THE ASYMPTOTIC DEPENDENCE STRUCTURE AND SOME FUNCTIONS OF GENERALIZED ORDER STATISTICS IN A STATIONARY GAUSSIAN SEQUENCE

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The main aim of this paper is to study the limit joint distribution function (df) of any two extreme, as well as central, m-generalized order statistics ($m-$gos) of a stationary Gaussian sequence under an equi-correlated set-up. It is shown that under this general set-up, any lower and upper extremes are asymptotically dependent unless the correlation is of order $\circ(\frac{1}{\log n})$, on the contrary of gos based on i.i.d random variables (rv's). Moreover, under this general framework of study, the classes of possible non-degenerate limit df's of the generalized quasi-ranges, quasi-mid-ranges, extremal quotient, extremal product and the ratio of the symmetric differences of $m-$gos are obtained. It is worth mentioning that, the results of this paper contribute not only to a critical assessment of existing statistical methodology, but also help to address their limitations within different contexts

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1. **INTRODUCTION**

The concept of generalized order statistics (gos) was introduced by [17], as a general framework for models of ordered rv's. In testing the strength of materials, reliability analysis, lifetime studies, etc., the realizations of experiments arise in nondecreasing order and therefore we need to consider several models of ascendingly ordered rv's. Many practical important models of ordered rv's, such as ordinary order statistics (oos), progressively type II censored order statistics, upper record values and sequential order statistics (sos), are seen to be particular cases of gos. These models can be effectively applied. For example, in reliability theory the rth order statistic in a sample of size n represents the life-length of a $(n-r+1)$-out of-n-system, which is an important technical structure. A more flexible and more adequate model for a $(n-r+1)$-out-of-n-system is sos, which has to take a specific dependence structure into consideration. Namely, if some component of the system fails, this may have an influence on the life-length distributions of the remaining components.

Let $\gamma_n = k > 0$, $\gamma_i = k + n - i + \sum_{j=i}^{n-1} m_j > 0$, $i = 1, 2, ..., n-1$, and $\tilde{m} = (m_1, m_2, \cdots, m_{n-1}) \in \Re^{n-1}$. Then, the rv's $X(1, n, \tilde{m}, k) \leq X(2, n, \tilde{m}, k) \leq ... \leq X(n, n, \tilde{m}, k)$ are called gos based on a continuous df F, with probability density function (pdf) f, which are defined via their pdf

$$f^{(\tilde{m},k)}_{1,2,...,n:n}(x_1, x_2, ..., x_n) = \left(\prod_{j=1}^{n} \gamma_j\right) \left(\prod_{j=1}^{n-1} (1 - F(x_j))^{\gamma_j - \gamma_{j+1} - 1} f(x_j)\right) \times$$
$$(1 - F(x_n))^{\gamma_n - 1} f(x_n),$$

where $F^{-1}(0) \leq x_1 \leq ... \leq x_n \leq F^{-1}(1)$. Particular choices of the parameters $\gamma_1, \gamma_2, ..., \gamma_n$ lead to different models, *e.g.*, $m-$gos ($\gamma_i = k + (n-i)(m+1), i = 1, 2, ..., n-1$), oos ($k = 1, \gamma_i = n - i + 1, i = 1, 2, ..., n-1$) and sos ($k = \alpha_n, \gamma_i = (n - i + 1)\alpha_i, i = 1, 2, ..., n-1$) (see [17]).

In this work, we consider a wide subclass of gos, specifically when $m_1 = m_2 = ... = m_{n-1} = m \neq -1$. This subclass is known as $m-$gos. Clearly, most of the known practical models of gos are included in this subclass such as oos, type II censored order statistics, sos and order statistics with non-integral sample size (the order statistics with non-integral sample size have been introduced as an extension of oos. Moreover, these quantities can be interpreted as certain sos, cf. [17].

Nasri-Roudsar [21] (see also [4]) derived the marginal df of the $(n-i+1)$th $m-$gos, in the form $\Phi^{(m,k)}_{n-i+1:n}(x) = I_{G_m(x)}(N_n - R_i + 1, R_i)$, where $G_m(x) = 1 - (1 - F(x))^{m+1} = 1 - \overline{F}^{m+1}(x)$, $N_n = \frac{k}{m+1} + n - 1$, $R_i = \frac{k}{m+1} + i - 1$ and $I_x(a, b) = \frac{1}{\beta(a,b)} \int_0^x t^{a-1}(1-t)^{b-1} dt$ is the incomplete beta ratio function. The possible non-degenerate limit distributions and the convergence rate of the upper extreme $m-$gos, *i.e.*, $(n-i+1)$th $m-$gos for fixed i, were discussed in [22]. The necessary and sufficient conditions of the weak convergence, as $n \to \infty$, as well as the form of the possible limit df's of extreme, intermediate and central $m-$gos were derived in [4].

The asymptotic theory of oos of stationary normal sequences has found many applications, as testified by many references, among them are [1, 9, 18]. Let $X_1, X_2, ..., X_n$ be a Gaussian sequence with zero expectation, unit variance and correlation $r_n = E(X_i X_j) \geq 0$, $i \neq j$. This sequence can be replaced, by the sequence $X_j = \sqrt{r_n}\ Y_0 + \sqrt{1 - r_n}\ Y_j$, $1 \leq j \leq n$, for the i.i.d standard normal variates $Y_0, Y_1, ..., Y_n$. Moreover, $X_j = Y_j$, for $r_n = 0$ (cf. [14]). Therefore, for any $0 \leq i \leq n$, we get

$$X(i, n, m, k) = \sqrt{r_n}\ Y_0 + \sqrt{1 - r_n}\ Y(i, n, m, k), \tag{1}$$

where $X(i,n,m,k)$ and $Y(i,n,m,k)$ are the ith $m-$gos based on $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$, respectively. In a recent paper [9], the limit df's of extreme, intermediate and central oos of a stationary Gaussian sequence under equi-correlated set-up, when the random sample size is assumed to converge weakly, were derived. These results were generalized by [10] for $m-$gos.

In Section 2 of this paper we will develop the limit theory for gos, by revealing the asymptotic dependence structural between the members of $m-$gos of the stationary Gaussian sequence $\{X(i,n,m,k)\}$. Namely, for any $1 \leq \ell < s < n$, the limit joint df's of the $m-$gos $X(n-s+1,n,m,k)$ and $X(n-\ell+1,n,m,k)$, or $X(\ell,n,m,k)$ and $X(s,n,m,k)$, or $X(\ell,n,m,k)$ and $X(n-s+1,n,m,k)$ (in this case $1 \leq \ell, s < n$), or $X(\ell_n,n,m,k)$ and $X(s_n,n,m,k)$, when $m \neq -1$, are derived in extreme case (*i.e.*, $1 \leq \ell < s \leq n$ are fixed with respect to n) and in the central case, where $\ell_n, s_n \to \infty$ and $\frac{\ell_n}{N_n} \to \lambda_1$, $\frac{s_n}{N_n} \to \lambda_2$, where $0 < \lambda_1 < \lambda_2 < 1$, as $N_n \to \infty$ (or equivalently, as $n \to \infty$). A remarkable example of the central oos is the pth sample quantile, where $\ell_n = [np], 0 < p < 1$, and $[x]$ denotes the largest integer not exceeding x. It is revealed that under this general set-up, any lower and upper extremes are asymptotically dependent, unless $r_n \log n \to 0$, as $n \to \infty$.

In Section 3, we will study the classes of possible non-degenerate limit df's of the suitably normalized generalized quasi-ranges, quasi-mid-ranges, extremal quotient and extremal product. These important functions when they based on oos were studied extensively in [3, 6, 7, 16]. Moreover, these functions, when they based on $m-$gos, which are arising from i.i.d rv's, were studied in [11]. Also, in Section 3, we will study the asymptotic behavior of the ratio of the symmetric differences of $m-$gos of a stationary Gaussian sequence $\underline{X} = (X_1, X_2, ..., X_n)$ defined in (1),

$$\triangle_n(m,k|\underline{X}) = \frac{X(\ell_{4;n},n,m,k) - X(\ell_{2;n},n,m,k)}{X(\ell_{3;n},n,m,k) - X(\ell_{1;n},n,m,k)},$$

where $q_i,\ i = 1,2,3,4\ , 0 < q_2 < q_1 < \frac{1}{2}$, $q_3 = (1-q_1) < q_4 = (1-q_2)$, $\ell_{3;n} = n - \ell_{1;n} + 1$ and $\ell_{4;n} = n - \ell_{2;n} + 1$. The statistic $\triangle_n(0,1|\underline{W})$, based on a general sequence of i.i.d rv's $W_1, W_2, ..., W_n$, was studied in [5], while the general statistic $\triangle_n(m,k|\underline{W})$ was recently studied in [12]. Section 4 is devoted to indicate some potential applications of the main results of the paper (Theorems 1-3 and 5).

Everywhere in what follows the symbols $\xrightarrow[n]{}$, $\xrightarrow[n]{w}$ and $\xrightarrow[n]{p}$ stand for convergence, converge weakly and converge in probability, as $n \to \infty$, respectively. Moreover, for every $i, x \geq 0$, $\Gamma_i(x) = \frac{1}{\Gamma(i)}\int_0^x t^{i-1}e^{-t}dt$ stands for the incomplete gamma ratio function (the gamma df), while $\bar{\Gamma}_i(x) = 1 - \Gamma_i(x)$ denotes its survivor function. Also the symbol "$*$" denotes the convolution operation. Finally, $\phi(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$ is the standard pdf and $\Phi(z)$ is its df (*i.e.*, the normal df), while $\Phi_{\mu;\sigma^2}(z)$ is normal df with mean μ and variance σ^2.

2. THE ASYMPTOTIC DEPENDENCE STRUCTURAL OF $m-$gos IN A STATIONARY GAUSSIAN SEQUENCE

The following theorems reveal the asymptotic dependence structural of $m-$gos, $m\geq -1$, in a stationary Gaussian sequence in the following cases: upper-upper extreme, lower-lower extreme, lower-upper extreme $m-$gos and central-central $m-$gos, respectively.

THEOREM 1 (the joint df of upper-upper extreme $m-$gos). *Let $m\geq -1$ and let $a_{n,m}=\frac{1}{b_{n,m}}-\frac{1}{2}b_{n,m}(\log\log n^{\frac{1}{m+1}}+\log 4\pi)$ and $b_{n,m}=(\frac{2}{m+1}\log n)^{\frac{-1}{2}}$. If $r_n\log n \underset{n}{\longrightarrow} \tau\in[0,\infty)$, then*

(2)
$$P(X(n-s+1,n,m,k)\leq x_{1;n},X(n-\ell+1,n,m,k)\leq x_{2;n})=$$
$$\Psi^{(m,k)}_{n-s+1,n-\ell+1:n}(x_{1;n},x_{2;n}) \underset{n}{\overset{w}{\longrightarrow}}$$
$$\begin{cases} \int_{-\infty}^{\infty}\bar{\Gamma}_{R_\ell}(e^{-(m+1)(x_2-z)-\tau})d\Phi(\sqrt{\frac{m+1}{2\tau}}\ z), & x_1\geq x_2,\\ \int_{-\infty}^{\infty}\bar{\Gamma}_{R_s}(e^{-(m+1)(x_1-z)-\tau})d\Phi(\sqrt{\frac{m+1}{2\tau}}\ z) & \\ -\frac{1}{\Gamma(R_s)}\int_{-\infty}^{\infty}\int_{e^{-(m+1)(x_1-z)-\tau}}^{\infty} I_{\frac{e^{-(m+1)(x_2-z)-\tau}}{t}}(R_\ell,R_s-R_\ell)\times & \\ \qquad t^{R_s-1}e^{-t}dtd\Phi(\sqrt{\frac{m+1}{2\tau}}\ z), & x_1\leq x_2,\end{cases}$$

where $x_{i;n}=b_{n,m}x_i+a_{n,m}, i=1,2$. Otherwise (i.e., if $r_n\log n \underset{n}{\longrightarrow}\infty$),

$$\Psi^{(m,k)}_{n-s+1,n-\ell+1:n}(x^\star_{1;n},x^\star_{2;n}) \underset{n}{\overset{w}{\longrightarrow}} \Phi(\min(x_1,x_2)), \tag{3}$$

where $x^\star_{i;n}=\sqrt{r_n}x_i+\sqrt{1-r_n}a_{n,m}, i=1,2$.

Proof. By using the representation (1) and in view of the independence of Y_0 and $Y_i, i=1,2,...,n$, we can write

$$\Psi^{(m,k)}_{n-s+1,n-\ell+1:n}(x_{1;n},x_{2;n})$$
$$=\int_{-\infty}^{\infty}P(X(n-s+1,n,m,k)\leq x_{1;n},X(n-\ell+1,n,m,k)\leq x_{2;n}|Y_0=y)\phi(y)dy$$
$$=\int_{-\infty}^{\infty}P(Y(n-s+1,n,m,k)\leq x_{1;n}(y),Y(n-\ell+1,n,m,k)\leq x_{2;n}(y))\phi(y)dy$$
$$=\int_{-\infty}^{\infty}\Phi^{(m,k)}_{n-s+1,n-\ell+1:n}(x_{1;n}(y),x_{2;n}(y))\phi(y)dy, \tag{4}$$

where, $x_{i,n}(y)=B_{n,m}x_i+A_{n,m}, i=1,2,\ B_{n,m}=\frac{b_{n,m}}{\sqrt{1-r_n}},\ A_{n,m}=\frac{a_{n,m}-\sqrt{r_n}y}{\sqrt{1-r_n}}$ and $\Phi^{(m,k)}_{n-s+1,n-\ell+1:n}\,(x_{1;n}(y),x_{2;n}(y))=P(Y(n-s+1,n,m,k)\leq x_{1;n}(y),Y(n-\ell+1,n,m,k)\leq x_{2;n}(y))$. On the other hand, by using Theorem 2.1, the relation (2.7) in [8] and by applying Theorem 2.1 in [4] on the normal upper extreme $m-$gos, we get

(5)

$$\Phi^{(m,k)}_{n-s+1,n-\ell+1:n}(x_{1;n},x_{2;n})\ \xrightarrow[n]{w}$$
$$\begin{cases}\bar{\Gamma}_{R_\ell}(e^{-(m+1)x_2}), & x_1\geq x_2,\\ \bar{\Gamma}_{R_s}(e^{-(m+1)x_1})-\frac{1}{\Gamma(R_s)}\int_{e^{-(m+1)x_1}}^{\infty}I_{\underline{\frac{e^{-(m+1)x_2}}{t}}}(R_\ell,R_s-R_\ell)t^{R_s-1}e^{-t}dt, & \\ & x_1\leq x_2.\end{cases}$$

Therefore, in view of Khinchin's type theorem and by using the transformation $z=\sqrt{\frac{2\tau}{m+1}}y$, the relations (4) and (5) yield (2), if we show that $\frac{A_{n,m}-a_{n,m}}{b_{n,m}}\ \xrightarrow[n]{}\ \frac{\tau}{m+1}-\sqrt{\frac{2\tau}{m+1}}y$ and $\frac{B_{n,m}}{b_{n,m}}\ \xrightarrow[n]{}\ 1$. The later is evident from the assumption that $r_n\log n\ \xrightarrow[n]{}\ \tau\geq 0$ (this assumption yields $r_n\ \xrightarrow[n]{}\ 0$). Hence, only the first relation needs proof. Applying that $\sqrt{1-r_n}=1-\frac{1}{2}r_n(1+\circ(1))$ and bearing in mind that $r_n\log\log n^{\frac{1}{m+1}}=(r_n\log n)(\frac{\log n-\log(m+1)}{\log n})\ \xrightarrow[n]{}\ 0$, we can verify that

$$\frac{A_{n,m}-a_{n,m}}{b_{n,m}}=\frac{-\sqrt{r_n}y+\frac{1}{2}r_n a_{n,m}(1+\circ(1))}{b_{n,m}}$$
$$=-\sqrt{\frac{2}{m+1}r_n\log n}\ y+\frac{r_n\log n}{m+1}-\frac{r_n}{4}\left[\log\log n^{\frac{1}{m+1}}+\log 4\pi\right]$$
$$\xrightarrow[n]{}\ \frac{\tau}{m+1}-\sqrt{\frac{2\tau}{m+1}}\ y.$$

Turning now to the case $r_n\log n\longrightarrow\infty$, for which we start with the relation (4), with $x^\star_{i;n}(y)=\sqrt{\frac{r_n}{1-r_n}}(x_i-y)+a_{n,m}, i=1,2$. Now, for every $\epsilon>0$, we have

$$P\left(\sqrt{\frac{1-r_n}{r_n}}|Y(n-j_i+1,n,m,k)-a_{n,m}|\geq\epsilon\right)$$
$$\leq P(|Y(n-j_i+1,n,m,k)-a_{n,m}|\geq\sqrt{r_n}\epsilon)$$
$$=P\left(|\frac{Y(n-j_i+1,n,m,k)-a_{n,m}}{b_{n,m}}|\geq(\frac{2}{m+1}r_n\log n)^{\frac{1}{2}}\epsilon\right)\ \xrightarrow[n]{}\ 0,$$

$i=1,2, j_1=\ell, j_2=s$. Thus, the df $P(|(\sqrt{\frac{1-r_n}{r_n}}|Y(n-j_i+1,n,m,k)-a_{n,m}|\leq x_i-y), i=1,2$, has a degenerate limit df at zero, *i.e.*, has the limit

$$\in(x_i-y)=\begin{cases}1, & x_i-y\geq 0,\\ 0, & x_i-y<0, i=1,2.\end{cases}$$

Consequently, by using the transformation $z=-y$, we get

$$\Psi^{(m,k)}_{n-s+1,n-\ell+1:n}(x^{\star}_{1;n},x^{\star}_{2;n}) \xrightarrow[n]{w} \int_{-\infty}^{\infty} \in (x_1+z,x_2+z)\phi(z)dz, \tag{6}$$

where

$$\in (x_1+z,x_2+z)=\begin{cases} 1, & z\geq \max(-x_1,-x_2)=-\min(x_1,x_2),\\ 0, & z<\max(-x_1,-x_2)=-\min(x_1,x_2). \end{cases} \tag{7}$$

The required relation (3) is now followed by combining (6) with (7). This completes the proof of Theorem 1. $\square$

Corollary 1. *Suppose the conditions in Theorem 1 hold with $x_1=x$ and $x_2\to\infty$. If $r_n\log n \xrightarrow[n]{} \tau\in[0,\infty)$, then the marginal df of $X(n-s+1,n,m,k)$ admits the following limiting representation*

$$P\left(\frac{X(n-s+1,n,m,k)-a_{n,m}}{b_{n,m}}\leq x\right) \xrightarrow[n]{w} \Psi^{(m,k)}_{s}(x)$$
$$=\bar{\Gamma}_{R_s}(e^{-(m+1)x-\tau})*\Phi(\sqrt{\frac{m+1}{2\tau}}x).$$

Otherwise (i.e., if $r_n\log n \xrightarrow[n]{} \infty$),

$$P\left(\frac{X(n-s+1,n,m,k)-\sqrt{1-r_n}a_{n,m}}{\sqrt{r_n}}\leq x\right) \xrightarrow[n]{w} \Phi(x)$$

(see [10]).

Theorem 2 (the joint df of lower-lower extreme $m-$gos). *Let $m\geq -1$ and let $\tilde{a}_{n,m}=-\tilde{b}^{-1}_{n,m}+\frac{1}{2}\tilde{b}_{n,m}(\log\log n(m+1))+\log 4\pi)$ and $\tilde{b}_{n,m}=(2\log n(m+1))^{\frac{-1}{2}}$. If $r_n\log n \xrightarrow[n]{} \tau\in[0,\infty)$, then*

$$P(X(\ell,n,m,k)\leq \tilde{x}_{1;n},X(s,n,m,k)\leq\tilde{x}_{2;n})=\Psi^{(m,k)}_{\ell,s:n}(\tilde{x}_{1;n},\tilde{x}_{2;n})$$
$$\xrightarrow[n]{w} \begin{cases} \int_{-\infty}^{\infty}\Gamma_s(e^{x_2-\tau-z})d\Phi(\frac{z}{\sqrt{2\tau}}), & x_1\geq x_2,\\ \frac{1}{(\ell-1)!}\int_{-\infty}^{\infty}\int_0^{e^{x_1-\tau-z}}\Gamma_{s-\ell}(e^{x_2-\tau-z}-t)t^{\ell-1}e^{-t}dtd\Phi(\frac{z}{\sqrt{2\tau}}), & x_1\leq x_2, \end{cases}$$

where $\tilde{x}_{i;n}=\tilde{b}_{n,m}x_i+\tilde{a}_{n,m}, i=1,2$. Otherwise (i.e., if $r_n\log n \xrightarrow[n]{} \infty$),

$$\Psi^{(m,k)}_{\ell,s:n}(\tilde{x}^{\star}_{1;n},\tilde{x}^{\star}_{2;n}) \xrightarrow[n]{w} \Phi(\min(x_1,x_2)),$$

where $\tilde{x}^{\star}_{i;n}=\sqrt{r_n}x_i+\sqrt{1-r_n}\tilde{a}_{n,m}, i=1,2.$

Proof. The proof is similar to the proof of Theorem 1 with only the exception of obvious changes, *e.g.*, we use the relation (2.6) of Theorem 2.1 in [8] instead of the relation (2.7) and by applying Theorem 2.1 in [4] on the normal lower extreme $m-$gos. This completes the proof of Theorem 2. $\square$

Corollary 2. *Suppose the conditions in Theorem 2 hold with* $x_1 = x$ *and* $x_2 \to \infty$. *If* $r_n \log n \xrightarrow[n]{} \tau \in [0, \infty)$, *then the marginal df of* $X(\ell, n, m, k)$ *admits the following limiting representation*

$$P\left(\frac{X(\ell,n,m,k)-\tilde{a}_{n,m}}{\tilde{b}_{n,m}} \le x\right) \xrightarrow[n]{w} \tilde{\Psi}_\ell^{(m,k)}(x) = \Gamma_\ell(e^{x-\tau}) * \Phi(\frac{x}{\sqrt{2\tau}}).$$

Otherwise (i.e., if $r_n \log n \xrightarrow[n]{} \infty$*),*

$$P\left(\frac{X(\ell,n,m,k)+(1-r_n)^{\frac{1}{2}}\tilde{a}_{n,m}}{\sqrt{r_n}} \le x\right) \xrightarrow[n]{w} \Phi(x)$$

(see [10]).

Theorem 3 (the joint df of lower-upper extreme $m-$gos). *Let* $m \ge -1$ *and let* $\tilde{x}_{1;n}$, $\tilde{x}^\star_{1;n}$, $x_{2;n}$, $x^\star_{2;n}$, $\tilde{x}_{1;n}(y)$, $\tilde{x}^\star_{1;n}(y)$, $x_{2;n}(y)$ *and* $x^\star_{2;n}(y)$ *be defined as in Theorems 1 and 2. If* $r_n \log n \xrightarrow[n]{} \tau \in [0, \infty)$, *then for all* $-\infty < x_1, x_2 < \infty$, *we have that*

$$\int_{-\infty}^{\infty} \Gamma_\ell(N_n G_m(\tilde{x}_{1n}(y)))\bar{\Gamma}_{R_s}(N_n \bar{G}_m(x_{2n}(y)))\phi(y)dy$$

$$\le P(X(\ell,n,m,k) \le \tilde{x}_{1;n}, X(n-s+1,n,m,k) \le x_{2;n}) = \Psi^{(m,k)}_{\ell,n-s+1:n}(\tilde{x}_{1;n}, x_{2;n})$$
$$\le \int_{-\infty}^{\infty} \Gamma_\ell(N_n G_m(\tilde{x}_{1n}(y)))(\Gamma_{R_s}(N_n) - \Gamma_{R_s}(N_n \bar{G}_m(x_{2n}(y))))\phi(y)dy$$

and

$$\Psi^{(m,k)}_{\ell,n-s+1:n}(\tilde{x}_{1;n}, x_{2;n}) \xrightarrow[n]{w} \int_{-\infty}^{\infty} \Gamma_\ell(e^{x_1-\tau-z\sqrt{m+1}})\bar{\Gamma}_{R_s}(e^{-(m+1)(x_2-z)-\tau})d\Phi(\sqrt{\frac{m+1}{2\tau}}\ z). \tag{8}$$

Otherwise, if $r_n \log n \xrightarrow[n]{} \infty$, *then*

$$\int_{-\infty}^{\infty} \Gamma_\ell(N_n G_m(\tilde{x}^\star_{1n}(y)))\bar{\Gamma}_{R_s}(N_n \bar{G}_m(x^\star_{2n}(y)))\phi(y)dy$$

$$\le P(X(\ell,n,m,k) \le \tilde{x}^\star_{1;n}, X(n-s+1,n,m,k) \le x^\star_{2;n}) = \Psi^{(m,k)}_{\ell,n-s+1:n}(\tilde{x}^\star_{1;n}, x^\star_{2;n})$$
$$\le \int_{-\infty}^{\infty} \Gamma_\ell(N_n \Gamma_m(\tilde{x}^\star_{1n}(y)))(\Gamma_{R_s}(N_n) - \Gamma_{R_s}(N_n \bar{G}_m(x^\star_{2n}(y))))\phi(y)dy$$

and

$$\Psi^{(m,k)}_{\ell,n-s+1:n}(\tilde{x}^\star_{1;n}, x^\star_{2;n}) \xrightarrow[n]{w} \Phi(\min(x_1, x_2)). \tag{9}$$

REMARK 1. Clearly, Theorem 3 (the relations (8) and (9)) reveals that the lower and upper extreme $m-$gos are asymptotically independent only if $r_n \log n \xrightarrow[n]{} \tau = 0$.

Proof. The proof of Theorem 3 is similar to the proof of Theorems 1 and 2, with only the exception of obvious changes, *e.g.* we use Lemma 2.5 besides the relations (2.6) and (2.7) of Theorem 2.1 in [8] and we apply Theorem 2.1 in [4] on the normal lower and upper extreme $m-$gos. This completes the proof of Theorem 3. $\square$

Let $0<\lambda_1<\lambda_2<1$ and x_{0i} be such that $\Phi(x_{0i})=\lambda_i, i=1,2$. Moreover, let ℓ_n and s_n be central rank sequences such that $\sqrt{n}(\frac{\ell_n}{n}-\lambda_1)\underset{n}{\longrightarrow}0$ and $\sqrt{n}(\frac{s_n}{n}-\lambda_2)\underset{n}{\longrightarrow}0$. It is known that (cf. [8], Lemma 3.1 and Theorem 3.1, see also Theorem 2.2 of [4])

$$P\left(\frac{Y(\ell_n,n,m,k)-x_{01}}{c_{1;n}}\leq x_1, \frac{Y(s_n,n,m,k)-x_{02}}{c_{2;n}}\leq x_2\right)$$

$$\underset{n}{\xrightarrow{w}} \text{BI}\mathcal{N}\left(\frac{c^*_{\lambda_1(m)}}{c^*_{\lambda_1}}(m+1)x_1, \frac{c^*_{\lambda_2(m)}}{c^*_{\lambda_2}}(m+1)x_2; R\right),$$

where $c_{i,n}=\frac{\sqrt{\lambda_i(1-\lambda_i)}}{\sqrt{n}\phi(x_{0i})}, i=1,2,\ c_{\lambda_i}=\sqrt{\lambda_i(1-\lambda_i)},\ \lambda_i(m)=1-(1-\lambda_i)^{\frac{1}{m+1}}$, $c^*_{\lambda_i}=\frac{c_{\lambda_i}}{1-\lambda_i}, i=1,2$, and $\text{BI}\mathcal{N}(x_1,x_2;R)$ is the standard bivariate normal df with correlation $R=\sqrt{\frac{\lambda_1(1-\lambda_2)}{\lambda_2(1-\lambda_1)}}$.

Under the above conditions concerning $\lambda_i, i=1,2, \ell_n$ and s_n, the following theorem gives the limit joint df of the (ℓ_n,s_n)th central m-gos of Gaussian sequence (1).

Theorem 4 (the joint df of central-central gos). *Let $m\geq -1$. If $nr_n\underset{n}{\longrightarrow}\tau\geq 0$, then*

$$P\left(\frac{X(\ell_n,n,m,k)-x_{01}}{c_{1;n}}\leq x_1, \frac{X(s_n,n,m,k)-x_{02}}{c_{2;n}}\leq x_2\right)\underset{n}{\xrightarrow{w}}$$

$$\int_{-\infty}^{\infty} BI\mathcal{N}\left(\frac{(m+1)c^*_{\lambda_1(m)}}{c^*_{\lambda_1}}\Big(x_1-\frac{\sqrt{\tau}\phi(x_{01})}{\sqrt{\lambda_1(1-\lambda_1)}}y\Big),\right.$$
$$\left.\frac{(m+1)c^*_{\lambda_2(m)}}{c^*_{\lambda_2}}\Big(x_2-\frac{\sqrt{\tau}\phi(x_{02})}{\sqrt{\lambda_2(1-\lambda_2)}}y\Big);R\right)\phi(y)dy. \tag{10}$$

Moreover, if $nr_n\underset{n}{\longrightarrow}\infty$, we have

$$P\left(\frac{X(\ell_n,n,m,k)-(1-r_n)^{\frac{1}{2}}x_{01}}{\sqrt{r_n}}\leq x_1, \frac{X(s_n,n,m,k)-(1-r_n)^{\frac{1}{2}}x_{02}}{\sqrt{r_n}}\leq x_2\right)$$
$$\underset{n}{\xrightarrow{w}}\Phi(\min(x_1,x_2)). \tag{11}$$

Proof. By using the representation (1) and in view of the independence of Y_0 and $Y_i, i, 1, 2, ..., n$, we can write

$$P\left(\frac{X(\ell_n, n, m, k) - x_{01}}{c_{1;n}} \le x_1, \frac{X(s_n, n, m, k) - x_{02}}{c_{2;n}} \le x_2\right)$$

$$= \int_{-\infty}^{\infty} P(X(\ell_n, n, m, k) \le c_{1;n}x_1 + x_{01},$$
$$X(s_n, n, m, k) \le c_{2;n}x_2 + x_{02}|Y_0 = y)\phi(y)dy$$

$$= \int_{-\infty}^{\infty} P(Y(\ell_n, n, m, k) \le C_{1;n}x_1 + D_{1;n}(y), \tag{12}$$
$$Y(s_n, n, m, k) \le C_{2;n}x_2 + D_{2;n}(y))\phi(y)dy,$$

where, $C_{i;n} = \frac{c_{i;n}}{\sqrt{1-r_n}}$ and $D_{i;n}(y) = \frac{x_{0i}-\sqrt{r_n}y}{\sqrt{1-r_n}}, i = 1, 2$. On the other hand, by applying Theorem 3.1 in [8] on the normal central $m-$gos, we get

$$P(\frac{Y(\ell_n, n, m, k) - x_{01}}{c_{1;n}} \le x_1, \frac{Y(s_n, n, m, k) - x_{02}}{c_{2;n}} \le x_2)$$

$$\xrightarrow[n]{w} \text{BI}\mathcal{N}\left(\frac{(m+1)c^*_{\lambda_1(m)}}{c^*_{\lambda_1}}x_1, \frac{(m+1)c^*_{\lambda_2(m)}}{c^*_{\lambda_2}}x_2; R\right). \tag{13}$$

Therefore, in view of Khinchin's type theorem, the relations (12) and (13) yield the relation (10), if we show that $\frac{D_{i;n}(y)-x_{0i}}{c_{i;n}} \xrightarrow[n]{} -\frac{\sqrt{\tau}\phi(x_{0i})y}{\sqrt{\lambda_i(1-\lambda_i)}}$ and $\frac{C_{i;n}}{c_{i;n}} \xrightarrow[n]{} 1$, $i = 1, 2$. The later is evident from the assumption that $nr_n \xrightarrow[n]{} \tau \ge 0$ (this assumption yields $r_n \xrightarrow[n]{} 0$). Hence, only the first relation needs proof. Applying that $\sqrt{1 - r_n} = 1 - \frac{1}{2}r_n(1 + \circ(1))$ and bearing in mind that $\frac{r_n}{c_{i,n}} = \frac{nr_n\phi(x_{0i})}{\sqrt{n\lambda_i(1-\lambda_i)}} \xrightarrow[n]{} 0$ and $\frac{\sqrt{r_n}}{c_{i,n}} \xrightarrow[n]{} \frac{\sqrt{\tau}\phi(x_{0i})}{\sqrt{\lambda_i(1-\lambda_i)}}$, we can easily verify that

$$\frac{D_{i;n} - x_{0i}}{c_{i;n}} = \frac{-\sqrt{r_n}y + \frac{1}{2}x_{0i}r_n(1 + \circ(1)) - \frac{1}{2}r_n^{\frac{3}{2}}y(1 + \circ(1))}{c_{i;n}}$$
$$\xrightarrow[n]{} -\frac{\sqrt{\tau}\phi(x_{0i})}{\sqrt{\lambda_i(1-\lambda_i)}}y, i = 1, 2.$$

Turning now to the case $r_n \log n \longrightarrow \infty$, for which we start with the relation (12). Now, for every $\epsilon > 0$, we have

$$P\left(\sqrt{\frac{1-r_n}{r_n}}|Y(i_n, n, m, k) - x_{0i}| \ge \epsilon\right) \le P\left(|Y(i_n, n, m, k) - x_{0i}| \ge \epsilon\sqrt{r_n}\right)$$

$$= P\left(\frac{|Y(i_n, n, m, k) - x_{0i}}{c_{i;n}} \ge \frac{\sqrt{nr_n}\phi(x_{0i})}{\sqrt{\lambda_i(1-\lambda_i)}}\epsilon\right) \xrightarrow[n]{w} 0, i = 1, 2, 1_n = \ell_n, 2_n = s_n$$

Thus, the df $P\left(\sqrt{\frac{1-r_n}{r_n}}|Y(i_n,n,m,k)-x_{0i}|\leq x_i-y\right)$ has a degenerate limit df at zero, *i.e.*, has the limit

$$\in(x_i-y)=\begin{cases}1, & x_i-y\geq 0,\\ 0, & x_i-y<0.\end{cases}$$

Consequently, by using the transformation $z=-y$, we get

$$P\left(\frac{X(\ell_n,n,m,k)-(1-r_n)^{\frac{1}{2}}x_{01}}{\sqrt{r_n}}\leq x_1,\frac{X(s_n,n,m,k)-(1-r_n)^{\frac{1}{2}}x_{02}}{\sqrt{r_n}}\leq x_2\right)$$

$$\xrightarrow[n]{w}\int_{-\infty}^{\infty}\in(x_1+z,x_2+z)\phi(z)dz, \tag{14}$$

where

$$\in(x_1+z,x_2+z)=\begin{cases}1, & z\geq\max(-x_1,-x_2)=-\min(x_1,x_2),\\ 0, & z<\max(-x_1,-x_2)=-\min(x_1,x_2).\end{cases} \tag{15}$$

The required relation (11) is now followed by combining (14) with (15). This completes the proof of Theorem 4. □

3. ASYMPTOTIC BEHAVIOR OD SOME FUNCTIONS OF $m-$gos IN A STATIONARY GAUSSIAN SEQUENCE

The generalized quasi-ranges and quasi-mid-ranges are linear functions (linear combination) of m-gos. The quasi range and quasi-mid-range are widely used, particularly, in statistical quality control as simple estimators of the dispersion and measure of central tendency, respectively. Many short-cut tests have been based on these statistics. The extremal quotient, as well as the generalized extremal quotient, is not affected by a change of scale. Thus, it is frequently used in cases, where the scale plays no role, *e.g.*, in climatic study. Moreover, the extremal quotient is used in several fields, most notably in life testing and the classical heterogeneity of variance situation. An important application on the study of the range and the extremal quotient is the statistic of ratio of the symmetric differences. This statistic is used as a test for kurtosis, or as a measure of tail thickness (see [2, 13]).

In this section, the classes of possible non-degenerate limit df's of the following suitably normalized generalized quasi-ranges, quasi-mid-ranges, extremal quotient and extremal product for the m-gos based on a Gaussian sequence of rv's $\underline{X}=(X_1,X_2,...,X_n)$ defined in (1):
$\mathcal{R}^*_{n;\ell,s}(m,k)=b^{-1}_{n,m:r}(\mathcal{R}_{n;\ell,s}(m,k)-a_{n,m:r}),\mathcal{V}^*_{n;\ell,s}(m,k)=b^{-1}_{n,m:v}(\mathcal{V}_{n;\ell,s}(m,k)-a_{n,m:v}),1\leq\ell,s<n,$ $\mathcal{Q}^*_n(m,k)=b^{-1}_{n,m:q}(\mathcal{Q}_n(m,k)-a_{n,m:q})$ and $\mathcal{P}^*_n(m,k)=$

$b^{-1}_{n,m:p}(\mathcal{P}_n(m,k)-a_{n,m:p})$, respectively, where $\mathcal{R}_{n;\ell,s}(m,k)=X(n-s+1,n,m,k)-X(\ell,n,m,k)$, $2\mathcal{V}_{n;\ell,s}(m,k)=X(\ell,n,m,k)+X(n-s+1,n,m,k)$, $\mathcal{Q}_n(m,k)=\frac{X(n,n,m,k)}{X(1,n,m,k)}$, $\mathcal{P}_n(m,k)=|X(n,n,m,k)X(1,n,m,k)|$. Moreover, the asymptotic behavior of the statistic $\triangle_n(m,k|\underline{X})=\frac{X(\ell_{4;n},n,m,k)-X(\ell_{2;n},n,m,k)}{X(\ell_{3;n},n,m,k)-X(\ell_{1;n},n,m,k)}$ is investigated. Throughout this section "$X_n \overset{w}{\underset{n}{=}} Y_n$" means that the rv's X_n and Y_n have the same limit df.

Theorem 5. *Let the conditions of Theorems 1-3 be satisfied. Then,*

$$P(\mathcal{R}^*_{n;\ell,s}(m,k)\le x)=P(b^{-1}_{n,m:r}(\mathcal{R}_{n;\ell,s}(m,k)-a_{n,m:r})\le x) \xrightarrow[n]{w} \bar{\Gamma}_{R_s}(e^{-(m+1)x})*\bar{\Gamma}_{\ell}(e^{-\sqrt{m+1}x}), \tag{16}$$

where $b_{n,m:r}=b_{n,m}$ *and* $a_{n,m:r}=a_{n,m}-\tilde{a}_{n,m}$, *if* $r_n\log n \xrightarrow[n]{} \tau\ge 0$, *while* $b_{n,m:r}=b_{n,m}\sqrt{1-r_n}$ *and* $a_{n,m:r}=a_{n,m}-\tilde{a}_{n,m}$, *if* $r_n\log n \xrightarrow[n]{} \infty$. *Furthermore,*

$$P(\mathcal{V}^*_{n;\ell,s}(m,k)\le x)=P(b^{-1}_{n,m:v}(\mathcal{V}_{n;\ell,s}(m,k)-a_{n,m:v})\le x) \xrightarrow[n]{w} \begin{cases} \Psi^{(m,k)}_s(x)*\tilde{\Psi}^{(m,k)}_{\ell}(\sqrt{m+1}x), \\ \text{where } b_{n,m:v}=\frac{1}{2}b_{n,m} \text{ and } a_{n,m:v}=\frac{1}{2}(a_{n,m}+\tilde{a}_{n,m}), \quad \text{if } r_n\log n \xrightarrow[n]{} \tau\ge 0, \\ \Phi_{0,2}(x), \\ \text{where } b_{n,m:v}=\sqrt{r_n} \text{ and } a_{n,m:v}=\frac{\sqrt{1-r_n}}{2}(a_{n,m}+\tilde{a}_{n,m}), \quad \text{if } r_n\log n \xrightarrow[n]{} \infty. \end{cases} \tag{17}$$

In addition,

$$P(\mathcal{Q}^*_n(m,k)\le x)=P(b^{-1}_{n,m:q}(\mathcal{Q}_n(m,k)-a_{n,m:q})\le x) \xrightarrow[n]{w} \bar{\Psi}^{(m,k)}_1(-x)*\bar{\bar{\tilde{\Psi}}}^{(m,k)}_1\left(\frac{-4(m+1)x}{4+(m+1)\log(m+1)}\right), \tag{18}$$

where $b_{n,m:q}=\frac{b_{n,m}}{\tilde{a}_{n,m}}$ *and* $a_{n,m:q}=\frac{a_{n,m}}{\tilde{a}_{n,m}}$, *if* $r_n\log n \xrightarrow[n]{} \tau\ge 0$, *and*

$$P(\mathcal{P}^*_n(m,k)\le x)=P(b^{-1}_{n,m:p}(\mathcal{P}_n(m,k)-a_{n,m:p})\le x) \xrightarrow[n]{w} \Psi^{(m,k)}_1(x)*\bar{\Psi}^{(m,k)}_1\left(\frac{-4(m+1)x}{4+(m+1)\log(m+1)}\right), \tag{19}$$

where $b_{n,m:p}=b_{n,m}\tilde{a}_{n,m}$ *and* $a_{n,m:p}=a_{n,m}\tilde{a}_{n,m}$, *if* $r_n\log n \xrightarrow[n]{} \tau\ge 0$. *Finally, let the condition of Theorem 4 be satisfied. Then, if* $nr_n \xrightarrow[n]{} \tau, 0\le\tau\le\infty$, *we have*

$$\triangle_n(m,k|\underline{X}) \overset{w}{\underset{n}{=}} \triangle_n(m,k|\underline{Y}). \tag{20}$$

Proof. The proof of the relations (16) and (20) follows immediately from the representation (1) and the definition of the statistics $\mathcal{R}_{n;\ell,s}(m,k)$ and

$\triangle_n(m,k|\underline{Y})$ and by using the results of [11] for the limit relation (16) and [12] for the limit relation (20). The relation (17) follows from the representation

$$2\mathcal{V}^{*}_{n;\ell,s}(m,k) \overset{w}{\underset{n}{=}} \begin{cases} \frac{X(n-s+1,n,m,k)-a_{n,m}}{b_{n,m}} + \eta^{-1}_{n,m}\left(\frac{X(\ell,n,m,k)-\tilde{a}_{n,m}}{\tilde{b}_{n,m}}\right), & \text{if } r_n \log n \underset{n}{\longrightarrow} \tau \geq 0, \\ \frac{X(n-s+1,n,m,k)-\sqrt{1-r_n}a_{n,m}}{\sqrt{r_n}} + \left(\frac{X(\ell,n,m,k)-\sqrt{1-r_n}\tilde{a}_{n,m}}{\sqrt{r_n}}\right), & \text{if } r_n \log n \underset{n}{\longrightarrow} \infty, \end{cases}$$

where $\eta_{n,m} = \frac{b_{n,m}}{\tilde{b}_{n,m}} \underset{n}{\longrightarrow} \sqrt{m+1}$, if $r_n \log n \underset{n}{\longrightarrow} \tau \geq 0$ and by applying Corollaries 1 and 2. For proving the relation (18), take $b_{n,m:q} = \frac{b_{n,m}}{\tilde{a}_{n,m}}$ and $a_{n,m:q} = \frac{a_{n,m}}{\tilde{a}_{n,m}}$, then

$$\begin{aligned} \mathcal{Q}^{\star}_n(m,k) &= \frac{X^{\star}(n,n,m,k) - (b^{-1}_{n,m}\tilde{b}_{n,m}|a_{n,m}\tilde{a}^{-1}_{n,m}|)X^{\star}(1,n,m,k)}{|\tilde{a}_{n,m}|^{-1}X(1,n,m,k)} \\ &= \frac{X^{\star}(n,n,m,k) - \zeta_{n,m}X^{\star}_n(1,n,m,k)}{|\tilde{a}_{n,m}|^{-1}X(1,n,m,k)}, \end{aligned}$$

where

$$X^{\star}(n,n,m,k) = b^{-1}_{n,m}(X(n,n,m,k) - a_{n,m})$$

and

$$X^{\star}(1,n,m,k) = \tilde{b}^{-1}_{n,m}(X(1,n,m,k) - \tilde{a}_{n,m}).$$

On the other hand, we have $a_{n,m} \uparrow \infty$ and $\tilde{a}_{n,m} \downarrow -\infty$, as $n \to \infty$ (cf. [15]). Thus, on account of Lemma 3.3 in [3], $|\tilde{a}_{n,m}|^{-1}X(1,n,m,k) \overset{p}{\underset{n}{\longrightarrow}} -1$, After some algebra, we get, for sufficiently large n, the following representation

$$\mathcal{Q}^{\star}_n(m,k) \overset{w}{\underset{n}{=}} -X^{\star}(n,n,m,k) - \zeta_{n,m}X^{\star}(1,n,m,k). \tag{21}$$

Now, the representation (21) combined with the results of Corollaries 1 and 2 and the fact that $\zeta_{n,m} \underset{n}{\longrightarrow} \frac{4+(m+1)\log(m+1)}{4(m+1)}$ prove the claimed relation (18). Finally, to prove the relation (19) we take $b_{n,m:p} = b_{n,m}|\tilde{a}_{n,m}|$, $a_{n,m:p} = |a_{n,m}\tilde{a}_{n,m}|$. After simple arrangements, we get, for sufficiently large n, the following representation

$$\begin{aligned} \mathcal{P}^{\star}_n(m,k) \overset{w}{\underset{n}{=}} & -\frac{\tilde{b}_{n,m}}{|\tilde{a}_{n,m}|}X^{\star}(n,n,m,k)X^{\star}(1,n,m,k) \\ & + X^{\star}(n,n,m,k) - \zeta_{n,m}X^{\star}(1,n,m,k). \end{aligned}$$

On the other hand, since $\frac{\tilde{b}_{n,m}}{|\tilde{a}_{n,m}|} \to 0$, we get the asymptotic relation $\mathcal{P}^{\star}_n(m,k) \overset{w}{\underset{n}{=}} X^{\star}(n,n,m,k) - \zeta_{n,m}X^{\star}(1,n,m,k)$, which yields the relation (19). This completes the proof of Theorem 5. □

Theorem 5 reveals an interesting fact that the limit df's of the generalized quasi-ranges, as well as the ratio of the symmetric differences, of $m-$gos, $m \geq$

-1, in i.i.d rv's and in a stationary Gaussian sequence of rv's are the same. In addition, the limiting form of the df of all statistics given in Theorem 5, with the exception of the statistic $\triangle_n(m,k|\underline{X})$ depends on the relation of r_n to $\log n$. On the other hand, if $\triangle_n(m,k|\underline{Y})$ weakly converges, then the convergence of the sequence nr_n (to a finite or infinite limit) guarantees the convergence of $\triangle_n(m,k|\underline{X})$ to the same limit as $\triangle_n(m,k|\underline{Y})$. The limit df of $\triangle_n(m,k|\underline{Y})$ was extensively studied in [12].

If we put $\tau = 0$ in Theorem 5, it easy to verify that $\mathcal{R}^*_{n;\ell,s}(0,k) \underset{n}{\overset{w}{=}} \mathcal{V}^*_{n;\ell,s}(0,k)$ and $\mathcal{Q}^*_{n;\ell,s}(0,k) \underset{n}{\overset{w}{=}} \mathcal{P}^*_{n;\ell,s}(0,k)$. Moreover, in this case, by virtue of Theorems 1 and 2 (and their Corollaries 1 and 2), we can deduce that the suitably normalized generalized quasi-ranges, quasi-mid-ranges, extremal quotient and extremal product are asymptotically equivalent to the limit of a linear combination of the lower and upper m-gos.

4. APPLICATIONS

Theorems 1-5 reduce the limitations of some statistical methodologies (*e.g.*, in survival analysis and clinical trials) in different contexts. Here we consider two practical examples.

In the first example, we consider the $(n-r+1)$-out-of-n system, where the life-length distribution of the remaining components may change after each failure of the components. In literature, one of the most efficient model that describes such system is the sos model, where we may take the original df of the ith component ($i = 1, 2, ..., n$), before beginning the test, as $F_i(x) = 1-(1-\Phi(x))^{\alpha_i}, \alpha_i > 0$. Clearly, this model is a m-gos model, with $m_i = (n-i+1)\alpha_i - (n-i)\alpha_{i+1} + 1$ (cf. [17]). On the other hand, all Theorems 1-5 study a more general situation, when these components constitute a Gaussian sequence with zero expectation, unit variance and correlation $r_n > 0$, *i.e.*, they are dependent (rather than independent).

Another important practical example is the type II censoring scheme, where in a life-testing experiment, n items are placed on the test. The failure times observed from such a life-test, $X(1,n,0,1) \leq X(2,n,0,1) \leq ... \leq X(n,n,0,1)$, are the oos based on i.i.d rv's from a continuous df F. However, one may not continue the experiment until the last failure since the waiting time for the final failure may be unbounded (cf. [20]). For this reason, in some cases, the life-testing experiment is usually terminated when the Mth failure $X(M,n,0,1)$ is observed, which is referred to as a type-II censoring scheme. We call this scheme "the classical type-II censoring scheme", whenever the basic rv's are i.i.d. Clearly, this censoring model saves time and cost.

This classical model is considered as the special case of m-gos model, with $\gamma_i = 2n + M - j + 1$, $M \leq n$ is an integer number and the test on the components terminates at the Mth failure. On the other hand, this model is also a special case of the progressive type II censored order statistics with censoring scheme $(R_1, ..., R_M)$, where $R_1 = ... = R_{M-1} = 0$ and $R_M = n - M$. Now, in this classical model, let all the lifetimes of the components be i.i.d normal variates (although the normal df is not used as often in reliability work, it can represent severe wear-out mechanism, rapidly increasing hazard function, *e.g.* filament bulbs, IC wire bonds, see [19]). Then, Theorem 2 reveals the asymptotic behavior of any two observed failures of order ℓ and s, where $1 \leq \ell < s \leq M \leq n$, in a general situation than the classical model, that the lifetimes of the components constitute a Gaussian sequence with correlation $r_n > 0$, *i.e.*, they are dependent. This situation happens practically, when the items in the test constitute a Gaussian sequence of large number of identical parts of a certain machine, where the inter-correlation between them depends only on their total number.

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