REMARKS ON THE RESTRICTED PARTITION FUNCTION

MIRCEA CIMPOEAŞ

Communicated by Lucian Beznea

Let $\mathbf{a} = (a_1, \ldots, a_r)$ be a vector of positive integers. In continuation of a previous paper we present other formulas for the restricted partition function $p_{\mathbf{a}}(n) :=$ the number of integer solutions $(x_1, ..., x_r)$ to $\sum_{j=1}^{r} a_j x_j = n$ with $x_1 \geq 0, ..., x_r \geq 0$.

AMS 2010 Subject Classification: restricted partition function, Sylvester waves, quasi-polynomial.

Key words: Primary 05A17, 11P83; Secondary 05A15, 11P82.

1. INTRODUCTION

Let $\mathbf{a} := (a_1, a_2, \ldots, a_r)$ be a sequence of positive integers, $r \geq 1$. The *restricted partition function* associated to $\mathbf{a}$ is $p_{\mathbf{a}} : \mathbb{N} \to \mathbb{N}$, $p_{\mathbf{a}}(n) :=$ the number of integer solutions $(x_1, \ldots, x_r)$ of $\sum_{i=1}^{r} a_i x_i = n$ with $x_i \geq 0$. Let D be a common multiple of $a_1, \ldots, a_r$.

Sylvester [15],[16] decomposed the restricted partition in a sum of "waves":

$$p_{\mathbf{a}}(n) = \sum_{j \geq 1} W_j(n, \mathbf{a}),$$

where the sum is taken over all distinct divisors j of the components of $\mathbf{a}$ and showed that for each such j, $W_j(n, \mathbf{a})$ is the coefficient of t^{-1} in

$$\sum_{0 \leq \nu < j,\ \gcd(\nu, j) = 1} \frac{\rho_j^{-\nu n} e^{nt}}{(1 - \rho_j^{\nu a_1} e^{-a_1 t}) \cdots (1 - \rho_j^{\nu a_r} e^{-a_r t})},$$

where $\rho_j = e^{\frac{2\pi i}{j}}$ and $\gcd(0, 0) = 1$ by convention.

Note that $W_j(n, \mathbf{a})$'s are quasi-polynomials of period j. (A *quasi-polynomial* of period j is a numerical function $f(n)$ such that there exists j polynomials $P_1(n), P_2(n), \ldots, P_j(n)$ such that $f(n) = P_i(n)$ if $n \equiv i(\bmod j)$.) The first wave $P_{\mathbf{a}}(n) := W_1(\mathbf{a}, n)$ is called the *polynomial part* of $p_{\mathbf{a}}(n)$.

Glaisher [7] made computations of the Sylvester waves in particular cases. Fel and Rubinstein [13] proved formulas for the Sylvester waves using Bernoulli

The author was supported by a grant of the Romanian National Authority for Scientific Research, CNCS - UEFISCDI, project number PN-II-ID-PCE-2011-1023.

and Euler polynomials of higher order. Rubinstein [12] showed that all Sylvester waves can be expressed in terms of Bernoulli polynomials only. Bayad and Beck [2, Theorem 3.1] proved an explicit expression of the partition function $p_{\mathbf{a}}(n)$ in terms of Bernoulli-Barnes polynomials and the Fourier Dedekind sums, in the case that $a_1, \ldots, a_r$ are pairwise coprime. Beck, Gessler and Komatsu [1, page 2], Dilcher and Vignat [6, Theorem 1.1] proved explicit formulas for the polynomial part of $p_{\mathbf{a}}(n)$.

As a continuation of [4] we present here other formulas for $p_{\mathbf{a}}(n)$ and for the Sylvester waves. Also, we reprove, using our method, several results. In Proposition 3.2 we prove that

$$p_{\mathbf{a}}(n) = \sum_{j=0}^{\lfloor \frac{n}{D} \rfloor} \binom{r+j-1}{j} f_{\mathbf{a}}(n-jD),$$

where $f_{\mathbf{a}}(n) = \#\{(j_1,\ldots,j_r) :\ a_1j_1+\cdots+a_rj_r = n,\ 0 \leq j_k \leq \frac{D}{a_k}-1,\ 1 \leq k \leq r\}$. This result is similar to Theorem 1 of Rodseth and Seller [11].

In Corollary 3.4 (compare [11, Theorem 2]) we prove the congruence

$$(r-1)!\, p_{\mathbf{a}}(n) \equiv 0 \mod (j+k+1)(j+k+2)\cdot \ldots \cdot (j+r-1),$$

where $k = \lfloor \frac{n}{D} \rfloor - \lceil \frac{n+a_1+\cdots+a_r}{D} \rceil + r$.

In Proposition 5.2 we prove that

$$W_j(n,\mathbf{a}) = \frac{1}{D(r-1)!} \sum_{m=1}^{r} \sum_{\ell=1}^{j} \rho_j^{\ell} \sum_{k=m-1}^{r-1} \begin{bmatrix} r \\ k+1 \end{bmatrix} (-1)^{k-m+1} \binom{k}{m-1} \cdot$$
$$\cdot \sum_{\substack{0\leq j_1 \leq \frac{D}{a_1}-1,\ldots,0\leq j_r\leq \frac{D}{a_r}-1 \\ a_1j_1+\cdots+a_rj_r \equiv \ell (\bmod j)}} D^{-k}(a_1j_1+\cdots+a_rj_r)^{k-m+1} n^{m-1},$$

where $\begin{bmatrix} r \\ k \end{bmatrix}$ are the unsigned Stirling numbers of the first kind.

The *Bernoulli numbers* are defined by the identity

$$\frac{t}{e^t-1} = \sum_{\ell=0}^{\infty} \frac{t^{\ell}}{\ell!} B_{\ell}.$$

If $\gcd(a_i,a_j) = 1$ for all $i \neq j$ we prove in Proposition 5.3 that

$$p_{\mathbf{a}}(n) = \sum_{m=1}^{r} \frac{(-1)^{r-m}}{(a_1\cdots a_r)(m-1)!} \sum_{i_1+\cdots+i_r=r-m} \frac{B_{i_1}\cdots B_{i_r}}{i_1!\cdots i_r!} a_1^{i_1}\cdots a_r^{i_r} n^{m-1} + \frac{1}{D(r-1)!} \times$$
$$\sum_{j\neq 1} \sum_{\ell=1}^{j} \rho_j^{\ell} \sum_{k=0}^{r-1} \frac{1}{D^k} \begin{bmatrix} r \\ k+1 \end{bmatrix} (-1)^k \sum_{\substack{0\leq j_1 \leq \frac{D}{a_1}-1,\ldots,0\leq j_r\leq \frac{D}{a_r}-1 \\ a_1j_1+\cdots+a_rj_r \equiv \ell (\bmod j)}} D^{-k}(a_1j_1+\cdots+a_rj_r)^{k},$$

where $j|a_i$ for some $1 \leq i \leq r$. Another formulas for $p_{\mathbf{a}}(n)$ in the case that $a_1, \ldots, a_r$ are pairwise coprimes were proved in [5, Theorem C, pag 113], [2, Theorem 3.1] and [8].

Let

$$\sum_{n=0}^{\infty} p_{\mathbf{a}}(n) z^n = \frac{1}{(1-z^{a_1}) \ldots (1-z^{a_r})} = \sum_{\lambda^D=1} \sum_{\ell=1}^{m(\lambda)} \frac{c_{\lambda,\ell}}{(\lambda - z)^{\ell}},$$

where $m(\lambda)$ is the multiplicity of λ as a root of $(1-z^{a_1}) \ldots (1-z^{a_r})$.

In Proposition 5.4 we prove that

$$c_{\rho_j,m} = \frac{\rho_j^m (m-1)!}{D} \sum_{t=m}^{m(\rho_j)} (-1)^{t-m} \left\{ {t \atop m} \right\} \sum_{\ell=1}^{j} \rho_j^{\ell} \sum_{k=m-1}^{r-1} \left[{r \atop k+1} \right] \times$$
$$(-1)^{k-m+1} \binom{k}{m-1} \sum_{\substack{0 \leq j_1 \leq \frac{D}{a_1}-1, \ldots, 0 \leq j_r \leq \frac{D}{a_r}-1 \\ a_1 j_1 + \cdots + a_r j_r \equiv \ell (\bmod j)}} D^{-k} (a_1 j_1 + \cdots + a_r j_r)^{k-m+1},$$

where $\left\{ {t \atop m} \right\}$ are Stirling numbers of the second kind.

In Proposition 5.5 we prove that

$$c_{1,m} = \frac{(m-1)!}{a_1 \cdots a_r} \sum_{\ell=m}^{r} (-1)^{\ell-m} \frac{\left\{ {\ell \atop m} \right\}}{(\ell-1)!} \sum_{i_1+\cdots+i_r=r-\ell} \frac{B_{i_1} \cdots B_{i_r}}{i_1! \cdots i_r!} a_1^{i_1} \cdots a_r^{i_r}.$$

In the case $\mathbf{a} = (1, 2, \ldots, r)$ we reprove O'Sullivan's formulas [9] for Rademacher's coefficients c_{01m}, see Corollary 5.6.

Given a sequence of positive integers $\mathbf{a} = (a_1, ..., a_r)$ with $gcd(a_1, ..., a_r) = 1$, the *Frobenius number* of $\mathbf{a}$, denoted by $F(\mathbf{a}) = F(a_1, \ldots, a_r)$ is the largest integer n with the property that $p_{\mathbf{a}}(n) = 0$. If $\gcd(a_i, a_j) = 1$ for all $i \neq j$, we prove in Corollary 6.2 that $F(A_1, \ldots, A_r) = D(r-1) - A_1 - \cdots - A_r$, where $A_1 := \frac{D}{a_1}, \ldots, A_r := \frac{D}{a_r}$. This is a particular case of [10, Theorem 2.7] and appears also in [17, Theorem 1(a)].

2. PRELIMINARIES

Let $\mathbf{a} := (a_1, a_2, \ldots, a_r)$ be a sequence of positive integers, $r \geq 1$. The *restricted partition function* associated to $\mathbf{a}$ is $p_{\mathbf{a}} : \mathbb{N} \to \mathbb{N}$, $p_{\mathbf{a}}(n) :=$ the number of integer solutions $(x_1, \ldots, x_r)$ of $\sum_{i=1}^r a_i x_i = n$ with $x_i \geq 0$.

Let D be a common multiple of $a_1, a_2, \ldots, a_r$. Bell [3] has proved that $p_{\mathbf{a}}(n)$ is a quasi-polynomial of degree $r-1$, with the period D, i.e.

$$p_{\mathbf{a}}(n) = d_{\mathbf{a},r-1}(n) n^{r-1} + \cdots + d_{\mathbf{a},1}(n) n + d_{\mathbf{a},0}(n),$$

where $d_{\mathbf{a},m}(n+D) = d_{\mathbf{a},m}(n)$ for $0 \leq m \leq r-1$ and $n \geq 0$, and $d_{\mathbf{a},r-1}(n)$ is not identically zero. In the following, we recall several results from our previous paper [4].

THEOREM 2.1 ([4, Theorem 2.8(1)]). *For $0 \leq m \leq r-1$ and $n \geq 0$ we have*

$$d_{\mathbf{a},m}(n) = \frac{1}{(r-1)!} \sum_{\substack{0 \leq j_1 \leq \frac{D}{a_1}-1, \ldots, 0 \leq j_r \leq \frac{D}{a_r}-1 \\ a_1 j_1 + \cdots + a_r j_r \equiv n (\bmod D)}} \sum_{k=m}^{r-1} \begin{bmatrix} r \\ k+1 \end{bmatrix} (-1)^{k-m} \binom{k}{m} \times$$
$$D^{-k}(a_1 j_1 + \cdots + a_r j_r)^{k-m}.$$

COROLLARY 2.2 ([4, Corollary 2.10]). *We have*

$$p_{\mathbf{a}}(n) = \frac{1}{(r-1)!} \sum_{\substack{0 \leq j_1 \leq \frac{D}{a_1}-1, \ldots, 0 \leq j_r \leq \frac{D}{a_r}-1 \\ a_1 j_1 + \cdots + a_r j_r \equiv n (\bmod D)}} \prod_{\ell=1}^{r-1} \left(\frac{n - a_1 j_1 - \cdots - a_r j_r}{D} + \ell\right).$$

COROLLARY 2.3 ([4, Corollary 2.12]). *For $n \geq 0$ we have $p_{\mathbf{a}}(n) = 0$ if and only if $n < a_1 j_1 + \cdots + a_r j_r$ for all $0 \leq j_1 \leq \frac{D}{a_1} - 1, \ldots, 0 \leq j_r \leq \frac{D}{a_r} - 1$ with $a_1 j_1 + \cdots + a_r j_r \equiv n (\bmod D)$.*

We also recall the following result of Beck, Gessler and Komatsu [1, page 2]. See also [4, Corollary 2.11].

THEOREM 2.4. *The polynomial part of $p_{\mathbf{a}}(n)$ is*

$$P_{\mathbf{a}}(n) := \frac{1}{a_1 \cdots a_r} \sum_{u=0}^{r-1} \frac{(-1)^u}{(r-1-u)!} \sum_{i_1 + \cdots + i_r = u} \frac{B_{i_1} \cdots B_{i_r}}{i_1! \cdots i_r!} a_1^{i_1} \cdots a_r^{i_r} n^{r-1-u}.$$

3. A FORMULA AND A CONGRUENCE FOR $p_{\mathbf{a}}(n)$

Let $\mathbf{a} := (a_1, a_2, \ldots, a_r)$ be a sequence of positive integers, $r \geq 1$. It holds that

$$\sum_{n=0}^{\infty} p_{\mathbf{a}}(n) z^n = \frac{1}{(1 - z^{a_1}) \cdots (1 - z^{a_r})},\ |z| < 1.$$

Let D be a common multiple of $a_1, \ldots, a_r$. Let

$$F_{\mathbf{a}}(z) := \frac{(1 - z^D)^r}{(1 - z^{a_1}) \cdots (1 - z^{a_r})} = \prod_{i=1}^{r} (1 + z^{a_i} + \cdots + z^{a_i(\frac{D}{a_i} - 1)}).$$

Let $d := rD - a_1 - \cdots - a_r$. Since $F_{\mathbf{a}}(z) = z^d F_{\mathbf{a}}(\frac{1}{z})$, it follows that

$$F_{\mathbf{a}}(z) =: f_{\mathbf{a}}(d) z^d + \cdots + f_{\mathbf{a}}(1) z + f_{\mathbf{a}}(0)$$

is a reciprocal polynomial, that is $f_{\mathbf{a}}(d-n) = f_{\mathbf{a}}(n)$ for $0 \leq n \leq d$. Let $f_{\mathbf{a}}(n) := 0$ for $n \geq d+1$. It holds that

$$f_{\mathbf{a}}(n) = \#\{(j_1,\ldots,j_r) :\ a_1j_1 + \cdots + a_rj_r = n,\ 0 \leq j_k \leq \frac{D}{a_k} - 1,\ 1 \leq k \leq r\}.$$

From the power series identity

$$\sum_{n=0}^{\infty} f_{\mathbf{a}}(n)z^n = F_{\mathbf{a}}(z) = (1-z^D)^r \sum_{n=0}^{\infty} p_{\mathbf{a}}(n)z^n = \sum_{n=0}^{\infty} p_{\mathbf{a}}(n) \sum_{j=0}^{r} \binom{r}{j}(-1)^j z^{n+jD}$$

it follows that

$$f_{\mathbf{a}}(n) = \sum_{j=0}^{\lfloor \frac{n}{D} \rfloor} \binom{r}{j}(-1)^j p_{\mathbf{a}}(n-jD),\ n \geq 0. \tag{3.1}$$

Proposition 3.1. *For $n \geq 0$ we have that*

$f_{\mathbf{a}}(n) =$

$$\frac{1}{(r-1)!} \sum_{j=0}^{\lfloor \frac{n}{D} \rfloor} \binom{r}{j}(-1)^j \sum_{\substack{0 \leq j_1 \leq \frac{D}{a_1}-1,\ldots,0 \leq j_r \leq \frac{D}{a_r}-1 \\ a_1j_1+\cdots+a_rj_r \equiv n (\bmod D)}} \prod_{\ell=1}^{r-1} \left(\frac{n - a_1j_1 - \cdots - a_rj_r}{D} + \ell - j\right).$$

Proof. It follows from Corollary 2.2 and (3.1). □

Proposition 3.2 (compare [11, Theorem 1]). *It holds that*

$$p_{\mathbf{a}}(n) = \sum_{j=0}^{\lfloor \frac{n}{D} \rfloor} \binom{r+j-1}{j} f_{\mathbf{a}}(n-jD),\ n \geq 0.$$

Proof. Denote $k := \lfloor \frac{n}{D} \rfloor$. From (3.1) we get the following system of linear equations in the indeterminates $p_{\mathbf{a}}(n-jD)$, $0 \leq j \leq k$

$$\sum_{j=t}^{k} \binom{r}{j-t}(-1)^{j-t} p_{\mathbf{a}}(n-jD) = f_{\mathbf{a}}(n-tD),\ 0 \leq t \leq k. \tag{3.2}$$

It follows that

$$p_{\mathbf{a}}(n) = \sum_{j=0}^{k} (-1)^j \Delta_j f_{\mathbf{a}}(n-jD),$$

where $\Delta_0 = 1$ and $\Delta_j = -\sum_{i=0}^{j-1} \binom{r}{j-i}\Delta_i$. Using induction on $j \geq 0$ it follows that $\Delta_j = (-1)^j \binom{r+j-1}{j}$ for all $0 \leq j \leq k$. Hence

$$p_{\mathbf{a}}(n) = \sum_{j=0}^{k} \binom{r+j-1}{j} f_{\mathbf{a}}(n-jD).$$

□

COROLLARY 3.3. *For $n \geq 0$ it holds that*

$$(r-1)!\, p_{\mathbf{a}}(n) = \sum_{j=\lceil \frac{n+\sigma}{D} \rceil - r}^{\lfloor \frac{n}{D} \rfloor} (j+1)\cdots(j+r-1) f_{\mathbf{a}}(n-jD),$$

where $\sigma = a_1 + \cdots + a_r$.

Proof. For $n - jD > rD - \sigma$ it holds that $f_{\mathbf{a}}(n-jD) = 0$. From Proposition 3.2 it follows that

$$(r-1)!\, p_{\mathbf{a}}(n) = \sum_{j=0}^{\lfloor \frac{n}{D} \rfloor} (r-1)! \binom{r+j-1}{j} f_{\mathbf{a}}(n-jD)$$

$$= \sum_{j=\lceil \frac{n+\sigma}{D} \rceil - r}^{\lfloor \frac{n}{D} \rfloor} (j+1)\cdots(j+r-1) f_{\mathbf{a}}(n-jD).$$

□

COROLLARY 3.4 (compare [11, Theorem 2]). *For $n \geq 0$ it holds that*

$$(r-1)!\, p_{\mathbf{a}}(n) \equiv 0 \mod (j+k+1)(j+k+2)\cdot \ldots \cdot (j+r-1),$$

where $k = \lfloor \frac{n}{D} \rfloor - \lceil \frac{n+\sigma}{D} \rceil + r$, $\sigma = a_1 + \ldots + a_r$.

Proof. For $\lceil \frac{n+\sigma}{D} \rceil - r \leq j \leq \lfloor \frac{n}{D} \rfloor$ it holds that

$$(j+1)\cdots(j+r-1) \equiv 0 \mod (j+k+1)\cdots(j+r-1).$$

Apply now Corollary 3.3. □

4. QUASI-POLYNOMIALS

Let $p : \mathbb{N} \to \mathbb{C}$ be a quasi-polynomial of degree $r - 1 \geq 0$,

$$p(n) := d_{r-1}(n)n^{r-1} + \cdots + d_1(n)n + d_0(n),$$

where $d_m(n)$'s are periodic functions with integral period $D > 0$ and $d_{r-1}(n)$ is not identically zero.

According to [14, Proposition 4.4.1], we have

$$\sum_{n=0}^{\infty} p(n)z^n = \frac{L(z)}{M(z)},$$

where $L(z), M(z) \in \mathbb{C}[z]$, every zero λ of $M(z)$ satisfies $\lambda^D = 1$ (provided $\frac{L(z)}{M(z)}$ has been reduced to lowest terms), and $\deg L(z) < \deg M(z)$. Moreover,

$$p(n) = \sum_{\lambda^D = 1} P_{\lambda}(n)\lambda^{-n},$$

where each $P_\lambda(n)$ is a polynomial function with $\deg P_\lambda(n) \leq m(\lambda) - 1$, where $m(\lambda)$ is the multiplicity of λ as a root of $M(z)$. We define the *polynomial part* of $p(n)$ to be the polynomial function $P(n) := P_1(n)$.

Let $\gamma \in \mathbb{C}$ with $\gamma^D = 1$. It holds that

$$p_\gamma(n) := \gamma^n p(n) = \sum_{\lambda^D=1} P_\lambda(n)(\gamma \cdot \lambda^{-1})^n,$$

hence $P_\gamma(n)$ is the polynomial part of $p_\gamma(n)$.

Proposition 4.1 ([4, Proposition 3.5]). *It holds that*

$$P_\gamma(n) = R_{\gamma,m(\gamma)} n^{r-1} + \cdots + R_{\gamma,2} n + R_{\gamma,1},$$

where $R_{\gamma,m} = \frac{1}{D}\sum_{v=0}^{D-1} \gamma^v d_{m-1}(v)$, $1 \leq m \leq m(\gamma)$.

Consider the decomposition

$$\sum_{n=0}^{\infty} p(n) z^n = \frac{L(z)}{M(z)} = \sum_{M(\lambda)=0} \sum_{\ell=1}^{m(\lambda)} \frac{c_{\lambda,\ell}}{(\lambda - z)^\ell}. \tag{4.1}$$

Let γ be a root of $M(z)$. Since the decomposition (4.1) is unique, it follows that

$$\sum_{n=0}^{\infty} P_\gamma(n)\gamma^{-n} z^n = \sum_{\ell=1}^{m(\gamma)} \frac{c_{\gamma,\ell}}{(\gamma - z)^\ell} = \sum_{\ell=1}^{m(\gamma)} \frac{c_{\gamma,\ell}}{\gamma^\ell}\left(\sum_{n=0}^{\infty} \gamma^{-n} z^n\right)^\ell$$
$$= \sum_{\ell=1}^{m(\gamma)} \frac{c_{\gamma,\ell}}{\gamma^\ell} \sum_{n=0}^{\infty} \binom{n+\ell-1}{\ell-1} \gamma^{-n} z^n.$$

It follows that

$$P_\gamma(n) = \sum_{\ell=1}^{m(\gamma)} \frac{c_{\gamma,\ell}}{\gamma^\ell} \binom{n+\ell-1}{\ell-1}. \tag{4.2}$$

The *Stirling numbers of the second kind*, denoted by $\left\{ n \atop k \right\}$, count the number of ways to partition a set of n labelled objects into k nonempty unlabelled subsets. They are related with the unsigned Stirling numbers of the first kind by

$$\sum_{k=0}^{n} \left\{ n \atop k \right\} \sum_{\ell=0}^{k} (-1)^\ell \left[k \atop \ell \right] = (-1)^n \tag{4.3}$$

Proposition 4.2. *For each* $1 \leq m \leq m(\gamma)$ *it holds that*

$$c_{\gamma,m} = \gamma^m (m-1)! \sum_{\ell=m}^{m(\gamma)} (-1)^{\ell-m} \left\{ \ell \atop m \right\} \frac{1}{D} \sum_{v=0}^{D-1} \gamma^v d_{\ell-1}(v).$$

Proof. From Proposition 4.1 and (4.2) it follows that $R_{\gamma,m} = 0$ for all $m > m(\gamma)$ and

$$R_{\gamma,m} = \sum_{\ell=1}^{m(\gamma)} \frac{c_{\gamma,\ell}}{\gamma^{\ell}(\ell-1)!} \begin{bmatrix} \ell \\ m \end{bmatrix},\ 1 \leq m \leq m(\gamma). \tag{4.4}$$

From (4.3) and (4.4) it follows that

$$c_{\gamma,m} = \gamma^m (m-1)! \sum_{\ell=m}^{m(\gamma)} (-1)^{\ell-m} \left\{ \begin{matrix} \ell \\ m \end{matrix} \right\} R_{\gamma,\ell}, \tag{4.5}$$

hence, the conclusion follows from Proposition 4.1. □

5. SYLVESTER WAVES AND THE PARTIAL FRACTION DECOMPOSITION OF $\sum_{n=0}^{\infty} p_{\mathbf{a}}(n) z^n$

Let $\mathbf{a} = (a_1, \ldots, a_r)$ a sequence of positive integers. We write $p_{\mathbf{a}}(n)$ as a sum of waves

$$p_{\mathbf{a}}(n) = \sum_j W_j(n, \mathbf{a}),$$

where the sum is taken over the $j \geq 1$ with $j | a_i$ for some $1 \leq i \leq r$. We have that

$$W_j(n, \mathbf{a}) = P_{\mathbf{a},\rho_j}(n) \rho_j^{-n}, \tag{5.1}$$

where $\rho_j := e^{\frac{2\pi i}{j}}$ and $P_{\mathbf{a},\rho_j}(n)$ is the polynomial part of the quasi-polynomial $\rho_j^n p_{\mathbf{a}}(n)$.

Proposition 5.1. *We have that*

$$W_j(n, \mathbf{a}) = \rho_j^{-n} (R_{j,m(j)} \cdot n^{m(j)-1} + \cdots + R_{j,2} \cdot n + R_{j,1}),$$

where $m(j) = \#\{i :\ j | a_i\}$ *and* $R_{j,m} = \frac{1}{D} \sum_{v=0}^{D-1} \rho_j^v d_{\mathbf{a},m-1}(v)$ *for* $1 \leq m \leq m(j)$.

Proof. It follows from Proposition 4.1 and (5.1). □

Proposition 5.2. *For any positive integer* j *with* $j | a_i$ *for some* $1 \leq i \leq r$, *we have that:*

$$W_j(n, \mathbf{a}) = \frac{1}{D(r-1)!} \sum_{m=1}^{r} \sum_{\ell=1}^{j} \rho_j^{\ell} \sum_{k=m-1}^{r-1} \begin{bmatrix} r \\ k+1 \end{bmatrix} (-1)^{k-m+1} \binom{k}{m-1} \cdot$$

$$\cdot \sum_{\substack{0 \leq j_1 \leq \frac{D}{a_1}-1, \ldots, 0 \leq j_r \leq \frac{D}{a_r}-1 \\ a_1 j_1 + \cdots + a_r j_r \equiv \ell (\bmod j)}} D^{-k} (a_1 j_1 + \cdots + a_r j_r)^{k-m+1} n^{m-1}.$$

Proof. It follows from Proposition 5.1 and Theorem 2.1. □

PROPOSITION 5.3. *If $a_1,\ldots,a_r$ are pairwise coprimes then*

$$p_{\mathbf{a}}(n)=\sum_{m=1}^{r}\frac{(-1)^{r-m}}{(a_1\cdots a_r)(m-1)!}\sum_{i_1+\cdots+i_r=r-m}\frac{B_{i_1}\cdots B_{i_r}}{i_1!\cdots i_r!}a_1^{i_1}\cdots a_r^{i_r}n^{m-1}+\frac{1}{D(r-1)!}\cdot$$

$$\cdot\sum_{j\neq 1}\sum_{\ell=1}^{j}\rho_j^{\ell}\sum_{k=0}^{r-1}\frac{1}{D^k}\begin{bmatrix} r \\ k+1\end{bmatrix}(-1)^k\sum_{\substack{0\leq j_1\leq\frac{D}{a_1}-1,\ldots,0\leq j_r\leq\frac{D}{a_r}-1\\ a_1j_1+\cdots+a_rj_r\equiv\ell(\bmod j)}}D^{-k}(a_1j_1+\cdots+a_rj_r)^k,$$

where $j|a_i$ for some $1\leq i\leq r$.

Proof. Since $a_1,\ldots,a_r$ are pairwise coprimes, it follows that $W_j(\mathbf{a},n)$ is a quasi-polynomial of degree 0. Hence, the conclusion follows from Proposition 4.1 and Proposition 4.2. □

Another formulas for $p_{\mathbf{a}}(n)$ in the case that $a_1,\ldots,a_r$ are pairwise coprimes were proved in [5, Theorem C, pag 113], [2, Theorem 3.1] and [8]. We consider the decomposition

$$\sum_{n=0}^{\infty}p_{\mathbf{a}}(n)z^n=\frac{1}{(1-z^{a_1})\cdots(1-z^{a_r})}=\sum_{\lambda}\sum_{\ell=1}^{m(\lambda)}\frac{c_{\lambda,\ell}}{(\lambda-z)^{\ell}},$$

where the sum it taken over the λ's with $\lambda^{a_i}=1$ for some $1\leq i\leq r$.

PROPOSITION 5.4. *Let $j\geq 1$. For $1\leq m\leq m(j)$ we have that*

$$c_{\rho_j,m}=\frac{\rho_j^m(m-1)!}{D}\sum_{t=m}^{m(j)}(-1)^{t-m}\begin{Bmatrix} t \\ m\end{Bmatrix}\sum_{\ell=1}^{j}\rho_j^{\ell}\sum_{k=m-1}^{r-1}\begin{bmatrix} r \\ k+1\end{bmatrix}(-1)^{k-m+1}\binom{k}{m-1}\cdot$$

$$\cdot\sum_{\substack{0\leq j_1\leq\frac{D}{a_1}-1,\ldots,0\leq j_r\leq\frac{D}{a_r}-1\\ a_1j_1+\cdots+a_rj_r\equiv\ell(\bmod j)}}D^{-k}(a_1j_1+\cdots+a_rj_r)^{k-m+1}.$$

Proof. It follows from Propositions 4.2, 5.1, and 5.2. □

PROPOSITION 5.5. *For $1\leq m\leq r$ it holds that*

$$c_m=\frac{(m-1)!}{a_1\cdots a_r}\sum_{\ell=m}^{r}(-1)^{\ell-m}\frac{\left\{{\ell \atop m}\right\}}{(\ell-1)!}\sum_{i_1+\cdots+i_r=r-\ell}\frac{B_{i_1}\cdots B_{i_r}}{i_1!\cdots i_r!}a_1^{i_1}\cdots a_r^{i_r},$$

where $c_m:=c_{1,m}$.

Proof. It follows from Theorem 2.4, Proposition 4.1 and Proposition 5.4. □

Let $r \geq 1$, $\mathbf{r} := (1, 2, \dots, r)$. *Rademacher's coefficients* $c_{hk\ell}(r)$ are defined by

$$\sum_{n=0}^{\infty} p_{\mathbf{r}}(n) z^n = \frac{1}{(1-z)(1-z^2)\cdots(1-z^r)} = \sum_{0 \leq h < k \leq r,\ (h,k)=1} \sum_{\ell=1}^{\lfloor \frac{r}{k} \rfloor} \frac{c_{hk\ell}(r)}{(z-\omega_{hk})^{\ell}},$$

where $\omega_{hk} := e^{2\pi i \frac{h}{k}}$. In the previous notations $c_{hk\ell}(r) = (-1)^{\ell} c_{\omega_{hk},\ell}$. As a direct consequence of Proposition 5.5 we get the following result of C. O'Sullivan [9]:

Corollary 5.6 ([9, Proposition 2.3]). *For $1 \leq m \leq r$ it holds that*

$$c_{01m}(r) = \frac{(-1)^r (m-1)!}{r!} \sum_{\ell=m}^{r} \frac{\left\{ {\ell \atop m} \right\}}{(\ell-1)!} \sum_{i_1+\cdots+i_r=r-\ell} \frac{B_{i_1}\cdots B_{i_r}}{i_1!\cdots i_r!} 1^{i_1} 2^{i_2} \cdots r^{i_r}.$$

6. FROBENIUS NUMBER

Given a sequence of positive integers $\mathbf{a} = (a_1, \dots, a_r)$ that satisfy $gcd(a_1, ..., a_r) = 1$, the *Frobenius number* of $\mathbf{a}$, denoted by $F(\mathbf{a}) = F(a_1, ..., a_r)$, is the largest integer n with the property that $p_{\mathbf{a}}(n) = 0$.

Proposition 6.1. *Let $\mathbf{a} = (a_1, \dots, a_r)$ with $gcd(a_1, \dots, a_r) = 1$ and $D = \operatorname{lcm}(a_1, \dots, a_r)$. We have that*

$$F(a_1, \dots, a_r) \leq D(r-1) - a_1 - \cdots - a_r.$$

Proof. Let n be an integer with $p_{\mathbf{a}}(n) = 0$. Since the map

$$\varphi : \mathbb{Z}/a_1\mathbb{Z} \times \cdots \times \mathbb{Z}/a_r\mathbb{Z} \to \mathbb{Z}/D\mathbb{Z},\ \varphi(\hat{j}_1, \dots, \hat{j}_r) := \overline{a_1 j_1 + \cdots + a_r j_r}$$

is a surjective morphism, it follows that there exists some integers $0 \leq j_1 \leq \frac{D}{a_1} - 1, \dots, 0 \leq j_r \leq \frac{D}{a_r} - 1$ such that $a_1 j_1 + \cdots + a_r j_r \equiv n (\bmod D)$.

From Corollary 2.3 it follows that $n < a_1 j_1 + \cdot + a_r j_r$, hence
$n \leq a_1 j_1 + \cdots + a_r j_r - D \leq (D-1)r - a_1 - \cdots - a_r$. $\square$

The following corollary is a particular case of [10, Theorem 2.7] and appears also in [17, Theorem 1(a)].

Corollary 6.2. *Let $\mathbf{a} = (a_1, \dots, a_r)$ such that $\gcd(a_i, a_j) = 1$ for all $i \neq j$, $D = \operatorname{lcm}(a_1, \dots, a_r) = a_1 \cdots a_r$, $A_i := \frac{D}{a_i}$, $1 \leq i \leq r$. It holds that*

$$F(A_1, \dots, A_r) = D(r-1) - A_1 - \cdots - A_r.$$

Proof. It holds that $D = \operatorname{lcm}(A_1, \dots, A_r)$. From Lemma 6.1 it follows that

$$F(A_1, \dots, A_r) \leq D(r-1) - A_1 - \cdots - A_r.$$

Suppose that $D(r-1) - A_1 - \cdots - A_r = A_1 j_1 + \cdots + A_r j_r$ with $j_k \geq 0$ for $1 \leq k \leq r$, hence

$$D(r-1) = A_1(j_1+1) + \cdots + A_r(j_r+1),\ r-1 = \frac{j_1+1}{a_1} + \cdots + \frac{j_r+1}{a_r}.$$

Since $\gcd(a_i, a_j) = 1$ it follows that $a_k|(j_k+1)$ for all $1 \leq k \leq r$. Since $j_k \geq 0$ we get

$$r-1 = \frac{j_1+1}{a_1} + \cdots + \frac{j_r+1}{a_r} \geq r,$$

a contradiction. So $F(A_1, \ldots, A_r) \geq D(r-1) - A_1 - \cdots - A_r$. $\square$

Acknowledgments. The author express his gratitude to Florin Nicolae for discussions and valuable suggestions which helped to improve this paper.

REFERENCES

[1] M. Beck, I. M. Gessel, and T. Komatsu, *The polynomial part of a restricted partition function related to the Frobenius problem.* Electronic Journal of Combinatorics **8** (2001), *1*, N 7.

[2] A. Bayad and M. Beck, *Relations for Bernoulli-Barnes Numbers and Barnes Zeta Functions.* International Journal of Number Theory **10** (2014), 1321–1335.

[3] E. T. Bell, *Interpolated denumerants and Lambert series.* Am. J. Math. **65** (1943), 382–386.

[4] M. Cimpoeaş and F. Nicolae, *On the restricted partition function.* Ramanujan J. **47** (2018), *3*, 565–588.

[5] L. Comtet, *Advanced combinatorics: The art of finite and infinite expansions.* D. Reidel Publishing Co., Dordrecht. Holland; Boston, Mass., 1974.

[6] K. Dilcher and C. Vignat, *An explicit form of the polynomial part of a restricted partition function.* Res. Number Theory **3** (2017), Paper No. 1.

[7] J. W. L. Glaisher, *Formulae for partitions into given elements, derived from Sylvesters theorem.* Quart. J. Pure Appl. Math. **40** (1908), 275–348.

[8] T. Komatsu, *On the number of solutions of the Diophantine equation of Frobenius: general case.* Math. Commun. **8** (2003), *2*, 195–206.

[9] C. O'Sullivan, *On the partial fraction decomposition of the restricted partition generating function.* Forum Mathematicum, **27** (2015), *2*, 735–766.

[10] M. Raczunas and P. Chrząstowski-Wachtel, *A diophantine problem of Frobenius in terms of the least common multiple.* Discrete Math. **150** (1996), 347–357.

[11] Ø. J. Rødseth and J. A. Sellers, *Partitions with parts in a finite set.* International Journal of Number Theory **02** (2006), *03*, 455–468.

[12] B. Y. Rubinstein, *Expression for restricted partition function through Bernoulli polynomials.* Ramanujan J. **15** (2008), *2*, 177–185.

[13] B. Y. Rubinstein and L. G. Fel, *Restricted partition functions as Bernoulli and Eulerian polynomials of higher order.* Ramanujan J. **11** (2006), *3*, 331–347.

[14] R. P. Stanley, *Enumerative Combinatorics*, Vol. 1. Wadsworth and Brooks/Cole, Monterey, California, 1986.

[15] J. J. Sylvester, *On the partition of numbers*. Quart. J. Pure Appl. Math. **1** (1857), 141–152.

[16] J. J. Sylvester, *On subinvariants, i.e. semi-invariants to binary quatities of an unlimited order with an excursus on rational fractions and partitions*. Am. J. Math. **5** (1882), 79–136.

[17] A. Tripathi, *On a linear diophantine problem of Frobenius*. Electronic Journal of Combinatorial Number Theory **6** (2006).

Received January 24, 2018

University "Politehnica" of Bucharest
Faculty of Applied Sciences
060042 Bucharest, Romania

and

"Simion Stoilow" Institute of Mathematics
of the Romanian Academy
Research unit 5,
P.O.Box 1-764, Bucharest 014700, Romania
mircea.cimpoeas@upb.ro, mircea.cimpoeas@imar.ro