GENERALIZED TRIBONACCI DIOPHANTINE QUADRUPLES

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Let $(t_n)_{n\geq 0}$ be defined by the recurrence $t_n = At_{n-1} + t_{n-2} + t_{n-3}$ with $t_0 = 0,\ t_1 = 1,\ t_2 = A$ and $A \geq 2$ integer. In this paper, we prove that there does not exist integers $1 \leq a_1 < a_2 < a_3 < a_4$ such that $a_1a_2 + 1$, $a_2a_3 + 1$, $a_3a_4 + 1$ and $a_1a_4 + 1$ are Tribonacci numbers.

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1. INTRODUCTION

A Diophantine $m-$tuple is a set $\{a_1, a_2, \ldots, a_n\}$ of positive integers such that $a_ia_j + 1$ is a square for all $1 \leq i < j \leq m$. Diophantus found a rational quadruple $\{\frac{1}{16}, \frac{33}{16}, \frac{68}{16}, \frac{105}{16}\}$. Fermat found the $\{1, 3, 8, 120\}$ as the first Diophantine quadruple. The numbers $1, 3, 8$ in Fermat's set can be viewed as three consecutive Fibonacci numbers with even subscript.

A famous conjecture is that no quintuple exists. A closely related theorem of Dujella states that there are only finitely many quintuples.

A variant of the problem is obtained if one replaces the squares by the terms of a given binary recurrence. Firstly, Luca and Szalay [7, 8] put Fibonacci and Lucas numbers instead of the squares and showed that there is no Fibonacci Diophantine triple and $\{1, 2, 3\}$ is the only Lucas Diophantine triples. Alp, Irmak and Szalay [1] proved afterwards that there are no balancing Diophantine triples. Fuchs, Luca and Szalay [3] gave the necessary conditions to have the finitely many solutions in the case of replacing the member of second order sequence instead of squares. Moreover, Irmak and Szalay [5] showed that there is no triple $0 < a < b < c$ satisfying the following system of equations

$$\begin{aligned} ab + 1 &= u_x \\ ac + 1 &= u_y \\ bc + 1 &= u_z \end{aligned}$$

such that $u_n = Au_{n-1} - u_{n-2}$ with $u_0 = 0$ and $u_1 = 1$ where $A \neq 2$ is an integer. In the sequel, Gómez and Luca [4] solved the following system of equations

$$a_1a_2 + 1 = T_x$$

$$\begin{aligned} a_2a_3+1 &= T_y \\ a_3a_4+1 &= T_z \\ a_1a_4+1 &= T_w \end{aligned}$$

and found no quadruple $0 < a_1 < a_2 < a_3 < a_4$ where T_x, T_y, T_z and T_w are Tribonacci numbers.

In this paper, we generalize the problem of Ruiz and Luca. Namely, we investigate the solutions of the following equation system

$$\begin{aligned} a_1a_2+1 &= t_x \\ a_2a_3+1 &= t_y \\ a_3a_4+1 &= t_z \\ a_1a_4+1 &= t_w \end{aligned} \tag{1}$$

where $0 < a_1 < a_2 < a_3 < a_4$ are integers. Before going further, we define the generalized Tribonacci sequence $(t_n)_{n\geq 0}$ by the following recurrence relation

$$t_n = At_{n-1} + t_{n-2} + t_{n-3} \tag{2}$$

with the initial conditions $t_0 = 0,\ t_1 = 1$ and $t_2 = A$ where A is a positive integer.

Our result is the following,

THEOREM 1.1. *There is no solution of the equation system (1) with $A \geq 2$.*

2. PRELIMINARIES

Before proceeding further, some considerations will be needed for the convenience of the reader. First lemma is about the Binet form of the terms of the sequence $(t_n)_{n\geq 0}$.

LEMMA 2.1. *The Binet-type formula of the generalized Tribonacci sequence $(t_n)_{n\geq 0}$ is*

$$t_n = \sum_{k=1}^{3} g(k, A)\,\alpha_k^n \tag{3}$$

where $g(k,A) = \frac{\alpha_k - A}{(A+3)\alpha_k^2 - A(A+5)\alpha_k + (A-1)(2A+1)}$ and the elements α_1, α_2 and α_3 are the roots of the characteristic equation $x^3 - Ax^2 - x - 1 = 0$.

Proof. The terms of the sequence $(t_n)_{n\geq 0}$ can be expressed by linear combination of the roots of the characteristic equation $x^3 - Ax^2 - x - 1 = 0$. Namely,

$$t_n = X\alpha_1^n + Y\alpha_2^n + Z\alpha_3^n. \tag{4}$$

Applying the initial conditions $t_0 = 0,\ t_1 = 1$ and $t_2 = A$ to the equation (4), we obtain the coefficient $X = \frac{\alpha_1}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}$. Then

$$\begin{aligned} X &= \frac{\alpha_1}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)} = \frac{\alpha_1}{\alpha_1^2-\alpha_1(\alpha_2+\alpha_3)+\alpha_2\alpha_3} \\ &= \frac{\alpha_1}{\alpha_1^2-\alpha_1(A-\alpha_1)+\left(-1-A\alpha_1+\alpha_1^2\right)} = \frac{1}{\alpha_2\alpha_3\left(3\alpha_1^2-2A\alpha_1-1\right)} \\ &= \frac{\alpha_1-A}{(A+3)\alpha_1^2-A(A+5)\alpha_1+(A-1)(2A+1)} \end{aligned}$$

follows. The other coefficients can be found by similar way. □

When we put $A = 1$ in the equation (3), it coincides with the Binet formula of 3−generalized Fibonacci sequence which is given by the paper of Dresden and Du [2].

LEMMA 2.2. *Let $A, s \in \mathbb{N}$, $s \geq 2$ and let*

$$f(x) = x^s - Ax^{s-1} - x^{s-2} - \cdots - x - 1.$$

Then

(a) *$f(x)$ has exactly one positive simple root $\alpha \in \mathbb{R}$ with $A < \alpha < A+1$*

(b) *the remaining $s-1$ roots of $f(x)$ lie within the unit circle in the complex plane.*

Proof. See Lemma 2 in [6]. □

Since $|\alpha_2| = |\alpha_3| < 1$, then the contribution of the roots α_2 and α_3 will quickly become trivial in the equation (3). That is,

$$t_n \cong g(1, A)\,\alpha_1^n \tag{5}$$

holds for n sufficiently large.

The next lemma presents the upper and lower bounds for the terms of the sequence $\{t_n\}$.

LEMMA 2.3. *Let $A \geq 2$ and $\alpha = \alpha_1$ be the dominant root of the equation $x^3 - Ax^2 - x - 1 = 0$. Then the following inequality*

$$\alpha^{n-2} \leq t_n < \alpha^{n-0.51} \tag{6}$$

holds for $n \geq 1$.

Proof. It is obvious that $\alpha^{n-2} \leq t_n$. Since $t_n = g(1, A)\,\alpha^n + g(2, A)\,\alpha_2^n + g(3, A)\,\alpha_3^n$, then

$$t_n = g(1, A)\,\alpha^n \left(1 + \frac{g(2, A)}{g(1, A)}\left(\frac{\alpha_2}{\alpha}\right)^n + \frac{g(3, A)}{g(1, A)}\left(\frac{\alpha_3}{\alpha}\right)^n\right) \leq 2g(1, A)\,\alpha^n$$

follows. So, we get that

$$t_n < 2g(1,A)\,\alpha^n = \alpha^{n+\log_\alpha 2g(1,A)}.$$

As the function $\log_\alpha 2g(1,A)$ is decreasing (since $\frac{d}{d\alpha}\log_\alpha 2g(1,A) < 0$), then it takes the maximum value at $A = 2$. So, we have

$$t_n < \alpha^{n-0.51}$$

as claimed. □

Lemma 2.4. *Let $A \geq 2$. For $n \geq 1$, the following inequality holds*

$$|t_n - g(1,A)\,\alpha^n| \leq 0.52.$$

Proof. Let $E_n = t_n - g(1,A)\,\alpha^n$. Since $t_n = At_{n-1} + t_{n-2} + t_{n-3}$ and $\alpha^n = A\alpha^{n-1} + \alpha^{n-2} + \alpha^{n-3}$ hold, then we have $E_n = AE_{n-1} + E_{n-2} + E_{n-3}$ for $n \geq 3$. Now, we will follow the method of the proof of Theorem 2 in the paper of Dresden of Du [2]. Assume that $|E_n| > 0.52$ for $n \geq 3$. Let n_0 be the smallest positive such integer n. Namely, $|E_{n_0}| > 0.52 > |E_{n_0-k}|$ holds for $k \geq 1$. Together with the equations

$$E_{n_0+1} = AE_{n_0} + E_{n_0-1} + E_{n_0-2}$$

and

$$E_{n_0} = AE_{n_0-1} + E_{n_0-2} + E_{n_0-3},$$

then we have

$$E_{n_0+1} = (A+1)\,E_{n_0} + (1-A)\,E_{n_0-1} - E_{n_0-3}.$$

This equation gives that

$$\begin{aligned}(A+1)\,|E_{n_0}| &= |E_{n_0+1} + (A-1)\,E_{n_0-1} + E_{n_0-3}| \\ &\leq |E_{n_0+1}| + (A-1)\,|E_{n_0-1}| + |E_{n_0-3}|. \end{aligned} \tag{7}$$

We obtain the followings by the inequality (7)

$$(A-1)\,(|E_{n_0}| - |E_{n_0-1}|) + |E_{n_0}| - |E_{n_0-3}| \leq |E_{n_0+1}| - |E_{n_0}|.$$

Since $|E_{n_0}| - |E_{n_0-1}| > 0$ and $|E_{n_0}| - |E_{n_0-3}| > 0$, then we obtain that $|E_{n_0+1}| - |E_{n_0}| > 0$. If we apply this argument repeatedly, we get

$$|E_{n_0+i}| > \cdots > |E_{n_0+1}| > |E_{n_0}| \geq 0.52.$$

But this contradicts with the observation from equation (5) since the error terms must converge to 0. We conclude that $|E_n| \leq 0.52$. □

Lemma 2.5. *Let $A \geq 2$ be an integer. For $n > m \geq 4$ and $k \in \{1,2\}$, t_n satisfies*

$$\frac{t_m - 1}{t_{m-k} - 1} > \frac{t_n - 1}{t_{n-k} - 1}.$$

Proof. Let $s(n,m) = (t_m - 1)(t_{n-k} - 1) - (t_{m-k} - 1)(t_n - 1)$. Our aim is to show that $s(n,m) > 0$. Applying induction on n, we have the followings

$$\begin{aligned} & s(n,m) + s(n+1,m) + A \times s(n+2,m) \\ = & \det\begin{pmatrix} t_m - 1 & t_{m-k} - 1 \\ t_n - 1 & t_{n-k} - 1 \end{pmatrix} + \det\begin{pmatrix} t_m - 1 & t_{m-k} - 1 \\ t_{n+1} - 1 & t_{n+1-k} - 1 \end{pmatrix} \\ & + A\det\begin{pmatrix} t_m - 1 & t_{m-k} - 1 \\ t_{n+2} - 1 & t_{n+2-k} - 1 \end{pmatrix} \\ = & \det\begin{pmatrix} t_m - 1 & t_{m-k} - 1 \\ t_n + t_{n+1} - 2 & t_{n-k} + t_{n-k+1} - 2 \end{pmatrix} \\ & + \det\begin{pmatrix} t_m - 1 & t_{m-k} - 1 \\ At_{n+2} - A & t_{n+2-k} - A \end{pmatrix} \\ = & \det\begin{pmatrix} t_m - 1 & t_{m-k} - 1 \\ t_{n+3} - (A+2) & t_{n+3-k} - (A+2) \end{pmatrix} > 0. \end{aligned}$$

We conclude that $(t_m - 1)(t_{n+3-k} - 1) - (t_{m-k} - 1)(t_{n+3} - 1) > (A+1)(t_m - t_{m-k})$. Since $(t_m - t_{m-k}) > 0$, then we deduce that $s(n+3,m) = (t_m - 1)(t_{n+3-k} - 1) - (t_{m-k} - 1)(t_{n+3} - 1) > 0$ as claimed. □

Now we are ready to deal with the proof of the theorems.

3. PROOF OF THEOREM 1.1

3.1. An upper bound for the integer z

Since $1 \le a_1 < a_2 < a_3 < a_4$, then $t_x = a_1 a_2 + 1 \ge 3$ follows. Define the function l_A as follows:

$$l_A = \begin{cases} 2, & \text{if } A \ge 3; \\ 3, & \text{if } A = 2. \end{cases}$$

So, we have that $l_A \le x$.

In the sequel, we obtain the equation

$$(t_x - 1)(t_z - 1) = (t_y - 1)(t_w - 1) \tag{8}$$

by the equation (1). Using the upper and lowers bounds for the terms of the sequence $(t_n)_{n\ge 0}$, we get the followings

$$\alpha^{x+z-4} \le (t_x - 1)(t_z - 1) < \alpha^{y+w-1.02}$$

and

$$\alpha^{y+w-4} \le (t_x - 1)(t_z - 1) < \alpha^{x+z-1.02}.$$

Combining these inequalities yields

$$|(x+z) - (y+w)| \le 2.$$

So, we have two possible cases $(x+z)=(y+w)$ and $(x+z)\neq(y+w)$. Note that, we can see easily that

$$l_A \leq x < \lambda = \min\{y,w\} \leq \delta = \max\{y,w\} < z.$$

since $1 \leq a_1 < a_2 < a_3 < a_4$.

Case I: $(x+z)=(y+w)$

The terms of the sequence $(t_n)_{n\geq 0}$ can be expressed by $t_n = g_1\alpha^n + h(n)$ where $g_1 = g(1,A)$ and $|h(n)| < 0.52$. Expanding the equation (8), the equation

$$t_x t_z - t_\lambda t_\delta = t_x + t_z - t_\lambda - t_\delta$$

gives that

$$\begin{aligned} \left|g_1\alpha^z h(x) - g_1\alpha^z\right| &= \Big|-g_1\alpha^x h(z) - h(x)h(z) + g_1\alpha^\lambda h(\delta) \\ &\quad + g_1\alpha^\delta h(\lambda) + h(\lambda)h(\delta) + h(z) + h(x) \\ &\quad - h(\lambda) - h(\delta) + g_1\alpha^x - g_1\alpha^\lambda - g_1\alpha^\delta\Big|. \end{aligned}$$

Divide both side by $g_1\alpha^z$. Then

$$\begin{aligned} |h(x)-1| &< \frac{|h(x)h(z)| + |h(\lambda)h(\delta)| + |h(z)| + |h(x)| + |h(\lambda)| + |h(\delta)|}{g_1\alpha^z} \\ &\quad + \frac{g_1\alpha^x|h(z)| + g_1\alpha^\lambda|h(\delta)| + g_1\alpha^\delta|h(\lambda)|}{g_1\alpha^z} \\ &\quad + \frac{g_1\alpha^x + g_1\alpha^\lambda + g_1\alpha^\delta}{g_1\alpha^z} \\ &< \frac{2.63}{g_1\alpha^z} + \frac{1.56}{\alpha^{z-\delta}} + \frac{3}{\alpha^{z-\delta}} < \frac{7.19}{\alpha^{z-\delta}} \end{aligned}$$

follows, where we used the facts $\frac{1}{g_1} < \alpha^2$ and $g_1 < 1$. The inequality $|h(x)-1| > 0.48$ yields that $z-\delta \leq 2$. So we can write that $z = \delta + k$ where $k \in \{1,2\}$. Applying this fact to the equation (8) together with $x+z = \delta+\lambda$, we obtain that

$$(t_x - 1)(t_{\delta+k} - 1) = (t_\delta - 1)(t_{x+k} - 1).$$

However, this contradicts with Lemma 2.5.

Case II: $(x+z)\neq(y+w)$

Combining the equation (8) together with $t_n = g_1\alpha^n + h(n)$, then we get

$$\begin{aligned} g_1^2\alpha^{x+z} - g_1^2\alpha^{\lambda+\delta} &= g_1\alpha^x(1-h(z)) + g_1\alpha^z(1-h(x)) \\ &\quad + g_1\alpha^\lambda(h(\delta)-1) + g_1\alpha^\delta(h(\lambda)-1) \\ &\quad + h(x) + h(z) - h(\lambda) - h(\delta) \\ &\quad + h(\lambda)h(\delta) - h(x)h(z). \end{aligned}$$

When we divide both sides by the term $g_1^2\alpha^{x+z}$, then

$$\begin{aligned}\left|1-\alpha^{-(x+z-\lambda-\delta)}\right| &< \frac{1.52}{g_1}\left(\frac{1}{\alpha^z}+\frac{1}{\alpha^x}+\frac{\alpha^{\lambda-z}}{\alpha^x}+\frac{\alpha^{\delta-z}}{\alpha^x}\right)+\frac{2.63}{g_1^2\alpha^{x+z}}\\ &< \frac{1}{g_1^2}\left(1.52g_1\frac{2+\alpha^{-1}+\alpha^{-2}}{\alpha^x}+2.63\frac{\alpha^{-4}}{\alpha^x}\right)<\frac{1.6}{g_1^2\alpha^x}<\frac{1.6}{\alpha^{x-4}}\end{aligned}$$

follows. We used the facts $\lambda-z\le-2$, $\delta-z\le-1$, $4\le z$, $\frac{1}{g_1^2}<\frac{1}{\alpha^{-4}}$ and $|g_1|<0.4$. So, the inequality

$$0.6<\min_{|x+z-\lambda-\delta|\le2}\left|1-\alpha^{-(x+z-\lambda-\delta)}\right|<\frac{1.6}{\alpha^{x-4}}$$

gives that $l_A\le x\le5$.

Now, we rewrite the equation (8) as follows:

$$(t_x-1)\,t_z-t_\lambda t_\delta=t_x-t_\lambda-t_\delta.$$

This equation yields that

$$\begin{aligned}(t_x-1)\,g_1\alpha^z-g_1^2\alpha^{\lambda+\delta} &= g_1\alpha^\lambda h(\delta)+g_1\alpha^\delta h(\lambda)+h(\lambda)\,h(\delta)\\ &\quad -h(z)\,(t_x-1)-g_1\alpha^\lambda-h(\lambda)\\ &\quad -g_1\alpha^\delta-h(\delta)+(t_x-1)+1\\ &= g_1\alpha^\lambda\,(h(\delta)-1)+g_1\alpha^\delta\,(h(\lambda)-1)\\ &\quad -(t_x-1)\,(h(z)-1)+h(\lambda)\,h(\delta)-h(\lambda)-h(\delta)+1.\end{aligned}$$

When we divide both sides by the term $(t_x-1)\,g_1\alpha^z$, we obtain the followings together with the fact $(t_x-1)\ge2$

$$\begin{aligned}\left|1-g_1\alpha^{\lambda+\delta-z}\,(t_x-1)^{-1}\right| &< \frac{|h(\delta)-1|}{(t_x-1)\,\alpha^{z-\lambda}}+\frac{|h(\lambda)-1|}{(t_x-1)\,\alpha^{z-\delta}}+\frac{|h(z)-1|}{g_1\alpha^z}\\ &\quad+\frac{|h(\lambda)\,h(\delta)-h(\lambda)-h(\delta)+1|}{g_1\,(t_x-1)\,\alpha^z}\\ &\le \frac{1.52}{\alpha^{z-\delta}}+\frac{1.52}{\alpha^{z-2}}+\frac{1.16}{\alpha^{z-2}}<\frac{4.2}{\alpha^{z-\delta}}.\end{aligned}$$

where we used the facts $\frac{1}{g_1}\le\alpha^2$ and $g_1>1$ for $A\ge2$ integer. Since $|\lambda+\delta-x-z|\le2$ and $y+w\ne x+z$ holds, then we write that $\lambda+\delta-z=x+\varepsilon$ where $\varepsilon\in\{\pm1,\pm2\}$. Then

$$\min_{\substack{\varepsilon\in\{\pm1,\pm2\}\\ l_A\le x\le5}}\left|1-g_1\alpha^{x+\varepsilon}\,(t_x-1)^{-1}\right|<\frac{4.2}{\alpha^{z-\delta}}$$

follows. If $A=2$, then $x=3$ yields that

$$\min_{\varepsilon\in\{\pm1,\pm2\}}\left|1-g_1\alpha^{3+\varepsilon}\,(t_3-1)^{-1}\right|>0.5.$$

If $A \geq 3$, then $x \geq 2$ gives that

$$\min_{\substack{\varepsilon\in\{\pm1,\pm2\}\\ 2\leq x\leq 5}} \left|1-g_1\alpha^{x+\varepsilon}\left(t_x-1\right)^{-1}\right| > 0.56.$$

Therefore, we have

$$0.5<\min_{\substack{\varepsilon\in\{\pm1,\pm2\}\\ l_A\leq x\leq 5}} \left|1-g_1\alpha^{x+\varepsilon}\left(t_x-1\right)^{-1}\right| < \frac{4.2}{\alpha^{z-\delta}}$$

follows. This fact yields that $z \leq \delta+2$.

By the equation

$$\frac{\left(t_x-1\right)\left(t_z-1\right)}{t_\delta-1}=t_\lambda-1$$

together with Lemma 2.3 and the fact $z\leq\delta+2$, we get

$$\begin{aligned}\alpha^{\lambda-2} &< t_\lambda-1=\frac{\left(t_x-1\right)\left(t_z-1\right)}{t_\delta-1}<\alpha^{z-\delta+1.49}+\alpha^{x-0.51}\\ &< \alpha^{3.49}+\alpha^{2.49}<\alpha^{3.49+2.48}=\alpha^{5.97}\end{aligned}$$

which yields that $3\leq\lambda\leq 7$.
The equation (8) gives that

$$\left(t_x-1\right)t_z-\left(t_\lambda-1\right)t_\delta=t_x-t_\lambda.$$

Put $t_n=g_1\alpha^n+h\left(n\right)$ for the indices δ and z, we have the followings

$$\left(t_x-1\right)g_1\alpha^z-\left(t_\lambda-1\right)g_1\alpha^\delta=\left(t_\lambda-1\right)\left(h\left(\delta\right)-1\right)+\left(t_x-1\right)\left(1-h\left(z\right)\right).$$

Divide both sides by $\left(t_\lambda-1\right)g_1\alpha^\delta$. Then

$$\text{(9)}\quad \left|1-\left(t_x-1\right)\left(t_\lambda-1\right)^{-1}\alpha^{z-\delta}\right|\leq\frac{\left|h\left(\delta\right)-1\right|}{g_1\alpha^\delta}+\frac{t_x-1}{t_\lambda-1}\frac{\left|h\left(z\right)-1\right|}{\alpha^{\delta-2}}<\frac{3.04}{\alpha^{\delta-2}}$$

follows. We used that facts $\frac{t_x-1}{t_\lambda-1}\leq 1$ and $\frac{1}{g_1}<\alpha^2$. Since $z-\delta=\lambda-x+\varepsilon$ and $\varepsilon\in\{\pm1,\pm2\}$, then we have that

$$\min_{\substack{l_A\leq x\leq 5\\ x+1\leq\lambda\leq 7\\ \varepsilon\in\{\pm1,\pm2\}}} \left|1-\left(t_x-1\right)\left(t_\lambda-1\right)^{-1}\alpha^{\lambda-x+\varepsilon}\right|<\frac{3.04}{\alpha^{\delta-2}}.$$

The case $A=2$ gives that $x=3$. Then

$$\text{(10)}\qquad 0.6<\min_{\substack{4\leq\lambda\leq 7\\ \varepsilon\in\{\pm1,\pm2\}}} \left|1-\left(t_3-1\right)\left(t_\lambda-1\right)^{-1}\alpha^{\lambda-3+\varepsilon}\right|$$

follows. If $A\geq 3$ integer, then it yields that

$$\text{(11)}\qquad 0.7<\min_{\substack{2\leq x\leq 5\\ x+1\leq\lambda\leq 7\\ \varepsilon\in\{\pm1,\pm2\}}} \left|1-\left(t_x-1\right)\left(t_\lambda-1\right)^{-1}\alpha^{\lambda-x+\varepsilon}\right|.$$

Together with the inequalities (10), (11) and (9), we obtain

$$0.6 < \min_{\substack{l_A \le x \le 5 \\ x+1 \le \lambda \le 7 \\ \varepsilon \in \{\pm 1, \pm 2\}}} \left|1 - (t_x - 1)(t_\lambda - 1)^{-1} \alpha^{\lambda - x + \varepsilon}\right| < \frac{3.04}{\alpha^{\delta - 2}}$$

gives that $\delta \le 3$.

By the equation system (1), one can see that

$$t_z - 1 < (t_\lambda - 1)(t_\delta - 1)$$

which gives that $z \le \lambda + \delta$. So, $z \le 10$ since $\lambda \le 7$ and $\delta \le 3$.

By the definition of the l_A function, there are two possibilities as follows:

(1) If $A \ge 3$, $2 \le x < \lambda \le \delta < z \le 10$ Since we find $\delta \le 3$, then only the case $x = 2$, $\delta = \lambda = 3$ and $4 \le z \le 10$ must hold.

(2) If $A = 2$, $x = 3$, $\lambda = 4$, $\delta = 5$, and $4 \le z \le 10$. However, this case impossible since $\delta \le 3$.

3.2. **An upper bound for the integer A**

Up to now, we prove that $x = 2$, $\lambda = \delta = 3$ and $4 \le z \le 10$ is the only possible case for the equation system (1). Now, we follow the key argument in the paper [5] to find the upper bound for A. The terms of the sequence $\{t_n\}$ are monic polynomials in A with integer coefficients. By the equation system (1), the term

$$a_1 a_2 = \sqrt{\frac{(t_2 - 1)(t_3 - 1)^2}{(t_z - 1)}}$$

must be integer. By the polynomial division, there uniquely exist polynomials $q(A)$ and $r(A)$ with integer coefficients such that

$$(t_2 - 1)(t_3 - 1)^2 = q(A)(t_z - 1) + r(A)$$

holds. For the case $4 \le z \le 10$, we obtain the following table.

z	$q(A)$	$r(A)$
4	$2A^2 + 4A$	$A^2 - A - 2$
5	$-3A^3 + A^2 + 2A$	$A - 1$
6	$-4A^3 - 3A^2 - 3A - 1$	1
7	0	$A^5 - A^4$
8	0	$A^5 - A^4$
9	0	$A^5 - A^4$
10	0	$A^5 - A^4$

Table 1: Remainder and quotient polynomials

Checking the eligible possibilities for $x = 2,\ \lambda = \ \delta = 3$ and $4 \leq z \leq 10$, we observe that $r(A)$ is never zero and there is no positive integer A such that $r(A) = 0$ for $A \geq 3$. Hence,

$$\frac{(t_x - 1)(t_\lambda - 1)(t_\delta - 1)}{t_z - 1} = q(A) + \frac{r(A)}{t_z - 1}, \tag{12}$$

the term $\frac{r(A)}{t_z - 1}$ never disappears in the equation (12). For some A, the right hand side of equation (12) is integer. But $\deg(r(A)) < \deg((t_z - 1))$, so A cannot be large since

$$\lim_{A\to\infty} \frac{r(A)}{t_z - 1} = 0.$$

Then, $|r(A)| > t_z - 1$ must hold which yields that $A \leq A_0$ for some positive integer A_0. We use Mathematica programme to find A_0 for $x = 2,\ \lambda = \delta = 3$ and $4 \leq z \leq 10$. Then, we find that $A_0 = 2$. But this contradicts with our assumption $A \geq 3$. Therefore, this gives the proof of Theorem 1.1.

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