} \in \mathfrak{A}_{11}$ and $V_{22} \in \mathfrak{A}_{22}$, we have

$$\phi([[U_{11} + V_{22}, Q], Q]) = [[\phi(U_{11} + V_{22}), Q], Q]$$

On the other hand, we have

$$\begin{aligned}\phi([[U_{11} + V_{22}, Q], Q]) &= \phi([[U_{11}, Q], Q]) + \phi([[V_{22}, Q], Q]) \\ &= [[\phi(U_{11}), Q], Q] + [[\phi(V_{22}), Q], Q \\ &= [[\Delta(U_{11}) + f_1(U_{11}), Q], Q] + [[\Delta(V_{22}) + f_2(U_{22}), Q], Q] \\ &= [[\Delta(U_{11}), Q], Q] + [[\Delta(V_{22}), Q], Q].\end{aligned}$$

On combining the above two identities, we get $[[\phi(U_{11} + V_{22}) - \Delta(U_{11}) - \Delta(V_{22}), Q], Q] = 0$, that is

$$\begin{aligned}0 &= P(\phi(U_{11} + V_{22}) - \Delta(U_{11}) - \Delta(V_{22}))Q \\ &\quad +Q(\phi(U_{11} + V_{22}) - \Delta(U_{11}) - \Delta(V_{22}))P.\end{aligned} \tag{2.5}$$

Now for any $W_{12} \in \mathfrak{A}_{12}$, we have

$$\begin{aligned}\phi([U_{11} + V_{22}, W_{12}]) &= \phi(U_{11}W_{12} - W_{12}V_{22}) \\ &= \phi(U_{11}W_{12}) - \phi(W_{12}V_{22}) = \Delta(U_{11}W_{12}) - \Delta(W_{12}V_{22}) \\ &= \Delta([[U_{11}, W_{12}], Q]) - \Delta([[W_{12}, V_{22}], Q]) \\ &= [[\Delta(U_{11}), W_{12}], Q] + [[U_{11}, \Delta(W_{12})], Q] \\ &- [[\Delta(W_{12}), V_{22}], Q] - [[W_{12}, \Delta(V_{22})], Q] \\ &= [[\Delta(U_{11}), W_{12}], Q] + [U_{11}, \Delta(W_{12})] \\ &+ [V_{22}, \Delta(W_{12})] + [[\Delta(V_{22}), W_{12}], Q].\end{aligned}$$

On the other hand, we see that

$$\begin{aligned}\phi([U_{11}+V_{22},W_{12}]) &= \phi([[U_{11}+V_{22},W_{12}],Q]) \\ &= [[\phi(U_{11}+V_{22}),W_{12}],Q]+[[U_{11}+V_{22},\Delta(W_{12})],Q] \\ &= [[\phi(U_{11}+V_{22}),W_{12}],Q]+[U_{11},\Delta(W_{12})]+[V_{22},\Delta(W_{12})].\end{aligned}$$

Comparing the above two identities, we obtain

$$[[\phi(U_{11}+V_{22})-\Delta(U_{11})-\Delta(V_{22}),W_{12}],Q] \quad = \quad 0.$$

In other words, we get

$$\begin{aligned}(2.6)\qquad & P\big(\phi(U_{11}+V_{22})-\Delta(U_{11})-\Delta(V_{22})\big)W_{12} \\ &= \quad W_{12}\big(\phi(U_{11}+V_{22})-\Delta(U_{11})-\Delta(V_{22})\big)Q.\end{aligned}$$

Equations (2.5) and (2.6), together with Lemma 2.3, yield that $\phi(U_{11}+U_{22})-\Delta(U_{11})-\Delta(U_{22})\in\mathbb{F}I$. $\square$

LEMMA 2.20. *Let $U_{12}\in\mathfrak{A}_{12}$ and $V_{21}\in\mathfrak{A}_{21}$. Then $\Delta(U_{12}V_{21})=\Delta(U_{12})V_{21}+U_{12}\Delta(V_{21})$ and $\Delta(U_{21}V_{12})=\Delta(U_{21})V_{12}+U_{21}\Delta(V_{12})$.*

Proof. For any $U_{12}\in\mathfrak{A}_{12}$ and $V_{21}\in\mathfrak{A}_{21}$,compute

$$\begin{aligned}\phi([U_{12},V_{21}])-\Delta([U_{12},V_{21}]) \quad &= \quad \phi([[P,U_{12}],V_{21}])-\Delta(U_{12}V_{21}-V_{21}U_{12}) \\ &= \quad [[P,\phi(U_{12})],V_{21}]+[[P,U_{12}],\phi(V_{21})] \\ &\qquad -\Delta(U_{12}V_{21})+\Delta(V_{21}U_{12}) \\ &= \quad \Delta(U_{12})V_{21}+U_{12}\Delta(V_{21})-\Delta(U_{12}V_{21}) \\ &\qquad -\Delta(V_{21})U_{12}-V_{21}\Delta(U_{12})+\Delta(V_{21}U_{12}).\end{aligned}$$

Since $\phi([U_{12},V_{21}])-\Delta([U_{12},V_{21}])=\phi(U_{12}V_{21}-V_{21}U_{12})-\Delta(U_{12}V_{21}-V_{21}U_{12})$, by Lemma 2.19, we have

$$\begin{aligned}\Delta(U_{12})V_{21}+U_{12}\Delta(V_{21})-\Delta(U_{12}V_{21})-\Delta(V_{21})U_{12}-V_{21}\Delta(U_{12}) \\ +\Delta(V_{21}U_{12})=\lambda I\in\mathbb{F}I.\end{aligned}$$

From the later relation we obtain the two identities

$$(2.7)\qquad \Delta(U_{12}V_{21}) \quad = \quad \Delta(U_{12})V_{21}+U_{12}\Delta(V_{21})-\lambda P,$$

and

$$(2.8)\qquad \Delta(V_{21}U_{12}) \quad = \quad \Delta(V_{21})U_{12}+V_{21}\Delta(U_{12})+\lambda Q.$$

Now it is sufficient to show that $\lambda=0$. Assume $\lambda\neq 0$. Then by using equations (2.7) and (2.8) together with Lemma 2.15, we have

$$\begin{aligned}\Delta(U_{12}V_{21}U_{12}) \quad &= \quad \Delta(U_{12})V_{21}U_{12}+U_{12}\Delta(V_{21}U_{12}) \\ &= \quad \Delta(U_{12})V_{21}U_{12}+U_{12}\Delta(V_{21})U_{12}+U_{12}V_{21}\Delta(U_{12})+\lambda U_{12},\end{aligned}$$

and

$$\begin{aligned}\Delta(U_{12}V_{21}U_{12}) &= \Delta(U_{12}V_{21})U_{12} + U_{12}V_{21}\Delta(U_{12}) \\ &= \Delta(U_{12})V_{21}U_{12} + U_{12}\Delta(V_{21})U_{12} + U_{12}V_{21}\Delta(U_{12}) - \lambda U_{12}.\end{aligned}$$

Comparing the above two identities, we obtain $\lambda U_{12} = 0$. Since $\mathbb{F}$ is a field, we have $U_{12} = 0$, a contradiction. Consequently, $\Delta(U_{12}V_{21}) = \Delta(U_{12})V_{21} + U_{12}\Delta(V_{21})$ and $\Delta(U_{21}V_{12}) = \Delta(U_{21})V_{12} + U_{21}\Delta(V_{12})$. □

Thus, we have shown that Δ is an additive derivation.

Proof of Theorem 2.1. Let us define $\tau : \mathfrak{A} \to \mathfrak{A}$ by $\tau(U) = \phi(U) - \Delta(U)$ for $U \in \mathfrak{A}$. For $i = j$, $\tau(U_{ij}) = f_i(U_{ij})I$; otherwise $\tau(U_{ij}) = 0$. We shall show that $\tau(U) \in \mathbb{F}I$ for all $U \in \mathfrak{A}$. For $T_{12} \in \mathfrak{A}_{12}$ and $U \in \mathfrak{A}$. Since

$$[[U, T_{12}], P] = [UT_{12} - T_{12}U, P] = T_{12}QUQ - PUPT_{12},$$

it follows

$$\begin{aligned}\phi(T_{12}QUQ - PUPT_{12}) &= \phi([[U, T_{12}], P]) \\ &= [[\phi(U), T_{12}], P] + [[U, \phi(T_{12})], P] \\ &= \phi(T_{12})QUQ - PUP\phi(T_{12}) + T_{12}Q\phi(U)Q \\ &\quad - P\phi(U)PT_{12}.\end{aligned}$$

On the other hand by Lemma 2.12, we have

$$\begin{aligned}\phi(T_{12}QUQ - PUPT_{12}) &= \phi(T_{12}QUQ) - \phi(PUPT_{12}) \\ &= \phi([[P, T_{12}], QUQ]) - \phi([[T_{12}, P], PUP]) \\ &= [[P, \phi(T_{12})], QUQ] + [[P, T_{12}], \phi(QUQ)] \\ &\quad - [[\phi(T_{12}), P], PUP] - [[T_{12}, P], \phi(PUP)] \\ &= \phi(T_{12})QUQ + T_{12}\phi(QUQ) - \phi(QUQ)T_{12} \\ &\quad - \phi(PUP)T_{12} + T_{12}\phi(PUP) - PUP\phi(T_{12}).\end{aligned}$$

Comparing the above two identities, we obtain

$$\big(P\phi(U)P - \phi(PUP) - \phi(QUQ)\big)T_{12} = T_{12}\big(Q\phi(U)Q - \phi(PUP) - \phi(QUQ)\big).$$

Hence for all $T_{12} \in \mathfrak{A}_{12}$,

$$\begin{aligned}&\big(P\phi(U)P - Q\phi(U)Q - \phi(PUP) - \phi(QUQ)\big)T_{12} \\ &= T_{12}\big(Q\phi(U)Q + Q\phi(U)Q - \phi(PUP) - \phi(QUQ)\end{aligned}$$

By using the Lemma 2.3, we get the desired result.

$$P\phi(U)P + Q\phi(U)Q - \phi(PUP) - \phi(QUQ) \in \mathbb{F}I. \tag{2.9}$$

Now by Lemmas 2.9 & 2.12, we have

$$P\phi(U)Q + Q\phi(U)P = \phi(PUQ + QUP) = \phi(PUQ) + \phi(QUP),$$

and hence

$$\begin{aligned}
&\phi(U) - \phi(PUP) + \phi(PUQ) + \phi(QUP) + \phi(QUQ) \\
&\quad = P\phi(U)P + Q\phi(U)Q + P\phi(U)Q + Q\phi(U)P \\
&\qquad -\phi(PUP) - \phi(QUQ) - \phi(PUQ) - \phi(QUP) \\
&\quad = P\phi(U)P + Q\phi(U)Q - \phi(PUP) - \phi(QUQ) \in \mathbb{F}I.
\end{aligned}$$

By equation (2.9) and by the definition of Δ and τ, we see that $\tau(U) \in \mathbb{F}I$ for all $U \in \mathfrak{A}$. Since Δ is an additive Lie triple derivation, it follows that for all $U, V, W \in \mathfrak{A}$

$$\begin{aligned}
\tau([[U,V],W]) &= \phi([[U,V],W]) - \Delta([[U,V],W]) \\
&= [[\phi(U),V],W] + [[U,\phi(V)],W] + [[U,V],\phi(W)] \\
&\quad - \Delta([[U,V],W]) \\
&= [[\Delta(U),V],W] + [[U,\Delta(V)],W] + [[U,V],\Delta(W)] \\
&\quad - \Delta([[U,V],W]) \\
&= 0.
\end{aligned}$$

Finally, let us define $\psi(U) = \Delta(U) - (TU - UT)$ for all $U \in \mathfrak{A}$, where $T = Pd(P)Q - Qd(P)P$. It is easy to check that ψ is an additive derivation on $\mathfrak{A}$. By the definitions of Δ and ϕ, we have $d(U) = \psi(U) + \tau(U)$ for all $U \in \mathfrak{A}$.

Furthermore, if d is linear , then ψ and τ are also linear. As any linear derivation on $\mathfrak{A}$ is inner, then there exists an operator $S \in \mathfrak{A}$ such that $\psi(U) = SU - US$ for all $U \in \mathfrak{A}$. Hence $d(U) = SU - US + \tau(U)$. This completes the proof. $\square$

3. MULTIPLICATIVE LIE TRIPLE HIGHER DERIVATION

This section of the paper is devoted to the study of characterization of multiplicative Lie triple higher derivations on some classical operator algebras. To the best of our knowledge, much less attention is paid on the characterization of Lie higher derivations and Lie triple higher derivation on operator algebras. There are no other articles dealing with Lie higher derivations of operator algebras except for [4] and [20]. The objective of this section is to describe the characterization of multiplicative Lie triple higher derivations on Banach space standard operator algebras. In particular, we have obtained the following result.

THEOREM 3.1. *Let $\mathcal{X}$ be a Banach space of dimension $n > 1$ and $\mathfrak{A} \subset \mathcal{B}(\mathcal{X})$ be a standard operator algebra. Suppose $\mathcal{D} = \{d_n\}_{n\in\mathbb{N}}$ be the sequence of* *mappings* $d_n : \mathfrak{A} \to \mathfrak{A}$ *such that*

$$d_n([[U,V],W]) = \sum_{i+j+k=n} [[d_i(U), d_j(V)], d_k(W)], \tag{3.1}$$

for all $U, V, W \in \mathfrak{A}$ *and for each* $n \in \mathbb{N}$. *Then there exists an additive higher derivation* $\Delta = \{\psi_n\}_{n\in\mathbb{N}}$ *and a mapping* τ_n *from* $\mathfrak{A}$ *into* $\mathbb{F}I$ *vanishing at second commutators* $[[U,V],W]$ *for all* $U, V, W \in \mathfrak{A}$ *such that* $d_n = \psi_n + \tau_n$ *for each* $n \in \mathbb{N}$.

In particular, if $\{d_n\}_{n\in\mathbb{N}}$ *is a sequence of linear mappings satisfying the equation (3.1), then* $d_n(U) = \sum_{i=0}^{n} X_i U Y_{n-i} + \tau_n(U)$ *for all* $U \in \mathfrak{A}$ *and for every* $n \in \mathbb{N}$, *where* $\{X_n\}_{n\in\mathbb{N}}$ *and* $\{Y_n\}_{n\in\mathbb{N}}$ *are two sequences in* $\mathfrak{A}$ *satisfying the conditions* $X_0 = Y_0 = 1$ *and* $\sum_{i=0}^{n} X_i Y_{n-i} = \delta_{n0} = \sum_{i=0}^{n} Y_i X_{n-i}$, δ_{n0} *is the Kronecker sign and* $\tau_n : \mathfrak{A} \to \mathbb{F}I$ *is a linear mapping satisfying* $\tau_n([[U,V],W]) = 0$ *for all* $U, V, W \in \mathfrak{A}$.

In the proof, it is assumed that $D = \{d_n\}_{n\in\mathbb{N}}$ is a multiplicative Lie triple higher derivation on $\mathfrak{A}$.

For $n = 1$, the expression of $d_n([[U,V],W])$ reduces to $d_1([[U,V],W]) = [[d_1(U),V],W]$
$+ [[U,d_1(V)],W] + [[U,V],d_1(W)]$ for all $U, V, W \in \mathfrak{A}$. We proceed by induction on $n \in \mathbb{N}$. In view of Theorem 2.1, it is clear that $d_1 = \psi_1 + \tau_1$, where ψ_1 is an additive derivation, τ_1 is a mapping from $\mathfrak{A}$ into $\mathbb{F}I$ satisfying $\tau_1([[U,V],W]) = 0$ for all $U, V, W \in \mathfrak{A}$. i.e.; Theorem 3.1 is true for $n = 1$.

By Lemmas 2.5-2.12 of Theorem 2.1, we have

$$\mathbf{B}_1 \begin{cases} Pd_1(P)P + Qd_1(P)Q \in \mathbb{F}I; \\ Pd_1(Q)P + Qd_1(Q)Q \in \mathbb{F}I; \\ \phi_1(PUQ + QUP) = P\phi_1(U)Q + Q\phi_1(U)P \text{ where} \\ \phi_1(U) = d_1(U) + d_{Pd(P)Q - Qd(P)P}(U); \\ \phi_1(P) \in \mathbb{F}I, \ \phi_1(Q) \in \mathbb{F}I; \\ \phi_1(U_{ij}) \subseteq \mathfrak{A}_{ij}, \ 1 \le i \ne j \le 2; \\ \phi_1(U_{ii}) - f_i(U_{ii})I \in \mathfrak{A}_{ii}; \\ \phi_1(U_{ii} + U_{ij}) - \phi_1(U_{ii}) - \phi_1(U_{ij}) \in \mathbb{F}I. \end{cases}$$

We now assume that $d_m(U) = \psi_m(U) + \tau_m(U)$ for all $U \in \mathfrak{A}$ and for all $m < n$, where $\tau_m : \mathfrak{A} \to \mathbb{F}I$ such that $\tau_m([[U,V],W]) = 0$ for all $U, V, W \in \mathfrak{A}$ and d_m

satisfies equation (3.1). Thus we have the following properties;

$$\mathbf{B}_m \begin{cases} Pd_m(P)P + Qd_m(P)Q \in \mathbb{F}I; \\ Pd_m(Q)P + Qd_m(Q)Q \in \mathbb{F}I; \\ \phi_m(PUQ + QUP) = P\phi_m(U)Q + Q\phi_m(U)P \text{ where} \\ \phi_m(U) = d_m(U) + d_{m(Pd_m(P)Q - Qd_m(P)P)}(U); \\ \phi_m(P) \in \mathbb{F}I,\ \phi_m(Q) \in \mathbb{F}I; \\ \phi_m(U_{ij}) \subseteq \mathfrak{A}_{ij},\ 1 \leq i \neq j \leq 2; \\ \phi_m(U_{ii}) - f_{mi}(U_{ii})I \in \mathfrak{A}_{ii}; \\ \phi_m(U_{ii} + U_{ij}) - \phi_m(U_{ii}) - \phi_m(U_{ij}) \in \mathbb{F}I. \end{cases}$$

Our aim is to show that d_n also satisfies the similar properties and $d_n(U) = \psi_n(U) + \tau_n(U)$ for all $U \in \mathfrak{A}$, where $\tau_n : \mathfrak{A} \to \mathbb{F}I$ such that $\tau_n([[U,V],W]) = 0$ for all $U, V, W \in \mathfrak{A}$. Now we shall prove the following Lemmas to obtain our main result.

Lemma 3.1. $Pd_n(P)P + Qd_n(P)Q \in \mathbb{F}I$.

Proof. Let $x \in \mathcal{X}$, $f \in \mathcal{X}^*$. Then

$$\begin{aligned} d_n(Px \otimes Q^*f) &= d_n([[Px \otimes Q^*f, P], P]) = [[d_n(Px \otimes Q^*f), P], P] \\ &\quad + [[Px \otimes Q^*f, d_n(P)], P] + [[Px \otimes Q^*f, P], d_n(P)] \\ &\quad + \sum_{\substack{r+s+t=n \\ 0 \leq r,s,t \leq n-1}} [[d_r(Px \otimes Q^*f), d_s(P)], d_t(P)] \\ &= Qd(Px \otimes Q^*f)P + Pd(Px \otimes Q^*f)Q - Px \otimes Q^*fd(P)Q \\ &\quad + Pd(P)Px \otimes Q^*f - Px \otimes Q^*fd(P) + d(P)Px \otimes Q^*f \\ &\quad + \sum_{\substack{r+s+t=n \\ 0 \leq r,s,t \leq n-1}} [[d_r(Px \otimes Q^*f), d_s(P)], d_t(P)]. \end{aligned}$$

On multiplying the above equation from the left by P and from the right by Q, we have

$$\begin{aligned} Px \otimes fQd_n(P)Q &= Pd_n(P)Px \otimes fQ \\ &\quad + P\Bigg(\sum_{\substack{r+s+t=n \\ 0 \leq r,s,t \leq n-1}} [[d_r(Px \otimes Q^*f), d_s(P)], d_t(P)] \Bigg) Q. \end{aligned}$$

By applying $\mathbf{B}_m$, it follows that

$$P\Bigg(\sum_{\substack{r+s+t=n \\ 0 \leq r,s,t \leq n-1}} [[d_r(Px \otimes Q^*f), d_s(P)], d_t(P)] \Bigg) Q = 0.$$

Thus, we have $Pd_n(P)P = \lambda P$ and $Qd_n(P)Q = \lambda Q$ for some $\lambda \in \mathbb{C}$. Hence $Pd_n(P)P+Qd_n(P)Q = \lambda I$. Similarly, we can prove $Pd_n(Q)P+Qd_n(Q)Q = \lambda I$. Now for all $U \in \mathfrak{A}$, we define

$$\phi_n(U) = d_n(U) + d_{n(Pd_n(P)Q-Qd_n(P)P)}(U),$$

where $d_{n(Pd_n(P)Q-Qd_n(P)P)}$ is an inner derivation determined by $Pd_n(P)Q - Qd_n(P)P$. It is easy to verify that

$$\phi_n([[U,V],W]) = \sum_{i+j+k=n} [[\phi_i(U), \phi_j(V)], \phi_k(W)]$$

for all $U, V, W \in \mathfrak{A}$. □

Lemma 3.2. *$\phi_n(P) \in \mathbb{F}I$.*

Proof. Using Lemma 3.1, we have

$$\begin{aligned} \phi_n(P) &= d_n(P) - Pd_n(P)Q - Qd_n(P)P \\ &= d_n(P)P + d_n(P)Q - Pd_n(P)Q - Qd_n(P)P \\ &= Pd_n(P)P + Qd_n(P)Q = \lambda I. \end{aligned}$$

Thus $\phi_n(P) \in \mathbb{F}I$. □

Lemma 3.3. *For $U \in \mathfrak{A}$, we have $\phi_n(PUQ + QUP) = P\phi_n(U)Q + Q\phi_n(U)P$.*

Proof. For any $U \in \mathfrak{A}$, we have

$$\begin{aligned} \phi_n(PUQ + QUP) &= \phi_n([[U,P],P]) \\ &= [[\phi_n(U),P],P] + [[U,\phi_n(P)],P] + [[U,P],\phi_n(P)] \\ &\quad + \sum_{\substack{r+s+t=n \\ 0\le r,s,t\le n-1}} [[\phi_r(U),\phi_s(P)],\phi_t(P)] \\ &= P\phi_n(U)Q + Q\phi_n(U)P + \sum_{\substack{r+s+t=n \\ 0\le r,s,t\le n-1}} [[\phi_r(U),\phi_s(P)],\phi_t(P)]. \end{aligned}$$

By Lemma 3.2, $\phi_n(P) \in \mathbb{F}I$ for all $n \in \mathbb{N}$, we see that

$$\phi_n(PUQ + QUP) = P\phi_n(U)Q + Q\phi_n(U)P.$$

□

Lemma 3.4. *$\phi_n(Q) \in \mathbb{F}I$.*

Proof. The proof is same as that of Lemma 2.7. □

Lemma 3.5. $\phi_n(U_{ij}) \subseteq \mathfrak{A}_{ij},\ 1 \leq i \neq j \leq 2.$

Proof. For any $U_{12} \in \mathfrak{A}_{12}$ by Lemma 3.2, we have

$$\begin{aligned}\phi_n(U_{12}) &= \phi_n([[U_{12}, P], P]) = [[\phi_n(U_{12}), P], P] \\ &\quad + \sum_{\substack{r+s+t=n \\ 0\leq r,s,t\leq n-1}} [[\phi_r(U_{12}), \phi_s(P)], \phi_t(P)] \\ &= P\phi_n(U_{12})Q + Q\phi_n(U_{12})P,\end{aligned} \tag{3.2}$$

from this we see that $P\phi_n(U_{12})P = Q\phi_n(U_{12})Q = 0$. Now if $U_{12}, V_{12} \in \mathfrak{A}_{12}$, then $[U_{12}, V_{12}] = 0$. Thus for any $W \in \mathfrak{A}$, we have

$$\begin{aligned}0 &= \phi_n([[U_{12}, V_{12}], W]) = [[\phi_n(U_{12}), V_{12}], W] + [[U_{12}, \phi_n(V_{12})], W] \\ &\quad + \sum_{\substack{r+s+t=n \\ 0\leq r,s,t\leq n-1}} [[\phi_r(U_{12}), \phi_s(V_{12})], \phi_t(W)].\end{aligned}$$

By applying $\mathbf{B}_m$, i.e.; $\phi_m(U_{ij}) \subseteq \mathfrak{A}_{ij},\ 1 \leq i \neq j \leq 2, m < n$, we have

$$[[\phi_n(U_{12}), V_{12}], W] + [[U_{12}, \phi_n(V_{12})], W] = 0 \text{ for all } W \in \mathfrak{A}.$$

Thus, we get

$$[\phi_n(U_{12}), V_{12}] + [U_{12}, \phi_n(V_{12})] \in \mathbb{F}I. \tag{3.3}$$

Since $U_{12} = [P, U_{12}]$, by using equation (3.3) and $\mathbf{B}_m$ again, we have

$$\begin{aligned}[\phi_n(U_{12}), V_{12}] &= [\phi_n([P, U_{12}]), V_{12}] = \lambda I - [[P, U_{12}], \phi_n(V_{12})] \\ &= \lambda I - \phi_n([[P, U_{12}], V_{12}]) + [[\phi_n(P), U_{12}], V_{12}] \\ &\quad + [[P, \phi_n(U_{12})], V_{12}] + \sum_{\substack{r+s+t=n \\ 0\leq r,s,t\leq n-1}} [[\phi_r(P), \phi_r(U_{12})], \phi_s(V_{12})] \\ &= \lambda I + [[P, \phi_n(U_{12})], V_{12}].\end{aligned}$$

Using equation (3.2), we see that

$$\begin{aligned}[P\phi_n(U_{12})Q + Q\phi_n(U_{12})P, V_{12}] &= \lambda I + [[P, P\phi_n(U_{12})Q + Q\phi_n(U_{12})P], V_{12}] \\ &= \lambda I + [P\phi_n(U_{12})Q - Q\phi_n(U_{12})P, V_{12}].\end{aligned}$$

Applying the same arguments as used in Lemma 2.9, we obtain $\phi_n(\mathfrak{A}_{12}) \subseteq \mathfrak{A}_{12}$.

Similarly, $\phi_n(U_{21}) = Q\phi_n(U_{21})P \in \mathfrak{A}_{21}$ for each $U_{21} \in \mathfrak{A}_{21}$ and therefore $\phi_n(\mathfrak{A}_{21}) \subseteq \mathfrak{A}_{21}$. $\square$

Lemma 3.6. *There is a functional $f_{ni} : \mathfrak{A}_{ii} \to \mathbb{F}I$ such that $\phi_n(U_{ii}) - f_{ni}(U_{ii})I \in \mathfrak{A}_{ii}$ for all $U_{ii} \in \mathfrak{A}_{ii}$, $i = 1, 2$.*

Proof. For $U_{11} \in \mathfrak{A}_{11}$, by Lemma 3.3, we have

$$P\phi_n(U_{11})Q + Q\phi_n(U_{11})P = \phi_n(PU_{11}Q + QU_{11}P) = 0.$$

Therefore, it can be assumed that $\phi_n(U_{11}) = A_{11} + A_{22}$ and $\phi_n(U_{22}) = B_{11} + B_{22}$, where $A_{ii}, B_{ii} \in \mathfrak{A}_{ii}, i = 1, 2$. Since $[U_{11}, U_{22}] = 0$, by $\mathbf{B}_m$, i.e.; $[\phi_r(U_{11}), \phi_s(U_{22})] = 0$ for $r, s < n$ and for any $W \in \mathfrak{A}$, we have

$$\begin{aligned} 0 &= \phi_n([[U_{11}, U_{22}], W]) = [[\phi_n(U_{11}), U_{22}], W] + [[U_{11}, \phi_n(U_{22})], W] \\ &+ \sum_{\substack{r+s+t=n \\ 0\leq r,s,t\leq n-1}} [[\phi_r(U_{11}), \phi_s(U_{22})], \phi_t(W)]. \end{aligned}$$

By simple calculation, it is easy to see that $[\phi_n(U_{11}), U_{22}] + [U_{11}, \phi_n(U_{22})] = \lambda I \in \mathbb{F}I$. Multiplying both sides by Q, we arrive at $[Q\phi_n(U_{11})Q, U_{22}] = \lambda Q$. Consequently, by Lemma 2.1, $[Q\phi_n(U_{11})Q, U_{22}] = 0$ for all $U_{22} \in \mathfrak{A}_{22}$. Similarly $[U_{11}, P\phi_n(U_{22})P] = 0$ for all $U_{11} \in \mathfrak{A}_{11}$. Equivalently, $[A_{22}, U_{22}] = 0$ for all $U_{22} \in \mathfrak{A}_{22}$ and $[U_{11}, B_{11}] = 0$ for all $U_{11} \in \mathfrak{A}_{11}$. Therefore, there exist scalars $f_{n1}(U_{11})$ and $f_{n2}(U_{22})$ such that $A_{22} = f_{n1}(U_{11})Q$ and $B_{11} = f_{n2}(U_{22})P$. Hence $\phi_n(U_{11}) - f_{n1}(U_{11})I \in \mathfrak{A}_{11}$ and $\phi_n(U_{22}) - f_{n2}(U_{22})I \in \mathfrak{A}_{22}$. □

Our next goal is to show that ϕ_n is additive on $\mathfrak{A}_{12}$ and $\mathfrak{A}_{21}$.

LEMMA 3.7. *Let $U_{ii} \in \mathfrak{A}_{ii}$ and $U_{ij} \in \mathfrak{A}_{ij}$, $1 \leq i \neq j \leq 2$. Then $\phi_n(U_{ii} + U_{ij}) - \phi_n(U_{ii}) - \phi_n(U_{ij}) \in \mathbb{F}I$.*

Proof. For $U_{11} \in \mathfrak{A}_{11}$ and $U_{12} \in \mathfrak{A}_{12}$, by Lemma 3.6, we find that

$$\begin{aligned} \phi_n([[U_{11} + U_{12}, P], P]) &= [[\phi_n(U_{11} + U_{12}), P], P] + [[U_{11} + U_{12}, \phi_n(P)], P] \\ &\quad + [[U_{11} + U_{12}, P], \phi_n(P)] \\ &\quad + \sum_{\substack{r+s+t=n \\ 0\leq r,s,t\leq n-1}} [[\phi_r(U_{11} + U_{12}), \phi_s(P)], \phi_t(P)] \\ &= [[\phi_n(U_{11} + U_{12}), P], P] \\ &\quad + \sum_{\substack{r+s+t=n \\ 0\leq r,s,t\leq n-1}} [[\phi_r(U_{11}) + \phi_r(U_{12}), \phi_s(P)], \phi_t(P)] \\ &= [[\phi_n(U_{11} + U_{12}), P], P]. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \phi_n([[U_{11} + U_{12}, P], P]) &= \phi_n([[U_{11}, P], P]) + \phi_n([[U_{12}, P], P]) \\ &= [[\phi_n(U_{11}), P], P] + \sum_{\substack{r+s+t=n \\ 0\leq r,s,t\leq n-1}} [[\phi_r(U_{11}), \phi_s(P)], \phi_t(P)] \end{aligned}$$

$$+ \quad [[\phi_n(U_{12}), P], P] + \sum_{\substack{r+s+t=n \\ 0\le r,s,t\le n-1}} [[\phi_r(U_{12}), \phi_s(P)], \phi_t(P)]$$
$$= \quad [[\phi_n(U_{11}), P], P] + [[\phi_n(U_{12}), P], P].$$

By combining the above two identities, we get

$$[[\phi_n(U_{11} + U_{12}) - \phi_n(U_{12}) - \phi_n(U_{12}), P], P] = 0,$$

that is

$$\begin{aligned} (3.4) \qquad 0 \;=\; & P(\phi_n(U_{11} + U_{12}) - \phi_n(U_{12}) - \phi_n(U_{12}))Q \\ & +Q(\phi_n(U_{11} + U_{12}) - \phi_n(U_{12}) - \phi_n(U_{12}))P. \end{aligned}$$

Now, for any $V_{12} \in \mathfrak{A}_{12}$, by $\mathbf{B}_m$, we have

$$\begin{aligned} \phi_n([[U_{11} + U_{12}, V_{12}], P]) &= [[\phi_n(U_{11} + U_{12}), V_{12}], P] + [[U_{11} + U_{12}, \phi_n(V_{12})], P] \\ &\quad + \sum_{\substack{r+s+t=n \\ 0\le r,s,t\le n-1}} [[\phi_r(U_{11} + U_{12}), \phi_s(V_{12})], \phi_t(P)] \\ &= [[\phi_n(U_{11} + U_{12}), V_{12}], P] + [[U_{11} + U_{12}, \phi_n(V_{12})], P] \\ &\quad + \sum_{\substack{r+s+t=n \\ 0\le r,s,t\le n-1}} [[\phi_r(U_{11}) + \phi_r(U_{12}), \phi_s(V_{12})], \phi_t(P)]. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \phi_n([[U_{11} + U_{12}, V_{12}], P]) &= \phi_n([[U_{11}, V_{12}], P]) + \phi_n([[U_{12}, V_{12}], P]) \\ &= [[\phi_n(U_{11}), V_{12}], P] + [[U_{11}, \phi_n(V_{12})], P] \\ &\quad + \sum_{\substack{r+s+t=n \\ 0\le r,s,t\le n-1}} [[\phi_r(U_{11}), \phi_s(V_{12})], \phi_t(P)] \\ &\quad +[[\phi_n(U_{12}), V_{12}], P] + [[U_{12}, \phi_n(V_{12})], P] \\ &\quad + \sum_{\substack{r+s+t=n \\ 0\le r,s,t\le n-1}} [[\phi_r(U_{12}), \phi_s(V_{12})], \phi_t(P)]. \end{aligned}$$

Combining the above two identities, we arrive at

$$[[\phi_n(U_{11} + U_{12}) - \phi_n(U_{11}) - \phi_n(U_{12}), V_{12}], P] \quad = \quad 0.$$

In other words,

$$\begin{aligned} (3.5) \qquad & P\phi_n(U_{11} + U_{12}) - \phi_n(U_{11}) - \phi_n(U_{12})PV_{12} \\ &= \quad V_{12}Q\phi_n(U_{11} + U_{12}) - \phi_n(U_{11}) - \phi_n(U_{12})Q \end{aligned}$$

Equations (3.4) and (3.5), together with Lemma 2.3, gives that $\phi_n(U_{11} + U_{12}) - \phi_n(U_{11}) - \phi_n(U_{12}) \in \mathbb{F}I$. Similarly, one can easily prove the other part. $\square$

Lemma 3.8. *ϕ_n is additive on $\mathfrak{A}_{12}$ and $\mathfrak{A}_{21}$.*

Proof. Let $U_{12}, V_{12} \in \mathfrak{A}_{12}$. By Lemmas 3.1, 3.4 & 3.7, we have

$$\begin{aligned}
\phi_n(U_{12}+V_{12}) &= \phi_n([[P+U_{12}, Q+V_{12}], Q]) \\
&= [[\phi_n(P+U_{12}), Q+V_{12}], Q] + [[P+U_{12}, \phi_n(Q+V_{12})], Q] \\
&\quad + [[P+U_{12}, Q+V_{12}], \phi_n(Q)] \\
&\quad + \sum_{\substack{r+s+t=n \\ 0\le r,s,t\le n-1}} [[\phi_r(P+U_{12}), \phi_s(Q+V_{12})], \phi_t(Q)] \\
&= [[\phi(P)+\phi(U_{12}), Q+V_{12}], Q] + [[P+U_{12}, \phi(Q)+\phi(V_{12})], Q] \\
&\quad + \sum_{\substack{r+s+t=n \\ 0\le r,s,t\le n-1}} [[\phi_r(P)+\phi_r(U_{12}), \phi_s(Q)+\phi_s(V_{12})], \phi_t(Q)].
\end{aligned}$$

Since $\sum\limits_{\substack{r+s+t=n \\ 0\le r,s,t\le n-1}} [[\phi_r(P)+\phi_r(U_{12}), \phi_s(Q)+\phi_s(V_{12})], \phi_t(Q)] = 0$, the above identity gives that $\phi_n(U_{12}+V_{12}) = \phi_n(U_{12})+\phi_n(V_{12})$. Hence ϕ_n is additive on $\mathfrak{A}_{12}$. Similarly ϕ_n is additive on $\mathfrak{A}_{21}$. □

Now for any $U \in \mathfrak{A}$, define $\delta_n(U) = \phi_n(PUP)+\phi_n(PUQ)+\phi_n(QUP)+\phi_n(QUQ)-(f_{n1}(PUP)+f_{n2}(QUQ))I$. By Lemmas 3.5 and 3.6, we have

Lemma 3.9. *For $U_{ij} \in \mathfrak{A}_{ij}$, $1 \le i \ne j \le 2$, we have*

(i) $\delta_n(U_{ij}) \in \mathfrak{A}_{ij}$, $1 \le i \ne j \le 2$,

(ii) $\delta_n(U_{12}) = \phi_n(U_{12})$ *and* $\delta_n(U_{21}) = \phi_n(U_{21})$,

(iii) $\delta_n(U_{11}+U_{12}+U_{21}+U_{22}) = \delta_n(U_{11})+\delta_n(U_{12})+\delta_n(U_{21})+\delta_n(U_{22})$.

The following lemma immediately follows from Lemma 3.8 and Lemma 3.9.

Lemma 3.10. *δ_n is additive on $\mathfrak{A}_{12}$ and $\mathfrak{A}_{21}$.*

Lemma 3.11. *Let $U_{ii} \in \mathfrak{A}_{ii}$, $U_{ij} \in \mathfrak{A}_{ij}$, $1 \le i \ne j \le 2$. Then*

(i) $\delta_n(U_{ii}V_{ij}) = \sum\limits_{r+s=n} \delta_r(U_{ii})\delta_s(V_{ij})$,

(ii) $\delta_n(V_{ij}U_{jj}) = \sum\limits_{r+s=n} \delta_r(U_{ij})\delta_s(V_{jj})$.

Proof. By Lemmas 3.4 & 3.9, we have

$$\delta_n(U_{11}V_{12}) = \phi_n(U_{11}V_{12}) = \phi_n([[U_{11}, V_{12}], Q])$$

$$
\begin{aligned}
&= [[\phi_n(U_{11}), V_{12}], Q] + [[U_{11}, \phi_n(V_{12})], Q] + [[U_{11}, V_{12}], \phi_n(Q)] \\
&\quad + \sum_{\substack{r+s+t=n \\ 0\le r,s,t\le n-1}} [[\phi_r(U_{11}), \phi_s(V_{12})], \phi_t(Q)] \\
&= [[\phi_n(U_{11}), V_{12}], Q] + [[U_{11}, \phi_n(V_{12})], Q] \\
&\quad + \sum_{\substack{r+s+t=n \\ 0\le r,s,t\le n-1}} [[\delta_r(U_{11}) + f_{r1}(U_{11}), \delta_s(V_{12})], \phi_t(Q)].
\end{aligned}
$$

Here it is to be noted that t runs from 0 and $n \in \mathbb{N}$, so using $\phi_0(Q) = Q$, we have

$$
\begin{aligned}
\delta_n(U_{11}V_{12}) &= [[\delta_n(U_{11}), V_{12}], Q] + [[U_{11}, \delta_n(V_{12})], Q] \\
&\quad + \sum_{\substack{r+s=n \\ 0\le r,s\le n-1}} [[\delta_r(U_{11}), \delta_s(V_{12})], Q] \\
&= [[\delta_n(U_{11}), V_{12}], Q] + [[U_{11}, \delta_n(V_{12})], Q] \\
&\quad + \sum_{\substack{r+s=n \\ 0\le r,s\le n-1}} \delta_r(U_{11})\delta_s(V_{12}) \\
&= \sum_{r+s=n} \delta_r(U_{11})\delta_s(V_{12}).
\end{aligned}
$$

Similarly, it is easy to prove the other three identities. □

Lemma 3.12. *δ_n is additive on $\mathfrak{A}_{11}$ and $\mathfrak{A}_{22}$.*

Proof. By Lemma 3.11, for any $U_{11}, V_{11} \in \mathfrak{A}_{11}$ and $W_{12} \in \mathfrak{A}_{12}$ we have

$$
\begin{aligned}
\delta_n((U_{11}+V_{11})W_{12}) &= \sum_{r+s=n} \delta_r(U_{11}+V_{11})\delta_s(W_{12}) \\
&= \delta_n(U_{11}+V_{11})W_{12} + (U_{11}+V_{11})\delta_n(W_{12}) \\
&\quad + \sum_{\substack{r+s=n \\ 0<r,s\le n-1}} \delta_r(U_{11}+V_{11})\delta_s(W_{12}) \\
&= \delta_n(U_{11}+V_{11})W_{12} + (U_{11}+V_{11})\delta_n(W_{12}) \\
&\quad + \sum_{\substack{r+s=n \\ 0<r,s\le n-1}} \delta_r(U_{11})\delta_s(W_{12}) + \delta_r(V_{11})\delta_s(W_{12}).
\end{aligned}
$$

On the other hand, by Lemmas 3.10 & 3.11, we have

$$
\begin{aligned}
\delta_n((U_{11}+V_{11})W_{12}) &= \delta_n(U_{11}W_{12} + V_{11}W_{12}) = \delta_n(U_{11}W_{12}) + \delta_n(V_{11}W_{12}) \\
&= \sum_{r+s=n} \delta_r(U_{11})\delta_s(W_{12}) + \sum_{r+s=n} \delta_r(V_{11})\delta_s(W_{12})
\end{aligned}
$$

$$= \delta_n(U_{11})W_{12} + U_{11}\delta_n(W_{12}) + \sum_{\substack{r+s=n \\ 0<r,s\leq n-1}} \delta_r(U_{11})\delta_s(W_{12})$$
$$+\delta_n(V_{11})W_{12} + V_{11}\delta_n(W_{12}) + \sum_{\substack{r+s=n \\ 0<r,s\leq n-1}} \delta_r(V_{11})\delta_s(W_{12}).$$

Comparing the above two equations , we get

$$(\delta_n(U_{11} + V_{11}) - \delta_n(U_{11}) - \delta_n(V_{11}))W_{12} = 0.$$

In other words$(\delta_n(U_{11}+V_{11})-\delta_n(U_{11})-\delta_n(V_{11}))P\mathfrak{A}Q = 0$. Since $\mathfrak{A}$ is prime, it follows that $(\delta_n(U_{11} + V_{11}) - \delta_n(U_{11}) - \delta_n(V_{11}))P = 0$. Hence, $\delta_n(U_{11} + V_{11}) = \delta_n(U_{11}) + \delta_n(V_{11})$ as $\delta_n(\mathfrak{A}_{11}) \subseteq \mathfrak{A}_{11}$. Similarly, δ_n is additive on $\mathfrak{A}_{22}$. □

LEMMA 3.13. *δ_n is additive.*

Proof. The proof is same as that of Lemma 2.17. □

In the sequel, we shall show that $\{\delta_n\}_{n\in\mathbb{N}}$ is a higher derivation.

LEMMA 3.14. *Let $U_{ii}, V_{ii} \in \mathfrak{A}_{\rm ii}, i = 1, 2$. Then*

$$\delta_n(U_{ii}V_{ii}) = \sum_{r+s=n} \delta_r(U_{ii})\delta_s(V_{ii}).$$

Proof. For any $W_{12} \in \mathfrak{A}_{12}$, by Lemma 3.11, we have

$$\delta_n(U_{11}V_{11}W_{12}) = \sum_{r+s=n} \delta_r(U_{11}V_{11})\delta_s(W_{12}).$$

On the other hand we have,

$$\begin{aligned}\delta_n(U_{11}V_{11}W_{12}) &= \sum_{r+s=n} \delta_r(U_{11})\delta_s(V_{11}W_{12}) \\ &= \sum_{r+s=n} \delta_r(U_{11}) \sum_{i+j=s} \delta_i(V_{11})\delta_j(W_{12}) \\ &= \sum_{r+l+m=n} \delta_r(U_{11})\delta_j(V_{11})\delta_j(W_{12}).\end{aligned}$$

Comparing the above two identities and noting that

$$\delta_m(U_{11}V_{11}) = \sum_{r+s=m} \delta_r(U_{11})\delta_s(V_{11}),$$

for all $m < n$, one can obtain that

$$\delta_n(U_{11}V_{11})W_{12} = \Big(\sum_{r+s=n} \delta_r(U_{11})\delta_s(V_{11})\Big)W_{12}.$$

In other words $\big(\delta_n(U_{11}V_{11})-\sum_{r+s=n}\delta_r(U_{11})\delta_s(V_{11})\big)P\mathfrak{A}Q=0$. Since $\mathfrak{A}$ is prime, it follows that

$$\big(\delta_n(U_{11}V_{11})-\sum_{r+s=n}\delta_r(U_{11})\delta_s(V_{11})\big)P \;=\; 0.$$

Hence, $\delta_n(U_{11}V_{11})=\sum_{r+s=n}\delta_r(U_{11})\delta_s(V_{11})$ as $\delta_n(\mathfrak{A}_{11})\subseteq\mathfrak{A}_{11}$. Similarly, $\delta_n(U_{22}V_{22})=\sum_{r+s=n}\delta_r(U_{22})\delta_s(V_{22})$. □

Lemma 3.15. *Let $U_{11}\in\mathfrak{A}_{11}$, $V_{22}\in\mathfrak{A}_{22}$. Then $\phi_n(U_{11}+V_{22})-\delta_n(U_{11})-\delta_n(V_{22})\in\mathbb{F}I$.*

Proof. Let $U_{11}\in\mathfrak{A}_{11}$, $V_{22}\in\mathfrak{A}_{12}$. Since $\phi_n(Q)\in\mathbb{F}I$, We have

$$\begin{aligned}\phi_n([[U_{11}+V_{22},Q],Q]) &= [[\phi_n(U_{11}+V_{22}),Q],Q]+[[U_{11}+V_{22},\phi_n(Q)],Q]\\ &\quad +[[U_{11}+V_{22},Q],\phi_n(Q)]\\ &\quad +\sum_{\substack{r+s+t=n\\0\le r,s,t\le n-1}}[[\phi_r(U_{11}+V_{22}),\phi_s(Q)],\phi_t(Q)]\\ &= [[\delta_n(U_{11}+V_{22}),Q],Q].\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\phi_n([[U_{11}+V_{22},Q],Q]) &= \phi_n([[U_{11},Q],Q])+\phi_n([[V_{22},Q],Q])\\ &= [[\phi_n(U_{11}),Q],Q]+[[\phi_n(V_{22}),Q],Q]\\ &\quad +\sum_{\substack{r+s+t=n\\0\le r,s,t\le n-1}}[[\phi_r(U_{11}),\phi_s(Q)],\phi_t(Q)]\\ &= [[\delta_n(U_{11})-f_{n1}(U_{11}),Q],Q]\\ &\quad +[[\phi_n(V_{22})-f_{n2}(U_{22}),Q],Q]\\ &= [[\delta_n(U_{11}),Q],Q]+[[\delta_n(V_{22}),Q],Q].\end{aligned}$$

On combining the above two identities, we see that

$$[[\phi_n(U_{11}+V_{22})-\delta_n(U_{11})-\delta_n(V_{22}),Q],Q] \;=\; 0,$$

that is

$$\begin{aligned}0 &= P(\phi_n(U_{11}+V_{22})-\delta_n(U_{11})-\delta_n(V_{22}))Q\\ &\quad +Q(\phi_n(U_{11}+V_{22})-\delta_n(U_{11})-\delta_n(V_{22}))P.\end{aligned}\tag{3.6}$$

For any $W_{12}\in\mathfrak{A}_{12}$, since $\delta_n(Q)=\phi_n(Q)-f_{n2}(Q)$, we have

$$\begin{aligned}\phi_n([U_{11}+V_{22},W_{12}]) &= \phi_n(U_{11}W_{12}-W_{12}V_{22})\\ &= \phi_n(U_{11}W_{12})-\phi_n(W_{12}V_{22})=\delta_n(U_{11}W_{12})-\delta_n(W_{12}V_{22})\end{aligned}$$

$$\begin{aligned}
&= \delta_n([[U_{11}, W_{12}], Q]) + \delta_n([[V_{22}, W_{12}], Q]) \\
&= [[\delta_n(U_{11}), W_{12}], Q] + [[U_{11}, \delta_n(W_{12})], Q] \\
&+ [[U_{11}, W_{12}], \delta_n(Q)] + \sum_{\substack{r+s+t=n \\ 0\leq r,s,t\leq n-1}} [[\delta_r(U_{11}), \delta_s(W_{12})], \delta_t(Q)] \\
&+ [[\delta_n(V_{22}), W_{12}], Q] + [[V_{22}, \delta_n(W_{12})], Q] \\
&+ [[V_{22}, W_{12}], \delta_n(Q)] + \sum_{\substack{r+s+t=n \\ 0\leq r,s,t\leq n-1}} [[\delta_r(V_{22}), \delta_s(W_{12})], \delta_t(Q)] \\
&= [[\delta_n(U_{11}) + \delta_n(V_{22}), W_{12}], Q] + [[U_{11} + V_{22}, \delta_n(W_{12})], Q] \\
&\quad + \sum_{\substack{r+s+t=n \\ 0\leq r,s,t\leq n-1}} [[\delta_r(U_{11}) + \delta_r(V_{22}), \delta_s(W_{12})], \phi_t(Q)].
\end{aligned}$$

On the other hand by Lemma 3.9, we have

$$\begin{aligned}
\phi_n([U_{11} + V_{22}, W_{12}]) &= \phi_n([[U_{11} + V_{22}, W_{12}], Q]) \\
&= [[\phi_n(U_{11} + V_{22}), W_{12}], Q] + [[U_{11} + V_{22}, \delta_n(W_{12})], Q] \\
&\quad + \sum_{\substack{r+s+t=n \\ 0\leq r,s,t\leq n-1}} [[\phi_r(U_{11} + V_{22}), \delta_s(W_{12})], \phi_t(Q)].
\end{aligned}$$

Comparing the above two identities, we obtain

$$[[\phi_n(U_{11} + V_{22}) - \delta_n(U_{11}) - \delta_n(V_{22}), W_{12}], Q] = 0.$$

In other words, we get

$$\begin{aligned}
(3.7) \qquad & P\big(\phi_n(U_{11} + V_{22}) - \delta_n(U_{11}) - \delta_n(V_{22})\big)W_{12} \\
&= W_{12}\big(\phi_n(U_{11} + V_{22}) - \delta_n(U_{11}) - \delta_n(V_{22})\big)Q.
\end{aligned}$$

Equations (3.6) and (3.7), together with Lemma 2.3, yields that

$$\phi_n(U_{11} + U_{22}) - \delta_n(U_{11}) - \delta_n(U_{22}) \in \mathbb{F}I.$$

□

LEMMA 3.16. *Let $U_{12} \in \mathfrak{A}_{12}$, $V_{21} \in \mathfrak{A}_{21}$. Then*

(*i*) $\delta_n(U_{12}V_{21}) = \sum_{r+s=n} \delta_r(U_{12})\delta_s(V_{21})$,

(*ii*) $\delta_n(U_{21}V_{12}) = \sum_{r+s=n} \delta_r(U_{ii})\delta_s(V_{ij})$

Proof. By using Lemma 3.13, we compute

$$\begin{aligned}
\phi_n([U_{12}, V_{21}]) - \delta_n([U_{12}, V_{21}]) &= \phi_n([[P, U_{12}], V_{21}]) - \delta_n(U_{12}V_{21} - V_{21}U_{12}) \\
&= [[P, \phi(U_{12})], V_{21}] + [[P, U_{12}], \phi(V_{21})]
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\substack{r+s+t=n\\0\leq r,s,t\leq n-1}}[[\phi_r(P),\phi_s(U_{12})],\phi_t(V_{21})]\\
&-\delta_n(U_{12}V_{21})+\delta_n(V_{21}U_{12})\\
=\;&\delta_n(U_{12})V_{21}+U_{12}\delta_n(V_{21})\\
&+\sum_{\substack{r+s=n\\0\leq r,s\leq n-1}}\delta_r(U_{12})\delta_s(V_{21})\\
&-\delta_n(U_{12}V_{21})-\delta_n(V_{21})U_{12}-V_{21}\delta_n(U_{12})\\
&-\sum_{\substack{r+s=n\\0\leq r,s\leq n-1}}\delta_r(V_{21})\delta_s(U_{12})+\delta_n(V_{21}U_{12})\\
=\;&\sum_{r+s=n}\delta_r(U_{12})\delta_s(V_{21})-\delta_n(U_{12}V_{21})\\
&-\sum_{r+s=n}\delta_r(V_{21})\delta_s(U_{12})+\delta_n(V_{21}U_{12}).
\end{aligned}
$$

Since
$\phi_n([U_{12},V_{21}])-\delta_n([U_{12},V_{21}])=\phi_n(U_{12}V_{21}-V_{21}U_{12})-\delta_n(U_{12}V_{21}-V_{21}U_{12})$,
by Lemma 3.15, we have
$\delta_n(V_{21}U_{12})-\sum_{r+s=n}\delta_r(V_{21})\delta_s(U_{12})+\sum_{r+s=n}\delta_r(U_{12})\delta_s(V_{21})-\delta_n(U_{12}V_{21})=\lambda I\in\mathbb{F}I.$

Thus we obtain the two identities

$$\delta_n(U_{12}V_{21}) = \sum_{r+s=n}\delta_r(U_{12})\delta_s(V_{21})-\lambda P,$$

and

$$\delta_n(V_{21}U_{12}) = \sum_{r+s=n}\delta_r(V_{21})\delta_s(U_{12})+\lambda Q.$$

Following the same arguments as used in the Lemma 2.20, we get the desired result. □

Thus we have shown that $\Delta=\{\delta_n\}_{n\in\mathbb{N}}$ is an additive higher derivation.

Proof of Theorem 3.1. Let us define $\tau_n:\mathfrak{A}\to\mathfrak{A}$ by $\tau_n(U)=\phi_n(U)-\delta_n(U)$ for $U\in\mathfrak{A}$. For $i=j$, $\tau_n(U_{ij})=f_{ni}(U_{ij})I$; otherwise $\tau_n(U_{ij})=0$. We shall show that $\tau_n(U)\in\mathbb{F}I$ for all $U\in\mathfrak{A}$. By following the same procedure as used in the proof of Theorem 2.1, it can be easily shown that

$$P\phi_n(U)P+Q\phi_n(U)Q-\phi_n(PUP)-\phi_n(QUQ)\in\mathbb{F}I \tag{3.8}$$

Now by using Lemmas 3.4 & 3.8, we have

$$P\phi_n(U)Q+Q\phi_n(U)P=\phi_n(PUQ+QUP)=\phi_n(PUQ)+\phi_n(QUP).$$

So,

$$\begin{aligned}
&\phi_n(U) - \phi_n(PUP) + \phi_n(PUQ) + \phi_n(QUP) + \phi_n(QUQ) \\
&= P\phi_n(U)P + Q\phi_n(U)Q + P\phi_n(U)Q + Q\phi_n(U)P \\
&\quad -\phi_n(PUP) - \phi_n(QUQ) - \phi_n(PUQ) - \phi_n(QUP) \\
&= P\phi_n(U)P + Q\phi_n(U)Q - \phi_n(PUP) - \phi_n(QUQ) \in \mathbb{F}I.
\end{aligned}$$

By equation (3.8) and by the definition of δ_n and τ_n, we see that $\tau_n(U) \in \mathbb{F}I$ for all $U \in \mathfrak{A}$. Since δ_n is an additive Lie triple higher derivation, it follows that for all $U, V, W \in \mathfrak{A}$

$$\begin{aligned}
\tau_n([[U,V],W]) &= \phi_n([[U,V],W]) - \delta_n([[U,V],W]) \\
&= \sum_{\substack{r+s+t=n \\ 0\leq r,s,t\leq n-1}} [[\phi_r(U), \phi_s(V)], \phi_t(W)] - \delta_n([[U,V],W]) \\
&= \sum_{\substack{r+s+t=n \\ 0\leq r,s,t\leq n-1}} [[\delta_r(U), \delta_s(V)], \delta_t(W)] - \delta_n([[U,V],W]) \\
&= 0.
\end{aligned}$$

Finally, let us define $\psi_n(U) = \delta_n(U) - (TU - UT)$ for all $U \in \mathfrak{A}$ and $n \in \mathbb{N}$, where $T = Pd_n(P)Q - Qd_n(P)P$. It is easy to check that $\{\psi_n\}_{n\in\mathbb{N}}$ is an additive higher derivation on $\mathfrak{A}$. By the definitions of Δ_n and ϕ_n, we have $d_n(U) = \psi_n(U) + \tau_n(U)$ for all $U \in \mathfrak{A}$.

Furthermore, if d_n is linear , then ψ_n and τ_n are also linear. It is to be noted that any linear derivation is an inner derivation (see [21]). Nowicki [19] proved that if every linear derivation of $\mathfrak{A}$ is inner, then every linear higher derivation of $\mathfrak{A}$ is also inner. So by Theorem 2.1, $\{\psi_n\}_{n\in\mathbb{N}}$ is an inner higher derivation, i.e., $\psi_n(U) = \sum_{i=0}^{n} X_i U Y_{n-i}$ for all $U \in \mathfrak{A}$ and for every $n \in \mathbb{N}$, where $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ are two sequences in $\mathfrak{A}$ satisfying the conditions $X_0 = Y_0 = 1$ and $\sum_{i=0}^{n} X_i Y_{n-i} = \delta_{n0} = \sum_{i=0}^{n} Y_i X_{n-i}$ and δ_{n0} is the Kronecker sign. This completes the proof. □

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REFERENCES

[1] W. Cheung, *Lie derivations of triangular algebras*. Linear Multilinear Algebra **51** (2003), 299–310.

[2] L. Chen and J. H. Zhang, *Nonlinear Lie derivations on upper triangular matrices*. Linear Multilinear Algebra **56** (2008), *6*, 725–730.

[3] M. N. Daif, *When is a multiplicative derivation additive?* International Journal of Mathematics and Mathematical Sciences **14** (1991), *3*, 615–618.

[4] S. Ebrahimi, *Lie higher deivation on $B(X)$*. Journal of Linear and Topological Algebra **4** (2015), *3*, 183–192.

[5] M. Ferrero and C. Haetinger, *Higher derivations of semiprime rings*. Comm. Algebra **30** (2002), 2321–2333.

[6] P. Halmos, *A Hilbert space Problem Book*, 2nd ed. Springer-Verlag, New York, 1982.

[7] D. Han, *Lie-type higher derivations on operator algebras*. Bull. Iranian Math. Soc. **40** (2014), *5*, 1169–1194.

[8] W. Jing and F. Lu, *Lie derivable mappings on prime rings*. Linear Multilinear Algebra **60** (2012), 167–180.

[9] P. Ji, R. Liu, and Y. Zhao, *Nonlinear Lie triple derivations of triangular algebras*. Linear Multilinear Algebra **60** (2012), 1155–1164.

[10] P. S. Ji and L. Wang, *Lie triple derivations of TUHF algebras*. Linear Algebra Appl. **403** (2005), 399–408.

[11] F. Lu, *Additivity of Jordan maps on standard operator algebras*. Linear Algebra Appl. **357** (2002), 123–131.

[12] F. Lu, *Lie triple derivations on nest algebras*. Math. Nachr. **280** (2007), *8*, 882–887.

[13] F. Lu and W. Jing, *Characterizations of Lie derivations of $\mathcal{B}(\mathcal{X})$*. Linear Algebra Appl. **432** (2010), *1*, 890-99.

[14] F. Lu and B. Liu, *Lie derivable maps on $\mathcal{B}(\mathcal{X})$*. Journal of Mathematical Analysis and Applications **372** (2010), 369–376.

[15] M. Mathieu and A. R. Villena, *The structure of Lie derivations on C^*-algebras*. J. Funct. Anal. **202** (2003), 504–525.

[16] W. S. Martindale III, *When are multiplicative mappings additive?* Proc. Amer. Math. Soc. **21** (1969), 695–698.

[17] C. R. Mires, *Lie derivations of von Neumann algebras*. Duke Math. J. **40** (1973), 403–409.

[18] C. R. Mires, *Lie triple derivations of von Neumann algebras*. Proc. Am. Math. Soc. **71** (1978), 57–61.

[19] A. Nowicki, *Inner derivations of higher orders*. Tsukuba J. Math. **8** (1984), *2*, 219–225.

[20] X. F. Qi and J. C. Hou, *Lie higher derivations on nest algebras*. Commun. Math. Res. **26** (2010), *2*, 131–143.

[21] P. Šemrl, *Additive derivations of some operator algebras*. Ilinois J. Math. **35** (1991), 234–240.

[22] A.R. Villena, *Lie derivations on Banach algebras*. J. Algebra **226** (2000), 390–409.

[23] F. Wei and Z. K. Xiao, *Higher derivations of triangular algebras and its generalizations*. Linear Algebra Appl. **435** (2011), 1034–1054.

[24] Z. K. Xiao and F. Wei, *Nonlinear Lie higher derivations on triangular algebras*. Linear Multilinear Algebra **60** (2012), *8*, 979–994.

[25] W. Yu and J. Zhang, *Nonlinear Lie derivations of triangular algebras*. Linear Algebra Appl. **432** (2010), *11*, 2953–2960.

[26] J. H. Zhang, B. W. Wu and H. X. Cao, *Lie triple derivations of nest algebras*. Linear Algebra Appl. **416** (2006), *2–3*, 559–567.

[27] F. Zhang and J. Zhang, *Nonlinear Lie derivations on factor von Neumann algebras*. Acta Mathematica Sinica. (Chin. Ser) **54** (2011), *5*, 791–802.

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