REMARKS ON THE RESTRICTED PARTITION FUNCTION

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Let \( \mathbf{a} = (a_1, \ldots, a_r) \) be a vector of positive integers. In continuation of a previous paper we present other formulas for the restricted partition function \( p_{\mathbf{a}}(n) := \) the number of integer solutions \( (x_1, \ldots, x_r) \) to \( \sum_{j=1}^{r} a_j x_j = n \) with \( x_1 \geq 0, \ldots, x_r \geq 0. \)

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1. INTRODUCTION

Let \( \mathbf{a} := (a_1, a_2, \ldots, a_r) \) be a sequence of positive integers, \( r \geq 1. \) The restricted partition function associated to \( \mathbf{a} \) is \( p_{\mathbf{a}} : \mathbb{N} \to \mathbb{N}, p_{\mathbf{a}}(n) := \) the number of integer solutions \( (x_1, \ldots, x_r) \) of \( \sum_{i=1}^{r} a_i x_i = n \) with \( x_i \geq 0. \) Let \( D \) be a common multiple of \( a_1, \ldots, a_r. \)

Sylvester \[15],[16\] decomposed the restricted partition in a sum of “waves”:

\[
p_{\mathbf{a}}(n) = \sum_{j \geq 1} W_j(n, \mathbf{a}),
\]

where the sum is taken over all distinct divisors \( j \) of the components of \( \mathbf{a} \) and showed that for each such \( j, \) \( W_j(n, \mathbf{a}) \) is the coefficient of \( t^{-1} \) in

\[
\sum_{0 \leq \nu < j, \gcd(\nu, j) = 1} \rho_j^{-\nu n} e^{nt}
\]

\[
x^{\nu a_1} e^{-a_1 t} \cdots (1 - \rho_j^{a_r} e^{-a_r t}),
\]

where \( \rho_j = e^{2\pi i/j} \) and \( \gcd(0, 0) = 1 \) by convention.

Note that \( W_j(n, \mathbf{a}) \)'s are quasi-polynomials of period \( j. \) (A quasi-polynomial of period \( j \) is a numerical function \( f(n) \) such that there exists \( j \) polynomials \( P_1(n), P_2(n), \ldots, P_j(n) \) such that \( f(n) = P_i(n) \) if \( n \equiv i (\mod j).) \) The first wave \( P_{\mathbf{a}}(n) := W_1(\mathbf{a}, n) \) is called the polynomial part of \( p_{\mathbf{a}}(n). \)

Glaisher \[7\] made computations of the Sylvester waves in particular cases. Fel and Rubinstein \[13\] proved formulas for the Sylvester waves using Bernoulli

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and Euler polynomials of higher order. Rubinstein [12] showed that all Sylvester waves can be expressed in terms of Bernoulli polynomials only. Bayad and Beck [2, Theorem 3.1] proved an explicit expression of the partition function \( p_a(n) \) in terms of Bernoulli-Barnes polynomials and the Fourier Dedekind sums, in the case that \( a_1, \ldots, a_r \) are pairwise coprime. Beck, Gessler and Komatsu [1, page 2], Dilcher and Vignat [6, Theorem 1.1] proved explicit formulas for the polynomial part of \( p_a(n) \).

As a continuation of [4] we present here other formulas for \( p_a(n) \) and for the Sylvester waves. Also, we reprove, using our method, several results. In Proposition 3.2 we prove that

\[
p_a(n) = \sum_{j=0}^{\left\lfloor \frac{n}{D} \right\rfloor} \binom{r+j-1}{j} f_a(n-jD),
\]

where \( f_a(n) = \#\{(j_1, \ldots, j_r) : a_1 j_1 + \cdots + a_r j_r = n, \ 0 \leq j_k \leq \frac{D}{a_k} - 1, \ 1 \leq k \leq r\} \). This result is similar to Theorem 1 of Rodseth and Seller [11].

In Corollary 3.4 (compare [11, Theorem 2]) we prove the congruence

\[(r-1)! p_a(n) \equiv 0 \mod (j+k+1)(j+k+2) \cdots (j+r-1),\]

where

\[k = \left\lfloor \frac{n}{D} \right\rfloor - \left\lfloor \frac{n+a_1+\cdots+a_r}{D} \right\rfloor + r.\]

In Proposition 5.2 we prove that

\[
W_j(n, a) = \frac{1}{D(r-1)!} \sum_{m=1}^{r} \sum_{\ell=1}^{j} \rho_j^\ell \sum_{k=m-1}^{r-1} \left[ \begin{array}{c} r \\ k+1 \end{array} \right] (-1)^{k-m+1} \binom{k}{m-1} D^{-k}(a_1 j_1 + \cdots + a_r j_r)^{k-m+1} n^{m-1},
\]

where \([r/k]\) are the unsigned Stirling numbers of the first kind.

The Bernoulli numbers are defined by the identity

\[
\frac{t}{e^t - 1} = \sum_{\ell=0}^{\infty} \frac{t^\ell}{\ell!} B_\ell.
\]

If \( \gcd(a_i, a_j) = 1 \) for all \( i \neq j \) we prove in Proposition 5.3 that

\[
p_a(n) = \sum_{m=1}^{r} \frac{(-1)^{r-m}}{(a_1 \cdots a_r)(m-1)!} \sum_{i_1+\cdots+i_r=r-m} B_{i_1} \cdots B_{i_r} a_1^{i_1} \cdots a_r^{i_r} n^{m-1} + \frac{1}{D(r-1)!} \times \sum_{j \neq 1} \sum_{\ell=1}^{j} \rho_j^\ell \sum_{k=0}^{r-1} \frac{1}{D^k} \left[ \begin{array}{c} r \\ k+1 \end{array} \right] (-1)^k \sum_{0 \leq j_1 \leq \frac{D}{a_1} - 1, \ldots, 0 \leq j_r \leq \frac{D}{a_r} - 1} D^{-k}(a_1 j_1 + \cdots + a_r j_r)^k,
\]

where

\[0 \leq j_1 \leq \frac{D}{a_1} - 1, \ldots, 0 \leq j_r \leq \frac{D}{a_r} - 1, a_1 j_1 + \cdots + a_r j_r \equiv \ell \mod j\]
where \( j \mid a_i \) for some \( 1 \leq i \leq r \). Another formulas for \( p_a(n) \) in the case that \( a_1, \ldots, a_r \) are pairwise coprimes were proved in [5, Theorem C, pag 113], [2; Theorem 3.1] and [8].

Let \[
\sum_{n=0}^{\infty} p_a(n)z^n = \frac{1}{(1-z^{a_1})\cdots(1-z^{a_r})} = \sum_{\lambda^\ell=1}^{m(\lambda)} \sum_{t=1}^\infty \frac{c_{\lambda,\ell}}{(\lambda-z)^\ell},
\]
where \( m(\lambda) \) is the multiplicity of \( \lambda \) as a root of \( (1-z^{a_1})\cdots(1-z^{a_r}) \).

In Proposition 5.4 we prove that

\[
c_{p_j,m} = \frac{\rho_j^n(m-1)!}{D} \sum_{t=m}^{m(p_j)} (-1)^{t-m} \binom{t}{m} \rho_j^\ell \sum_{\lambda^\ell=1}^{r-1} \left[ \frac{r}{k+1} \right] \times
\]

\[
(-1)^{k-m+1} \binom{k}{m-1} \sum_{0 \leq j_1 \leq \frac{D}{a_1}-1, \ldots, 0 \leq j_r \leq \frac{D}{a_r}-1} \frac{D^{-k}(a_1j_1 + \cdots + a_rj_r)^{k-m+1}},
\]

where \( \binom{t}{m} \) are Stirling numbers of the second kind.

In Proposition 5.5 we prove that

\[
c_{1,m} = \frac{(m-1)!}{a_1 \cdots a_r} \sum_{\ell=m}^{r} (-1)^{\ell-m} \binom{\ell}{m-1} \sum_{i_1 + \cdots + i_r = r-\ell} \frac{B_{i_1} \cdots B_{i_r} a_1^{i_1} \cdots a_r^{i_r}}{i_1! \cdots i_r!}.
\]

In the case \( a = (1,2,\ldots,r) \) we reprove O’Sullivan’s formulas [9] for Rademacher’s coefficients \( c_{01m} \), see Corollary 5.6.

Given a sequence of positive integers \( a = (a_1,\ldots,a_r) \) with \( \gcd(a_1,\ldots,a_r) = 1 \), the Frobenius number of \( a \), denoted by \( F(a) = F(a_1,\ldots,a_r) \) is the largest integer \( n \) with the property that \( p_a(n) = 0 \). If \( \gcd(a_i,a_j) = 1 \) for all \( i \neq j \), we prove in Corollary 6.2 that \( F(A_1,\ldots,A_r) = D(r-1) - A_1 - \cdots - A_r \), where \( A_1 := \frac{D}{a_1}, \ldots, A_r := \frac{D}{a_r} \). This is a particular case of [10, Theorem 2.7] and appears also in [17, Theorem 1(a)].

2. PRELIMINARIES

Let \( a := (a_1,a_2,\ldots,a_r) \) be a sequence of positive integers, \( r \geq 1 \). The restricted partition function associated to \( a \) is \( p_a : \mathbb{N} \to \mathbb{N} \), \( p_a(n) := \) the number of integer solutions \((x_1,\ldots,x_r)\) of \( \sum_{i=1}^{r} a_ix_i = n \) with \( x_i \geq 0 \).

Let \( D \) be a common multiple of \( a_1,a_2,\ldots,a_r \). Bell [3] has proved that \( p_a(n) \) is a quasi-polynomial of degree \( r - 1 \), with the period \( D \), i.e.

\[
p_a(n) = d_{a,r-1}(n)n^{r-1} + \cdots + d_{a,1}(n)n + d_{a,0}(n),
\]
where $d_{a,m}(n+D) = d_{a,m}(n)$ for $0 \leq m \leq r-1$ and $n \geq 0$, and $d_{a,r-1}(n)$ is not identically zero. In the following, we recall several results from our previous paper [4].

**Theorem 2.1 ([4, Theorem 2.8(1)])**. For $0 \leq m \leq r-1$ and $n \geq 0$ we have

$$d_{a,m}(n) = \frac{1}{(r-1)!} \sum_{0 \leq j_1 \leq \frac{D}{a_1} - 1, \ldots, 0 \leq j_r \leq \frac{D}{a_r} - 1} \sum_{a_1j_1 + \cdots + a_rj_r \equiv n \pmod{D}} r \sum_{k=m}^{r-1} \binom{r}{k+1} (-1)^{k-m} \binom{k}{m} \times D^{-k}(a_1j_1 + \cdots + a_rj_r)^{k-m}.$$

**Corollary 2.2 ([4, Corollary 2.10])**. We have

$$p_a(n) = \frac{1}{(r-1)!} \sum_{0 \leq j_1 \leq \frac{D}{a_1} - 1, \ldots, 0 \leq j_r \leq \frac{D}{a_r} - 1} \prod_{\ell=1}^{r-1} \left( \frac{n-a_1j_1 - \cdots - a_rj_r}{D} + \ell \right).$$

**Corollary 2.3 ([4, Corollary 2.12])**. For $n \geq 0$ we have $p_a(n) = 0$ if and only if $n < a_1j_1 + \cdots + a_rj_r$ for all $0 \leq j_1 \leq \frac{D}{a_1} - 1, \ldots, 0 \leq j_r \leq \frac{D}{a_r} - 1$ with $a_1j_1 + \cdots + a_rj_r \equiv n \pmod{D}$.

We also recall the following result of Beck, Gessler and Komatsu [1, page 2]. See also [4, Corollary 2.11].

**Theorem 2.4**. The polynomial part of $p_a(n)$ is

$$P_a(n) := \frac{1}{a_1 \cdots a_r} \sum_{u=0}^{r-1} \frac{(-1)^u}{(r-1-u)!} \sum_{i_1+\cdots+i_r=u} B_{i_1} \cdots B_{i_r} a_1^{i_1} \cdots a_r^{i_r} n^{r-1-u}.$$

### 3. A Formula and A Congruence For $p_a(n)$

Let $a := (a_1, a_2, \ldots, a_r)$ be a sequence of positive integers, $r \geq 1$. It holds that

$$\sum_{n=0}^{\infty} p_a(n) z^n = \frac{1}{(1 - z^{a_1}) \cdots (1 - z^{a_r})}, \quad |z| < 1.$$ 

Let $D$ be a common multiple of $a_1, \ldots, a_r$. Let

$$F_a(z) := \frac{(1 - z^D)^r}{(1 - z^{a_1}) \cdots (1 - z^{a_r})} = \prod_{i=1}^{r} (1 + z^{a_i} + \cdots + z^{a_i(\frac{D}{a_i} - 1)}) \cdot \frac{(1-z^D)^r}{(1 - z^{a_1}) \cdots (1 - z^{a_r})}.$$

Let $d := rD - a_1 - \cdots - a_r$. Since $F_a(z) = z^d F_a(\frac{1}{z})$, it follows that

$$F_a(z) = f_a(d) z^d + \cdots + f_a(1) z + f_a(0).$$
is a reciprocal polynomial, that is \( f_a(d - n) = f_a(n) \) for \( 0 \leq n \leq d \). Let \( f_a(n) := 0 \) for \( n \geq d + 1 \). It holds that

\[
    f_a(n) = \# \{(j_1, \ldots, j_r) : a_1 j_1 + \cdots + a_r j_r = n, \ 0 \leq j_k \leq \frac{D}{a_k} - 1, \ 1 \leq k \leq r \}. \]

From the power series identity

\[
    \sum_{n=0}^{\infty} f_a(n) z^n = F_a(z) = (1 - z^D)^r \sum_{n=0}^{\infty} p_a(n) z^n = \sum_{n=0}^{\infty} p_a(n) \sum_{j=0}^{\infty} \binom{r}{j} (-1)^j z^{n+jD}
\]

it follows that

\[
    f_a(n) = \binom{n}{D} \sum_{j=0}^{\lfloor \frac{n}{D} \rfloor} \binom{r}{j} (-1)^j p_a(n - jD), \ n \geq 0.
\]

**Proposition 3.1.** For \( n \geq 0 \) we have that

\[
    f_a(n) = \frac{1}{(r-1)!} \sum_{j=0}^{\lfloor \frac{n}{D} \rfloor} \binom{r}{j} (-1)^j \sum_{0 \leq j_1 \leq \frac{D}{a_1} - 1, \ldots, 0 \leq j_r \leq \frac{D}{a_r} - 1, a_1 j_1 + \cdots + a_r j_r \equiv n \pmod{D}}^{r-1} \prod_{\ell=1}^{r-1} \left( \frac{n - a_1 j_1 - \cdots - a_r j_r}{D} + \ell - j \right).
\]

**Proof.** It follows from Corollary 2.2 and (3.1). \( \square \)

**Proposition 3.2 (compare [11, Theorem 1]).** It holds that

\[
    p_a(n) = \sum_{j=0}^{\lfloor \frac{n}{D} \rfloor} \binom{r+j-1}{j} f_a(n - jD), \ n \geq 0.
\]

**Proof.** Denote \( k := \lfloor \frac{n}{D} \rfloor \). From (3.1) we get the following system of linear equations in the indeterminates \( p_a(n - jD), \ 0 \leq j \leq k \)

\[
    \sum_{j=t}^{k} \binom{r}{j-t} (-1)^{j-t} p_a(n - jD) = f_a(n - tD), \ 0 \leq t \leq k.
\]

It follows that

\[
    p_a(n) = \sum_{j=0}^{k} (-1)^j \Delta_j f_a(n - jD),
\]

where \( \Delta_0 = 1 \) and \( \Delta_j = - \sum_{i=0}^{j-1} \binom{r}{j-i} \Delta_i \). Using induction on \( j \geq 0 \) it follows that \( \Delta_j = (-1)^j \binom{r+j-1}{j} \) for all \( 0 \leq j \leq k \). Hence

\[
    p_a(n) = \sum_{j=0}^{k} \binom{r+j-1}{j} f_a(n - jD).
\]

\( \square \)
Corollary 3.3. For \( n \geq 0 \) it holds that
\[
(r - 1)! p_a(n) = \sum_{j=\left[\frac{n}{D}\right] - r}^{\left[\frac{n+\sigma}{D}\right]} (j + 1) \cdots (j + r - 1) f_a(n - jD),
\]
where \( \sigma = a_1 + \cdots + a_r \).

Proof. For \( n - jD > rD - \sigma \) it holds that \( f_a(n - jD) = 0 \). From Proposition 3.2 it follows that
\[
(r - 1)! p_a(n) = \sum_{j=0}^{\left[\frac{n}{D}\right]} (r - 1)! \binom{r + j - 1}{j} f_a(n - jD)
\]
\[
= \sum_{j=\left[\frac{n+\sigma}{D}\right] - r}^{\left[\frac{n+\sigma}{D}\right]} (j + 1) \cdots (j + r - 1) f_a(n - jD).
\]

Corollary 3.4 (compare [11, Theorem 2]). For \( n \geq 0 \) it holds that
\[
(r - 1)! p_a(n) \equiv 0 \mod (j + k + 1)(j + k + 2) \cdots \cdot (j + r - 1),
\]
where \( k = \left[\frac{n}{D}\right] - \left[\frac{n+\sigma}{D}\right] + r, \sigma = a_1 + \cdots + a_r \).

Proof. For \( \left[\frac{n+\sigma}{D}\right] - r \leq j \leq \left[\frac{n}{D}\right] \) it holds that
\[
(j + 1) \cdots (j + r - 1) \equiv 0 \mod (j + k + 1) \cdots (j + r - 1).
\]
Apply now Corollary 3.3. \( \Box \)

4. QUASI-POLYNOMIALS

Let \( p : \mathbb{N} \to \mathbb{C} \) be a quasi-polynomial of degree \( r - 1 \geq 0 \),
\[
p(n) := d_{r-1}(n)n^{r-1} + \cdots + d_1(n)n + d_0(n),
\]
where \( d_m(n) \)'s are periodic functions with integral period \( D > 0 \) and \( d_{r-1}(n) \) is not identically zero.

According to [14, Proposition 4.4.1], we have
\[
\sum_{n=0}^{\infty} p(n)z^n = \frac{L(z)}{M(z)},
\]
where \( L(z), M(z) \in \mathbb{C}[z] \), every zero \( \lambda \) of \( M(z) \) satisfies \( \lambda^D = 1 \) (provided \( \frac{L(z)}{M(z)} \) has been reduced to lowest terms), and \( \deg L(z) < \deg M(z) \). Moreover,
\[
p(n) = \sum_{\lambda^D=1} P_\lambda(n)\lambda^{-n},
\]
where each $P_\lambda(n)$ is a polynomial function with $\deg P_\lambda(n) \leq m(\lambda) - 1$, where $m(\lambda)$ is the multiplicity of $\lambda$ as a root of $M(z)$. We define the polynomial part of $p(n)$ to be the polynomial function $P(n) := P_1(n)$.

Let $\gamma \in \mathbb{C}$ with $\gamma^D = 1$. It holds that

$$p_\gamma(n) := \gamma^n p(n) = \sum_{\lambda^D = 1} P_\lambda(n)(\gamma \cdot \lambda^{-1})^n,$$

hence $P_\gamma(n)$ is the polynomial part of $p_\gamma(n)$.

**Proposition 4.1** ([4, Proposition 3.5]). It holds that

$$P_\gamma(n) = R_{\gamma,m(\gamma)} n^{r-1} + \cdots + R_{\gamma,2} n + R_{\gamma,1},$$

where $R_{\gamma,m} = \frac{1}{D} \sum_{v=0}^{D-1} \gamma^v d_{m-1}(v)$, $1 \leq m \leq m(\gamma)$.

Consider the decomposition

$$\sum_{n=0}^{\infty} p(n) z^n = \frac{L(z)}{M(z)} = \sum_{M(\lambda) = 0} \sum_{\ell = 1}^{m(\lambda)} \frac{c_{\gamma,\ell}}{(\lambda - z)^\ell}.$$ 

Let $\gamma$ be a root of $M(z)$. Since the decomposition (4.1) is unique, it follows that

$$\sum_{n=0}^{\infty} P_\gamma(n) \gamma^{-n} z^n = \sum_{\ell = 1}^{m(\gamma)} \frac{c_{\gamma,\ell}}{\gamma^\ell} \left( \sum_{n=0}^{\infty} \gamma^{-n} z^n \right)^\ell = \sum_{\ell = 1}^{m(\gamma)} \frac{c_{\gamma,\ell}}{\gamma^\ell} \sum_{n=0}^{\infty} \left( n + \ell - 1 \choose \ell - 1 \right) \gamma^{-n} z^n.$$

It follows that

$$P_\gamma(n) = \sum_{\ell = 1}^{m(\gamma)} \frac{c_{\gamma,\ell}}{\gamma^\ell} \left( n + \ell - 1 \choose \ell - 1 \right).$$

The Stirling numbers of the second kind, denoted by $\{n \atop k\}$, count the number of ways to partition a set of $n$ labelled objects into $k$ nonempty unlabelled subsets. They are related with the unsigned Stirling numbers of the first kind by

$$\sum_{k=0}^{n} \left\{ n \atop k \right\} \sum_{\ell=0}^{k} (-1)^\ell \left[ k \atop \ell \right] = (-1)^n \quad (4.3)$$

**Proposition 4.2**. For each $1 \leq m \leq m(\gamma)$ it holds that

$$c_{\gamma,m} = \gamma^m (m - 1)! \sum_{\ell = m}^{m(\gamma)} (-1)^{m-\ell} \left\{ \ell \atop m \right\} \frac{1}{D} \sum_{v=0}^{D-1} \gamma^v d_{\ell-1}(v).$$
Proof. From Proposition 4.1 and (4.2) it follows that \( R_{\gamma,m} = 0 \) for all \( m > m(\gamma) \) and

\[
R_{\gamma,m} = \sum_{\ell=1}^{m(\gamma)} \frac{c_{\gamma,\ell}}{\gamma^\ell (\ell - 1)!} \left[ \ell \atop m \right], \quad 1 \leq m \leq m(\gamma).
\]

From (4.3) and (4.4) it follows that

\[
c_{\gamma,m} = \gamma^m (m - 1)! \sum_{\ell=m}^{m(\gamma)} (-1)^{\ell-m} \left\{ \ell \atop m \right\} R_{\gamma,\ell},
\]

hence, the conclusion follows from Proposition 4.1. \( \square \)

5. SYLVESTER WAVES AND THE PARTIAL FRACTION DECOMPOSITION OF \( \sum_{n=0}^{\infty} p_\mathbf{a}(n)z^n \)

Let \( \mathbf{a} = (a_1, \ldots, a_r) \) a sequence of positive integers. We write \( p_\mathbf{a}(n) \) as a sum of waves

\[ p_\mathbf{a}(n) = \sum_j W_j(n, \mathbf{a}), \]

where the sum is taken over the \( j \geq 1 \) with \( j|a_i \) for some \( 1 \leq i \leq r \). We have that

\[
W_j(n, \mathbf{a}) = P_{\mathbf{a},\rho_j}(n)\rho_j^{-n},
\]

where \( \rho_j := e^{\frac{2\pi i}{j}} \) and \( P_{\mathbf{a},\rho_j}(n) \) is the polynomial part of the quasi-polynomial \( \rho_j^n p_\mathbf{a}(n) \).

**PROPOSITION 5.1.** We have that

\[
W_j(n, \mathbf{a}) = \rho_j^{-n}(R_{j,m(j)} \cdot n^{m(j)-1} + \cdots + R_{j,2} \cdot n + R_{j,1}),
\]

where \( m(j) = \# \{ i : j|a_i \} \) and \( R_{j,m} = \frac{1}{D} \sum_{v=0}^{D-1} \rho_j^v d_{a,m-1}(v) \) for \( 1 \leq m \leq m(j) \).

**Proof.** It follows from Proposition 4.1 and (5.1). \( \square \)

**PROPOSITION 5.2.** For any positive integer \( j \) with \( j|a_i \) for some \( 1 \leq i \leq r \), we have that:

\[
W_j(n, \mathbf{a}) = \frac{1}{D(r-1)!} \sum_{m=1}^{r} \sum_{\ell=1}^{j} \rho_j^\ell \sum_{k=m-1}^{r-1} \left[ \binom{r}{k+1} \right] (-1)^{k-m+1} \left( \binom{k}{m-1} \right) \times \\
\sum_{0 \leq j_1 \leq \frac{D}{a_1} - 1, \ldots, 0 \leq j_r \leq \frac{D}{a_r} - 1} D^{-k}(a_1j_1 + \cdots + a_rj_r)^{k-m+1}n^{m-1}.
\]
Proof. It follows from Proposition 5.1 and Theorem 2.1. \qed

**Proposition 5.3.** If $a_1, \ldots, a_r$ are pairwise coprimes then

$$p_a(n) = \sum_{m=1}^{r} (-1)^{r-m} \frac{(a_1 \cdot a_r)(m-1)!}{(a_1 \cdot a_r)(m-1)!} \sum_{i_1+\cdots+i_r = r-m} \frac{B_{i_1} \cdots B_{i_r}}{i_1! \cdots i_r!} a_1^{i_1} \cdots a_r^{i_r} n^{m-1} + \frac{1}{D(r-1)!}.$$ 

\[ \cdot \sum_{j \neq 1} \sum_{\ell=1}^{j} \rho_j^{r-1} \sum_{k=0}^{r} \frac{1}{D^k} \sum_{k+1}^{r} (-1)^{r-k} \sum_{0 \leq j_1 \leq \frac{D}{a_1} - 1, \ldots, 0 \leq j_r \leq \frac{D}{a_r} - 1} a_1 j_1 + \cdots + a_r j_r \equiv \ell \mod j \]

where $j \mid a_i$ for some $1 \leq i \leq r$.

Proof. Since $a_1, \ldots, a_r$ are pairwise coprimes, it follows that $W_j(a, n)$ is a quasi-polynomial of degree 0. Hence, the conclusion follows from Proposition 4.1 and Proposition 4.2. \qed

Another formulas for $p_a(n)$ in the case that $a_1, \ldots, a_r$ are pairwise coprimes were proved in [5, Theorem C, pag 113], [2, Theorem 3.1] and [8]. We consider the decomposition

$$\sum_{n=0}^{\infty} p_a(n) z^n = \frac{1}{(1 - z^{a_1}) \cdots (1 - z^{a_r})} = \sum_{\lambda} \sum_{\ell=1}^{m(\lambda)} c_{\lambda, \ell} (\lambda - z)^{\ell},$$

where the sum it taken over the $\lambda$'s with $\lambda^{a_i} = 1$ for some $1 \leq i \leq r$.

**Proposition 5.4.** Let $j \geq 1$. For $1 \leq m \leq m(j)$ we have that

$$c_{p_{j,m}} = \frac{\rho_j^m (m-1)!}{D} \sum_{t=m}^{j} (-1)^{t-m} \binom{t}{m} \sum_{\ell=1}^{j} \rho_j^r \sum_{k=m-1}^{r-1} \left[ \frac{1}{k+1} \right] (-1)^{k-m+1} \binom{k}{m-1}.\]

\[ \cdot \sum_{0 \leq j_1 \leq \frac{D}{a_1} - 1, \ldots, 0 \leq j_r \leq \frac{D}{a_r} - 1} a_1 j_1 + \cdots + a_r j_r \equiv \ell \mod j \]

Proof. It follows from Propositions 4.2, 5.1, and 5.2. \qed

**Proposition 5.5.** For $1 \leq m \leq r$ it holds that

$$c_m = \frac{(m-1)!}{a_1 \cdots a_r} \sum_{\ell=m}^{r} (-1)^{\ell-m} \binom{\ell}{m} \sum_{i_1+\cdots+i_r = r-\ell} \frac{B_{i_1} \cdots B_{i_r}}{i_1! \cdots i_r!} a_1^{i_1} \cdots a_r^{i_r},$$

where $c_m := c_{1,m}$.

Proof. It follows from Theorem 2.4, Proposition 4.1 and Proposition 5.4. \qed
Let \( r \geq 1 \), \( \mathbf{r} := (1, 2, \ldots, r) \). Rademacher’s coefficients \( c_{\omega h k \ell}(r) \) are defined by

\[
\sum_{n=0}^{\infty} p_r(n) z^n = \frac{1}{(1 - z)(1 - z^2) \cdots (1 - z^r)} = \sum_{0 \leq h < r, (h, k) = 1} \sum_{\ell = 1}^{\lfloor \frac{r}{k} \rfloor} \frac{c_{\omega h k \ell}(r)}{(z - \omega h k)^\ell},
\]

where \( \omega_{h k} := e^{2\pi i \frac{h}{k}} \). In the previous notations \( c_{\omega h k \ell}(r) = (-1)^\ell c_{\omega h k, \ell} \). As a direct consequence of Proposition 5.5 we get the following result of C. O’Sullivan [9]:

**Corollary 5.6 ([9, Proposition 2.3])**. For \( 1 \leq m \leq r \) it holds that

\[
c_{01m}(r) = \frac{(-1)^r (m-1)!}{r!} \sum_{\ell=m}^{r} \frac{\{\ell\}_m}{(\ell-1)!} \sum_{i_1 + \cdots + i_r = r - \ell} \frac{B_{i_1} \cdots B_{i_r} 1_{i_1} 2_{i_2} \cdots r_{i_r}}{i_1! \cdots i_r!}
\]

### 6. Frobenius Number

Given a sequence of positive integers \( \mathbf{a} = (a_1, \ldots, a_r) \) that satisfy \( \gcd(a_1, \ldots, a_r) = 1 \), the Frobenius number of \( \mathbf{a} \), denoted by \( F(\mathbf{a}) = F(a_1, \ldots, a_r) \), is the largest integer \( n \) with the property that \( p_\mathbf{a}(n) = 0 \).

**Proposition 6.1.** Let \( \mathbf{a} = (a_1, \ldots, a_r) \) with \( \gcd(a_1, \ldots, a_r) = 1 \) and \( D = \text{lcm}(a_1, \ldots, a_r) \). We have that

\[
F(a_1, \ldots, a_r) \leq D(r-1) - a_1 - \cdots - a_r.
\]

**Proof.** Let \( n \) be an integer with \( p_\mathbf{a}(n) = 0 \). Since the map

\[
\varphi : \mathbb{Z}/a_1 \mathbb{Z} \times \cdots \times \mathbb{Z}/a_r \mathbb{Z} \to \mathbb{Z}/D \mathbb{Z}, \quad \varphi(\hat{a}_1, \ldots, \hat{a}_r) := \frac{a_1 j_1 + \cdots + a_r j_r}{n \mod D}
\]

is a surjective morphism, it follows that there exists some integers \( 0 \leq j_1 \leq \frac{D}{a_1} - 1, \ldots, 0 \leq j_r \leq \frac{D}{a_r} - 1 \) such that \( a_1 j_1 + \cdots + a_r j_r \equiv n \pmod{D} \).

From Corollary 2.3 it follows that \( n < a_1 j_1 + \cdots + a_r j_r \), hence

\[
n \leq a_1 j_1 + \cdots + a_r j_r - D \leq (D-1)r - a_1 - \cdots - a_r.
\]

The following corollary is a particular case of [10, Theorem 2.7] and appears also in [17, Theorem 1(a)].

**Corollary 6.2.** Let \( \mathbf{a} = (a_1, \ldots, a_r) \) such that \( \gcd(a_i, a_j) = 1 \) for all \( i \neq j \), \( D = \text{lcm}(a_1, \ldots, a_r) = a_1 \cdots a_r \), \( A_i := \frac{D}{a_i} \), \( 1 \leq i \leq r \). It holds that

\[
F(A_1, \ldots, A_r) = D(r-1) - A_1 - \cdots - A_r.
\]

**Proof.** It holds that \( D = \text{lcm}(A_1, \ldots, A_r) \). From Lemma 6.1 it follows that

\[
F(A_1, \ldots, A_r) \leq D(r-1) - A_1 - \cdots - A_r.
\]
Suppose that $D(r - 1) - A_1 - \cdots - A_r = A_1 j_1 + \cdots + A_r j_r$ with $j_k \geq 0$ for $1 \leq k \leq r$, hence

$$D(r - 1) = A_1 (j_1 + 1) + \cdots + A_r (j_r + 1), \quad r - 1 = \frac{j_1 + 1}{a_1} + \cdots + \frac{j_r + 1}{a_r}.$$ 

Since $\gcd(a_i, a_j) = 1$ it follows that $a_k | (j_k + 1)$ for all $1 \leq k \leq r$. Since $j_k \geq 0$ we get

$$r - 1 = \frac{j_1 + 1}{a_1} + \cdots + \frac{j_r + 1}{a_r} \geq r,$$

a contradiction. So $F(A_1, \ldots, A_r) \geq D(r - 1) - A_1 - \cdots - A_r$. 

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