ON THE NONEXISTENCE OF BLOW UP SOLUTIONS TO $\Delta^{\frac{\alpha}{2}} u = u^\gamma$ IN THE UNIT BALL

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We investigate the nonexistence of positive blow up boundary solutions to $\Delta^{\frac{\alpha}{2}} u = u^\gamma$ in the unit ball of $\mathbb{R}^d$.

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1. INTRODUCTION

We consider, for $\gamma > 0$, the fractional semilinear elliptic problem

$$(1)\qquad \begin{cases} \Delta^{\frac{\alpha}{2}} u = u^\gamma \ \text{ in } \ B \\ u > 0 \ \text{ on } \ B \\ u = 0 \ \text{ on } \ B^c := \mathbb{R}^d \backslash B \\ \lim\limits_{|x|\to 1} (1-|x|)^{1-\frac{\alpha}{2}} u(x) = \infty, \end{cases}$$

where $\Delta^{\frac{\alpha}{2}} := -(-\Delta)^{\frac{\alpha}{2}}$, $0 < \alpha < 2$, is the fractional power of the classical Laplacian and B is the unit ball of $\mathbb{R}^d$, $d > \alpha$. Solutions of this problem are understood in the distributional sense and are called *blow up (boundary) solutions.*

Formally taking $\alpha = 2$, it is well known that problem (1) possesses at least one solution if and only if $\gamma > 1$. However, for $0 < \alpha < 2$, this problem does not fully resolved as yet. By way of illustration, we give a brief account of the results obtained. The existence of blow up solutions has recently been investigated in [6], see also [1, 2]. The authors proved that if $1 + \alpha < \gamma < \frac{2+\alpha}{2-\alpha}$ then problem (1) has at least one solution. For $0 < \gamma < 1 + \frac{\alpha}{2}$, it was proved in [1] that problem (1) has no solutions. The ranges $1 + \frac{\alpha}{2} \le \gamma \le 1 + \alpha$ and $\gamma \ge \frac{2+\alpha}{2-\alpha}$ are still open. Due to the nonlocal character of $\Delta^{\frac{\alpha}{2}}$, classical techniques used in the study of problem (1) for $\alpha = 2$; see for instance [9, 13], are not applicable for $0 < \alpha < 2$ in general. This obstacle makes the above open ranges encouraging enough to merit further investigation.

Our contribution in this direction is to prove that problem (1) has no solutions for

$$0 < \gamma < 1 + \alpha \quad \text{or} \quad \frac{2 + 2\alpha}{2 - \alpha} \leq \gamma.$$

The question of whether problem (1) has a solution when $\gamma = 1 + \alpha$ or $\frac{2+\alpha}{2-\alpha} \leq \gamma < \frac{2+2\alpha}{2-\alpha}$ remains unanswered. Our proofs make substantial use of explicit formulas of the Green function G_B^α and the Poisson kernel K_B^α of the unit ball B. This method has the advantage of being easily explained. However, it seems to be of little help when we replace the unit ball B by an arbitrary domain of $\mathbb{R}^d$. We think that for general domains refinements of the ideas exploited in this paper will essentially still work to give similar results, but we have no proof of this.

2. PRELIMINARIES AND MAIN RESULTS

Let $0 < \alpha < 2$ and $d > \alpha$. We denote by $\mathcal{L}_\alpha$ the set of all Borel measurable functions $u : \mathbb{R}^d \to [-\infty, +\infty]$ such that

(2)
$$\int_{\mathbb{R}^d} \frac{|u(y)|}{(1 + |y|^2)^{\frac{d}{2}+\frac{\alpha}{2}}}\, \mathrm{d}y < \infty.$$

The fractional Laplacian $\Delta^{\frac{\alpha}{2}}$ on $\mathbb{R}^d$ is defined, for $u \in \mathcal{C}^2(\mathbb{R}^d) \cap \mathcal{L}_\alpha$, by

$$\begin{aligned}\Delta^{\frac{\alpha}{2}} u(x) &= \mathcal{A}_{d,-\alpha} P.V \int_{\mathbb{R}^d} \frac{u(x+y) - u(x)}{|y|^{d+\alpha}}\, \mathrm{d}y \\ &= \mathcal{A}_{d,-\alpha} \lim_{\varepsilon \to 0} \int_{\{|y| \geq \varepsilon\}} \frac{u(x+y) - u(x)}{|y|^{d+\alpha}}\, \mathrm{d}y,\end{aligned}$$

where

$$\mathcal{A}_{d,-\alpha} = 2^\alpha \Gamma(\frac{d+\alpha}{2}) / (\pi^{d/2} |\Gamma(-\frac{\alpha}{2})|).$$

For $u \in \mathcal{L}_\alpha$, we define $\Delta^{\frac{\alpha}{2}} u$ as a distribution on the space $\mathcal{C}_c^\infty(\mathbb{R}^d)$ of all real-valued infinitely differentiable functions on $\mathbb{R}^d$ with compact support by

$$\Delta^{\frac{\alpha}{2}} u(\varphi) := \int_{\mathbb{R}^d} u(x) \Delta^{\frac{\alpha}{2}} \varphi(x)\, \mathrm{d}x.$$

Definition 1. Let $u \in \mathcal{L}_\alpha \cap L^\infty_{loc}(B)$. We say that u is a solution of $\Delta^{\frac{\alpha}{2}} u = u^\gamma$ in B if

(3)
$$\int_{\mathbb{R}^d} u(x)\, \Delta^{\frac{\alpha}{2}} \varphi(x)\, \mathrm{d}x = \int_B u^\gamma(x)\, \varphi(x)\, \mathrm{d}x$$

holds for every nonnegative function $\varphi \in \mathcal{C}_c^\infty(B)$. Supersolution and subsolution have to be understood in the same way replacing " $=$ " in (3) by " $\leq$ " and " $\geq$ " respectively.

Remark 1. 1. If $u = 0$ on B^c then the condition $u \in \mathcal{L}_\alpha$ simply means that $u \in L^1(B)$ the set of all Lebesgue integrable functions on B. So, solutions of problem (1), if there are any, should be in $L^1(B)$.

2. In the above definition, the conditions $u \in \mathcal{L}_\alpha$ and $u \in L^\infty_{loc}(B)$ are necessary to make sense of left and right integrals in (3) respectively.

Let D be a regular bounded open set. For every nonnegative function $f \in \mathcal{C}(D^c) \cap \mathcal{L}_\alpha$, we denote by $H^\alpha_D f$ the unique nonnegative continuous extension of f on $\mathbb{R}^d$ such that $\Delta^{\frac{\alpha}{2}} u = 0$ on D, see [11]. The α-harmonic measure relative to x and D, which will be denoted by $H^\alpha_D(x, \cdot)$, is defined to be the positive Radon measure on D^c given by the mapping $f \mapsto H^\alpha_D f(x)$. It was proved in [5] that $H^\alpha_D(x, \cdot)$, $x \in D$, is concentrated on $\overline{D}^c$ and is absolutely continuous with respect to the Lebesgue measure on D^c. Furthermore, the corresponding density function $K^\alpha_D(x, y)$, $x \in D$, $y \in D^c$, is continuous in $(x, y) \in D \times \ \overline{D}^c$. The explicit formula of the Poisson kernel $K^\alpha_{B_r}$ for balls $B_r := \{x \in \mathbb{R}^d;\ |x| < r\}$ is given in [4] by

$$K^\alpha_{B_r}(x, y) = C_{d,\alpha} \left(\frac{r^2 - |x|^2}{|y|^2 - r^2} \right)^{\frac{\alpha}{2}} \frac{1}{|x - y|^d}\,;\ \ |x| < r \ \text{ and } \ |y| > r, \tag{4}$$

where

$$C_{d,\alpha} = \pi^{-1-d/2}\Gamma(d/2)\sin(\pi\alpha/2).$$

The Riesz kernel $G^\alpha_{\mathbb{R}^d}$ is given by

$$G^\alpha_{\mathbb{R}^d}(x, y) = \frac{\mathcal{A}_{d,\alpha}}{|x - y|^{d-\alpha}}.$$

The Green kernel G^α_D of D is defined by $G^\alpha_D(x, y) = 0$ if x or y belongs to D^c and

$$G^\alpha_D(x, y) = G^\alpha_{\mathbb{R}^d}(x, y) - \int_{D^c} G^\alpha_{\mathbb{R}^d}(z, y) K^\alpha_D(x, z)\,\mathrm{d}z\,;\ \ x, y \in D.$$

It is known explicitly only for few choices of D, namely, for the ball B_r :

$$G^\alpha_{B_r}(x, y) = \frac{\kappa_{d,\alpha}}{|x - y|^{d-\alpha}} \int_0^{\frac{(r^2-|x|^2)(r^2-|y|^2)}{|x-y|^2}} \frac{s^{\frac{\alpha}{2}-1}}{(1 + s)^{\frac{d}{2}}}\,\mathrm{d}s\,;\ x, y \in B_r, \tag{5}$$

where $\kappa_{d,\alpha} = \Gamma(d/2)/\left(2^\alpha \pi^{d/2}[\Gamma(\alpha/2)]^2\right)$, see [4, 10, 12]. Furthermore, the following scaling property holds

$$G^\alpha_{B_r}(x, y) = r^{\alpha-d} G^\alpha_{B_1}\left(\frac{x}{r}, \frac{y}{r}\right);\ x, y \in B_r. \tag{6}$$

However, many important properties of $G^\alpha_D(x, y)$ are well known. We record some of them which can already be found in [4, 10, 11]. The mapping $(x, y) \longmapsto G^\alpha_D(\cdot, \cdot)$ is symmetric, positive and continuous except along the diagonal as a mapping from $D \times D$ into $]0, \infty]$. For every $y \in D$ and every $z \in \partial D$, $\lim_{x \to z} G_D^\alpha(x, y) = 0$. Furthermore,

$$\Delta^{\frac{\alpha}{2}} G_D^\alpha(x, \cdot) = -\varepsilon_x, \tag{7}$$

where ε_x is the Dirac measure at the point $x \in D$.

Lemma 1. *Let $u \in \mathcal{L}_\alpha \cap L^\infty_{loc}(B)$. Then u is a solution of $\Delta^{\frac{\alpha}{2}} u = u^\gamma$ in B if and only if*

$$u(x) + G_{B_r}^\alpha(u^\gamma)(x) = H_{B_r}^\alpha u(x) \tag{8}$$

for every $x \in B_r := \{x \in \mathbb{R}^d;\ |x| < r\}$ and every $0 < r < 1$.

Proof. Let $0 < r < 1$ and define $h(x) := u(x) + G_{B_r}^\alpha(u^\gamma)(x)$. $G_{B_r}^\alpha(u^\gamma) \in \mathcal{C}_0(B_r)$ since u is bounded on B_r. This implies that $h = u$ on B_r^c and hence $H_{B_r}^\alpha h = H_{B_r}^\alpha u$. On the other hand, using (7), for every $\varphi \in \mathcal{C}_c^\infty(B_r)$, we have

$$\begin{aligned}\int_{\mathbb{R}^d} h(x)\Delta^{\frac{\alpha}{2}}\varphi(x)\,\mathrm{d}x &= \int_{\mathbb{R}^d} u(x)\Delta^{\frac{\alpha}{2}}\varphi(x)\,\mathrm{d}x + \int_{B_r} G_{B_r}^\alpha(u^\gamma)(x)\Delta^{\frac{\alpha}{2}}\varphi(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} u(x)\Delta^{\frac{\alpha}{2}}\varphi(x)\,\mathrm{d}x - \int_{B_r} u^\gamma(x)\varphi(x)\,\mathrm{d}x.\end{aligned}$$

Therefore, $\Delta^{\frac{\alpha}{2}} u = u^\gamma$ in B_r if and only if $\Delta^{\frac{\alpha}{2}} h = 0$ in B_r, and hence $h = H_{B_r}^\alpha h = H_{B_r}^\alpha u$ as desired. □

Remark 2. Solutions of $\Delta^{\frac{\alpha}{2}} u = u^\gamma$ in B are continuous in B since the functions $H_{B_r}^\alpha u$, $G_{B_r}^\alpha(u^\gamma) \in \mathcal{C}(B_r)$ and , by (8), $u = H_{B_r}^\alpha u - G_{B_r}^\alpha(u^\gamma)$ for every $0 < r < 1$. So, in virtue of this remark and Remark 1, solutions of problem (1) have to be understood as functions in $\mathcal{C}(B) \cap L^1(B)$.

Before giving our first nonexistence result, we first need the following preparatory technical lemma. The proof use basic properties of the Gaussian hypergeometric function $F(a, b; c; \cdot)$ which can be found in [8].

Lemma 2. *Let v be the function defined on B by $v(x) = (1 - |x|^2)^{-1-\alpha}$. Then there exists a constant $C > 0$ such that, for every $0 < r < 1$,*

$$G_{B_r}^\alpha(v)(0) \geq C\, r^\alpha\, (1 - r^2)^{-\frac{\alpha}{2}}. \tag{9}$$

Proof. By (5), we have

$$G_B^\alpha(0, y) = \kappa_{d,\alpha}\, |y|^{\alpha - d} \int_0^{\frac{1-|y|^2}{|y|^2}} \frac{\lambda^{\frac{\alpha}{2}-1}}{(\lambda+1)^{\frac{d}{2}}}\mathrm{d}\lambda;\ \ |y| < 1.$$

By changing the variable $s = |y|\sqrt{1+\lambda}$, we get

$$G_B^\alpha(0, y) = 2\kappa_{d,\alpha} \int_{|y|}^1 s^{1-d}(s^2 - |y|^2)^{\frac{\alpha}{2}-1}\,\mathrm{d}s.$$

Then, using the scaling property (6) and the spherical coordinates, we obtain

$$\begin{aligned} G^{\alpha}_{B_r}(v)(0) &= r^{\alpha-d}\int_{B_r} G_B(0,\frac{y}{r})v(y)\mathrm{d}y \\ &= C_1\, r^{\alpha-d}\int_0^r t^{d-1}v(t)\int_{\frac{t}{r}}^1 s^{1-d}(s^2-\frac{t^2}{r^2})^{\frac{\alpha}{2}-1}\mathrm{d}s\mathrm{d}t \\ &= C_1\int_0^r t^{d-1}v(t)\int_t^r s^{1-d}(s^2-t^2)^{\frac{\alpha}{2}-1}\mathrm{d}s\mathrm{d}t \\ &= C_1\int_0^r s^{1-d}\int_0^s (s^2-t^2)^{\frac{\alpha}{2}-1}t^{d-1}v(t)\,\mathrm{d}t\mathrm{d}s, \end{aligned} \tag{10}$$

where

$$C_1 := \frac{4\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}\kappa_{d,\alpha}.$$

On the other hand,

$$\begin{aligned} \int_o^s t^{d-1}\,(s^2-t^2)^{\frac{\alpha}{2}-1}v(t)\,\mathrm{d}t &= \frac{1}{2}s^{\alpha+d-2}\int_0^1 t^{\frac{d}{2}-1}(1-t)^{\frac{\alpha}{2}-1}(1-s^2t)^{-1-\alpha}\,\mathrm{d}t \\ &= C_2\, s^{d+\alpha-2}F(1+\alpha,\frac{d}{2};\frac{d+\alpha}{2};s^2) \\ &= C_2\, s^{d+\alpha-2}(1-s^2)^{-\frac{\alpha}{2}-1}F(\frac{d-\alpha-2}{2},\frac{\alpha}{2};\frac{d+\alpha}{2};s^2) \\ &\geq C_2\,C_3\, s^{d+\alpha-2}(1-s^2)^{-\frac{\alpha}{2}-1}, \end{aligned}$$

where

$$C_2 := \frac{\Gamma(\frac{d}{2})\Gamma(\frac{\alpha}{2})}{2\Gamma(\frac{d+\alpha}{2})} \quad\text{and}\quad C_3 := \inf_{0\leq s\leq 1} F(\frac{d-\alpha-2}{2},\frac{\alpha}{2};\frac{d+\alpha}{2};s) > 0.$$

Now, plugging the last inequality into (10), we obtain

$$\begin{aligned} G^{\alpha}_{B_r}(v)(0) &\geq C_1C_2C_3\int_0^r s^{\alpha-1}(1-s^2)^{-\frac{\alpha}{2}-1}\mathrm{d}s \\ &= \frac{2C_1C_2C_3}{\alpha}\, r^{\alpha}\, F(1+\frac{\alpha}{2},\frac{\alpha}{2};1+\frac{\alpha}{2};r^2) \\ &= \frac{2C_1C_2C_3}{\alpha}\, r^{\alpha}\,(1-r^2)^{-\frac{\alpha}{2}}. \end{aligned}$$

This completes the proof by taking $C := 2\alpha^{-1}C_1C_2C_3$. □

We now give our first nonexistence result.

THEOREM 3. *If $\gamma \geq \frac{2+2\alpha}{2-\alpha}$ then problem (1) has no solution.*

Proof. Aiming for a contradiction, suppose that problem (1) admits a solution $u \in \mathcal{C}(B)\cap L^1(B)$. The fact that the function $(1-|x|^2)^{1-\frac{\alpha}{2}}u(x)$ is continuous on B and blow up at the boundary ∂B asserts that its overall minimum on B, which we denote by m, is attained. Moreover, $m > 0$ since $u > 0$ on B. Thus, for every $x \in B$,

$$\begin{aligned} u^\gamma(x) &\geq \left(\frac{m}{(1-|x|^2)^{1-\frac{\alpha}{2}}} \right)^\gamma \\ &\geq \frac{m^\gamma}{(1-|x|^2)^{1+\alpha}} =: m^\gamma\, v(x). \end{aligned} \tag{11}$$

Let $0 < r < 1$. By applying the Green operator $G^\alpha_{B_r}$ on both sides of (11), we get $G^\alpha_{B_r}(u^\gamma) \geq G^\alpha_{B_r}(v)$ on B_r. In particular,

$$G^\alpha_{B_r}(u^\gamma)(0) \geq G^\alpha_{B_r}(v)(0)$$

which leads using (9) to

$$G^\alpha_{B_r}(u^\gamma)(0) \geq m^\gamma\, C\, r^\alpha (1-r^2)^{-\frac{\alpha}{2}}.$$

Then, taking $x = 0$ in (8), there holds

$$H^\alpha_{B_r} u(0) \geq m^\gamma\, C\, r^\alpha (1-r^2)^{-\frac{\alpha}{2}}.$$

On the other hand, using (4) and the spherical coordinates,

$$H^\alpha_{B_r} u(0) = C_{d,\alpha}\, r^\alpha \int_r^1 \frac{w(s)}{s(s^2-r^2)^{\frac{\alpha}{2}}} \mathrm{d}s,$$

where

$$w(s) := \int_{\partial B} u(sy)\, \sigma(\mathrm{d}y).$$

Thus,

$$C_{d,\alpha} \int_r^1 \frac{w(s)}{s(s^2-r^2)^{\frac{\alpha}{2}}} \mathrm{d}s \geq m^\gamma\, C\, (1-r^2)^{-\frac{\alpha}{2}}.$$

Now, multiplying both sides by $1/r(r^2-\lambda^2)^{1-\alpha/2}$ and integrating from λ to 1, we obtain

$$C_{d,\alpha} \int_\lambda^1 \frac{w(s)}{s^{1+\alpha}} \mathrm{d}s \geq m^\gamma\, C.$$

Here, we used the fact that, for $0 < a < b$ and $0 < \nu < 1$,

$$\int_a^b \frac{\mathrm{d}t}{t(t^2-a^2)^\nu (b^2-t^2)^{1-\nu}} = \frac{\Gamma(\nu)\Gamma(1-\nu)}{2} b^{2\nu-2} a^{-2\nu}.$$

Therefore, for every $0 < \lambda < 1$,

$$\int_\lambda^1 \frac{w(s)}{s^{1+\alpha}} \mathrm{d}s \geq \frac{m^\gamma\, C}{C_{d,\alpha}}.$$

This implies, in particular, that

$$\lim_{\lambda \to 1} \int_\lambda^1 \frac{w(s)}{s^{1+\alpha}} \mathrm{d}s \geq \frac{m^\gamma\, C}{C_{d,\alpha}}$$

which is a contradiction since

$$\lim_{\lambda\to 1}\int_\lambda^1 \frac{w(s)}{s^{1+\alpha}}\mathrm{d}s = \lim_{\lambda\to 1}\int_{B\setminus B_\lambda} \frac{u(x)}{|x|^{d+\alpha}}\,\mathrm{d}x = 0$$

justified by the fact that $u \in L^1(B)$. □

Our next investigation is about the nonexistence of solutions to problem (1) for $0 < \gamma < 1+\alpha$. In all the following, we assume that

$$0 < \gamma < 1+\alpha.$$

It was proved in [2] that, for every integer $n \geq 1$, the problem

$$\begin{cases} \Delta^{\frac{\alpha}{2}} u = u^\gamma \ \text{ in } \ B \\ u = 0 \ \text{ on } \ B^c \\ \lim\limits_{|x|\to 1} (1-|x|^2)^{1-\frac{\alpha}{2}} u(x) = n \end{cases}$$

admits one and only one solution $u_n \in \mathcal{C}^+(B) \cap L^1(B)$. Furthermore, for every $x \in B$,

$$u_n(x) + G_B^\alpha(u_n^\gamma)(x) = n\,(1-|x|^2)^{\frac{\alpha}{2}-1}. \tag{12}$$

Lemma 4. *Let u be a solution of problem (1). Then, for every $n \geq 1$,*

$$u_n \leq u. \tag{13}$$

Proof. Let $n \geq 1$. Define $w_n = u - u_n$ and suppose that the open set

$$V = \{x \in B; w_n(x) < 0\}$$

is not empty. Then $\Delta^{\frac{\alpha}{2}} w_n = u^\gamma - u_n^\gamma \leq 0$ on V which means that w_n is α-superharmonic on V. By the blow up boundary conditions on u and u_n, we have

$$\lim_{|x|\to 1} (1-|x|^2)^{\frac{\alpha}{2}-1} w_n(x) = \infty$$

from which we deduce the existence of $0 < r < 1$ such that $w_n(x) \geq 0$ for every $r \leq |x| < 1$. This entails in particular that $V \subset B_r$ and hence w_n is continuous on $\overline{V}$. Furthermore, $w_n \geq 0$ on B^c since $u = u_n = 0$ on B^c. Then, the minimum principle for α-superharmonic functions as stated in [14, Proposition 2.17] yields $w_n \geq 0$ in V, a contradiction. Therefore V is empty and hence $u_n \geq u$ in B. □

By (12), we have $u_n(x) \leq n(1-|x|^2)^{\frac{\alpha}{2}-1}$ and thus

$$\Delta^{\frac{\alpha}{2}} u_n = u_n^\gamma \leq n^{\gamma-1}(1-|x|^2)^{(\gamma-1)(\frac{\alpha}{2}-1)}\,u.$$

This means that u_n is a supersolution of the Schrödinger equation

$$\Delta^{\frac{\alpha}{2}} u = n^{\gamma-1}\,q(x)\,u \ \text{ on } \ B,$$

where

$$q(x) := (1-|x|^2)^{(\gamma-1)(\frac{\alpha}{2}-1)}.$$

For every integer $n \geq 1$, we consider the Schrödinger problem

$$\begin{cases} \Delta^{\frac{\alpha}{2}} u = n^{\gamma-1}\, q(x)\, u \;\; \text{in} \;\; B \\ u = 0 \;\; \text{on} \;\; B^c \\ \lim\limits_{|x|\to 1} (1-|x|^2)^{1-\frac{\alpha}{2}} u(x) = n-1, \end{cases}$$

The function q is in the Kato class $K^\alpha(B)$ since $(\gamma-1)(\frac{\alpha}{2}-1) < \alpha$, see [3, Example 1]. Then, it follows from [3, Theorem A] that the above Schrödinger problem has one and only one solution $v_n \in \mathcal{C}^+(B) \cap L^1(B)$. Furthermore,

$$(n-1)\,(1-|x|^2)^{\frac{\alpha}{2}-1} e^{-n^{\gamma-1}\, S1(x)} \leq v_n(x)\;;\;\; x \in B, \tag{14}$$

where

$$\begin{aligned} S1(x): &= (1-|x|^2)^{1-\frac{\alpha}{2}} \int_B G_B^\alpha(x,y) q(y) (1-|y|^2)^{\frac{\alpha}{2}-1}\, \mathrm{d}y \\ &= (1-|x|^2)^{1-\frac{\alpha}{2}} \int_B G_B^\alpha(x,y) (1-|y|^2)^{\gamma(\frac{\alpha}{2}-1)}\, \mathrm{d}y. \end{aligned}$$

It should be noted that (a more general version of) (14) is given at the end of the proof of Theorem A and not in the statements.

LEMMA 5. *For every integer $n \geq 1$,*

$$v_n \leq u_n. \tag{15}$$

Proof. Let $n \geq 1$. Define $w_n = u_n - v_n$ and suppose that the open set

$$V = \{x \in B; w_n(x) < 0\}$$

is not empty. Then $\Delta^{\frac{\alpha}{2}} w_n \leq n^{\gamma-1}\, q(x)\, u_n - n^{\gamma-1}\, q(x)\, v_n \leq 0$ on V which means that w_n is α-superharmonic on V. On the other hand, the fact that

$$\lim_{|x|\to 1} (1-|x|^2)^{1-\frac{\alpha}{2}} w_n(x) = 1$$

yields $\overline{V} \subset B$, and hence w_n is continuous on $\overline{V}$. Since $w_n \geq 0$ on V^c, the minimum principle for α-superharmonic functions entails $w_n \geq 0$ in V, a contradiction. Therefore V is empty and hence $v_n \leq u_n$. □

Combining (13), (14) and (15), we obtain the following result which is essential in the proof of our next nonexistence result.

PROPOSITION 1. *Let u be a solution of problem (1). Then, for every integer $n \geq 1$,*

$$(n-1)\,(1-|x|^2)^{\frac{\alpha}{2}-1} e^{-n^{\gamma-1}\, S1(x)} \leq u(x)\;;\;\; x \in B. \tag{16}$$

The following lemma provide an important sharp estimates of $S1(x)$.

LEMMA 6. *There exist two constants* $0 < c_1 < c_2$ *such that, for every* $x \in B$,

$$c_1\,(1-|x|^2)^{1+\frac{\alpha}{2}-\gamma(1-\frac{\alpha}{2})} \le S1(x) \le c_2\,(1-|x|^2)^{1+\frac{\alpha}{2}-\gamma(1-\frac{\alpha}{2})}. \tag{17}$$

Proof. Let $\beta := \gamma(1-\frac{\alpha}{2})-\alpha$ and let u_β be the function defined on $\mathbb{R}^d$ by

$$u_\beta(x) = (1-|x|^2)^{-\beta} \text{ if } x \in B \quad \text{and} \quad u_\beta(x) = 0 \text{ if } x \in B^c.$$

By [7, Theorem 1], for every $x \in B$,

$$\Delta^{\frac{\alpha}{2}} u_\beta(x) = (\alpha+2\beta-2)\frac{\mathcal{A}_{d,-\alpha}\Gamma(1-\frac{\alpha}{2})\Gamma(1-\beta)\pi^{\frac{d}{2}}}{\alpha\Gamma(\frac{d}{2})\Gamma(2-\beta-\frac{\alpha}{2})} F(\beta+\frac{\alpha}{2},\frac{d+\alpha}{2};\frac{d}{2};|x|^2).$$

Using the hypothesis $\gamma < 1+\alpha$, it is easy to check that $u_\beta \in L^1(B)$. Moreover, $\alpha+2\beta-2<0$, and hence

$$C(\alpha,\beta,d) := -(\alpha+2\beta-2)\frac{\mathcal{A}_{d,-\alpha}\Gamma(1-\frac{\alpha}{2})\Gamma(1-\beta)\pi^{\frac{d}{2}}}{\alpha\Gamma(\frac{d}{2})\Gamma(2-\beta-\frac{\alpha}{2})} > 0.$$

By the Euler transformation of the Gaussian hypergeometric functions, we obtain

$$\Delta^{\frac{\alpha}{2}} u_\beta(x) = -C(\alpha,\beta,d)(1-|x|^2)^{-\alpha-\beta} F(\frac{d}{2}-\beta-\frac{\alpha}{2},-\frac{\alpha}{2};\frac{d}{2};|x|^2).$$

Since $F(\frac{d}{2}-\beta-\frac{\alpha}{2},-\frac{\alpha}{2};\frac{d}{2};\cdot)$ is monotone on the interval $[0,1]$, there holds

$$-C_2\,(1-|x|^2)^{-\alpha-\beta} \le \Delta^{\frac{\alpha}{2}} u_\beta(x) \le -C_1\,(1-|x|^2)^{-\alpha-\beta}, \tag{18}$$

where

$$C_1 := C(\alpha,\beta,d)\min\left(F(\frac{d}{2}-\beta-\frac{\alpha}{2},-\frac{\alpha}{2};\frac{d}{2};0)\,;\,F(\frac{d}{2}-\beta-\frac{\alpha}{2},-\frac{\alpha}{2};\frac{d}{2};1)\right)$$

and

$$C_2 := C(\alpha,\beta,d)\max\left(F(\frac{d}{2}-\beta-\frac{\alpha}{2},-\frac{\alpha}{2};\frac{d}{2};0)\,;\,F(\frac{d}{2}-\beta-\frac{\alpha}{2},-\frac{\alpha}{2};\frac{d}{2};1)\right).$$

Now, we apply the Green operator G_B^α in (18) to obtain

$$\frac{1}{C_2}\,u_\beta(x) \le \int_B G_B^\alpha(x,y)(1-|y|^2)^{-\alpha-\beta}\mathrm{d}y \le \frac{1}{C_1}\,u_\beta(x).$$

The proof of (17) concludes by multiplying by $(1-|x|^2)^{1-\frac{\alpha}{2}}$ and by observing that $\alpha+\beta = \gamma(1-\frac{\alpha}{2})$. $\square$

We now give our second nonexistence result.

THEOREM 7. *If* $0 < \gamma < 1 + \alpha$ *then problem (1) has no solutions.*

Proof. Suppose, towards a contradiction, that (1) has a solution u. Then (16) combined with (17) gives, for $x \in B$ and $n \geq 1$,

$$(n-1)\,(1-|x|^2)^{\frac{\alpha}{2}-1} \exp\Big(-n^{\gamma-1}\,c_2\,(1-|x|^2)^{1+\frac{\alpha}{2}-\gamma(1-\frac{\alpha}{2})}\Big) \leq u(x). \tag{19}$$

We denote by $k_n(x)$ the left hand side of (19). If $0 < \gamma \leq 1$ then $\lim_{n\to\infty} k_n(x) = \infty$, and hence $u \equiv \infty$ on B, a contradiction. If $1 < \gamma < 1 + \alpha$ then

$$\int_B k_n(x)\,\mathrm{d}x \longrightarrow \infty \quad \text{as} \quad n \to \infty, \tag{20}$$

which leads, using (19), to the contradiction $u \notin L^1(B)$. So, it remains to prove (20). Let

$$\beta := 1 + \frac{\alpha}{2} - \gamma(1 - \frac{\alpha}{2}).$$

Then, using spherical coordinates, we obtain

$$\begin{aligned}
\int_B k_n(x)\,\mathrm{d}x &= \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}(n-1)\int_0^1 t^{d-1}(1-t^2)^{\frac{\alpha}{2}-1}\, e^{-n^{\gamma-1}c_2(1-t^2)^{\beta}}\,\mathrm{d}t \\
&= \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}(n-1)\int_0^1 (1-s)^{\frac{d}{2}-1}\, s^{\frac{\alpha}{2}-1}\, e^{-n^{\gamma-1}c_2\, s^{\beta}}\,\mathrm{d}s \\
&\geq \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}(n-1)\int_0^{n^{-\frac{\gamma-1}{\beta}}} (1-s)^{\frac{d}{2}-1}\, s^{\frac{\alpha}{2}-1}\, e^{-n^{\gamma-1}c_2\, s^{\beta}}\,\mathrm{d}s \\
&\geq \frac{e^{-c_2}\,\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}\,(1-n^{-\frac{\gamma-1}{\beta}})(n-1)\int_0^{n^{-\frac{\gamma-1}{\beta}}} s^{\frac{\alpha}{2}-1} \\
&= \frac{2\,e^{-c_2}\,\pi^{\frac{d}{2}}}{\alpha\,\Gamma(\frac{d}{2})}\,(1-n^{-\frac{\gamma-1}{\beta}})(n-1)\,n^{-\frac{\alpha(\gamma-1)}{2\beta}}.
\end{aligned}$$

This leads to (20) by observing that

$$(1-n^{-\frac{\gamma-1}{\beta}}) \longrightarrow 1 \quad \text{and} \quad (n-1)\,n^{-\frac{\alpha(\gamma-1)}{2\beta}} \longrightarrow \infty \quad \text{as} \quad n \longrightarrow \infty,$$

since $1 < \gamma$, $0 < \beta$ and

$$\begin{aligned}
1 - \frac{\alpha(\gamma-1)}{2\beta} &= \frac{2\beta - \alpha(\gamma-1)}{2\beta} \\
&= \frac{2 + \alpha - \gamma(2-\alpha) - \alpha\gamma + \alpha}{2\beta} \\
&= \frac{1+\alpha-\gamma}{\beta} > 0
\end{aligned}$$

by hypothesis. This completes the proof of the theorem. □

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