Dedicated to Marius Tucsnak on the occasion of his 60^{th} anniversary

OPTIMAL BOUNDARY CONTROL OF THE WAVE EQUATION: THE FINITE-TIME TURNPIKE PHENOMENON

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It is well-known that vibrating strings can be steered to a position of rest in finite time by suitably defined boundary control functions, if the time horizon is sufficiently long. In optimal control problems, the desired terminal state is often enforced by terminal conditions, that add an additional difficulty to the optimal control problem. In this paper we present an optimal control problem for the wave equation with a time-dependent weight in the objective function such that for a sufficiently long time horizon, the optimal state reaches a position of rest in finite time without prescribing a terminal constraint. This situation can be seen as a realization of the finite-time turnpike phenomenon that has been studied recently in [3].

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1. INTRODUCTION

Often in dynamic optimal control problems with a long time horizon, in a large neighbourhood of the middle of the time interval the optimal control and the optimal state are very close to the solution of a control problem that is derived from the dynamic optimal control problems by omitting the information about the initial state and possibly a desired terminal state. Often, the solution of this auxiliary problem is simpler than the solution of the original dynamic problem since it has a solution that is independent of time. We refer to this solution as the *turnpike*. The situation that the dynamic optimal state approaches the turnpike in the interior of the time interval is referred to as the turnpike phenomenon. The turnpike phenomenon has been studied in economics for a long time (see for example [2]). Recently, it has also been investigated for optimal control problems for systems governed by partial differential equations, see for example [4] for a turnpike result for a boundary control system governed by the wave equation or [5] for a turnpike result for linear quadratic optimal control of general evolution equations. The turnpike phenomenon for optimal boundary control problems with hyperbolic systems has also been studied in [6].

If the turnpike is reached after finite time, the situation is referred to as the finite-time turnpike phenomenon. The finite-time turnpike phenomenon has been studied for example in [7], [3].

Optimal boundary control problems for systems governed by the wave equation that include terminal constraints have been the subject of numerous studies, see for example [8] for Dirichlet controls, [9] and the references therein. Let us observe that under reasonable simplifications also models from engineering applications lead to systems governed by the wave equation, see for example the models for gas pipeline flow that are discussed in [10].

One way to avoid the terminal constraint is to replace it by a non-smooth penalty term in the objective function, that can enforce the desired terminal state for a sufficiently large penalty parameter, see [11]. Apart from the non-smoothness this requires a penalty term that depends only on the state at the terminal time, so this leads to additional difficulties in the solution of the optimal control problem.

In this paper we study an optimal control problem with a differentiable objective function that is given by a weighted L^2-norm with a time-dependent periodic weight function. We show that if the time-horizon is sufficiently long, the optimal state reaches the desired position of rest after finite time, although there is no terminal constraint in the optimal control problem and there is no terminal penalty term in the objective function.

2. STATEMENT OF THE OPTIMAL CONTROL PROBLEM

Let a length $L > 0$, the wave speed $c > 0$ and a time $T_0 > 2\frac{L}{c}$, $k \in \{1, 2, 3, ...\}$ and $T = k\,T_0$ be given. We introduce the T_0-periodic weight function $w(t)$ with $w(t) = T_0 - t$ for $t \in [0, T_0]$. So if $T_0 = 2\,\pi$, we have

$$w(t) = \pi + \sum_{k=1}^{\infty} \frac{2}{k} \sin(k\,t).$$

Figure 1 shows the graph of the T_0-periodic weight function $w(t)$ for $T_0 = 1$.

For $y \in C((0,T),\, H^1(0,\, L))$ and $t \in [0,T]$ let $E(t)$ denote the energy

$$E(t) = \frac{1}{2} \int_0^L (y_x(t,\, x))^2 + \frac{1}{c^2}\,(y_t(t,\, x))^2 \,\mathrm{d}x. \tag{1}$$

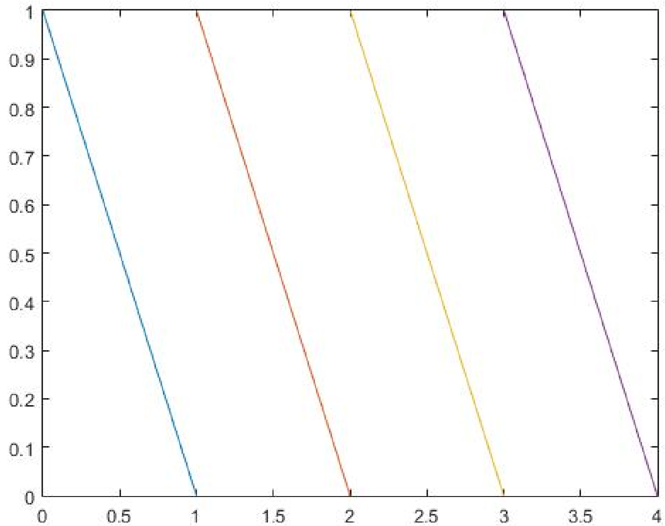

Fig. 1 – Shows the graph of the T_0-periodic weight function $w(t)$ for $T_0 = 1$.

For a parameter $\gamma \geq 0$, an initial position $y_0 \in H^1(0,\, L)$ with $y_0(0) = 0$ and an initial velocity $y_1 \in L^2(0,\, L)$ we consider the following problem of optimal Neumann-boundary control:

$$\mathbf{OCP}(T,\,\gamma)\begin{cases} \min\limits_{u\in L^2(0,T)} \int\limits_0^T E(t) + \frac{\gamma}{2}\, w(t)\, u^2(t)\,\mathrm{d}t \\ \text{subject to} \\ y(0,x) = y_0(x),\ y_t(0,x) = y_1(x),\ x \in (0,\, L) \\ y(t,0) = 0,\ y_x(t,L) = u(t),\ t \in (0,T) \\ y_{tt}(t,x) = c^2\, y_{xx}(t,x),\ (t,x) \in (0,T) \times (0,\, L). \end{cases}$$

Note that problem $\mathbf{OCP}(T,\,\gamma)$ has a quadratic objective functional. For the system that governs the state in $\mathbf{OCP}(T,\,\gamma)$, exact boundary controllability is only possible for $T > 2\frac{L}{c}$. This follows for example from Theorem 3.1 in [9], where the case that the control acts on both ends is considered.

Let $X(T) = H^1(0,\, L) \times L^2(0,\, L) \times L^2(0,T)$. The corresponding problem with free initial state and free terminal state is

$$\mathbf{Free}(T,\,\gamma)\begin{cases} \min\limits_{(y_0,y_1,u)\in X(T):y_0(0)=0} \int\limits_0^T E(t) + \frac{\gamma}{2}\, w(t)\, u^2(t)\,\mathrm{d}t \\ \text{subject to} \\ y(0,x) = y_0(x),\ y_t(0,x) = y_1(x),\ x \in (0,\, L) \\ y(t,0) = 0,\ y_x(t,L) = u(t),\ t \in (0,T) \\ y_{tt}(t,x) = c^2\, y_{xx}(t,x),\ (t,x) \in (0,T) \times (0,\, L). \end{cases}$$

The solution of $\mathbf{Free}(T,\gamma)$ is the position of rest with $u(t)=0$ for all $t\in[0,T]$ and $y(t,x)=0$ for $(t,x)\in(0,T)\times(0,L)$. Now we state our main result:

THEOREM 2.1. *For all $\gamma\geq 0$, $\mathbf{OCP}(T,\gamma)$ determines a unique optimal state where the energy $E(t)$ decays exponentially fast.*

For $\gamma=0$ and $T\geq T_0>2\,\frac{L}{c}$, the optimal state of $\mathbf{OCP}(T,\gamma)$ reaches a position of rest in the finite time $2\,\frac{L}{c}$, that is the optimal state has a finite-time turnpike property.

For the proof of Theorem 2.1 we need the following Lemma.

LEMMA 2.2. *Let $k\in\{1,2,3,...\}$, $T=k\,T_0$ and $u\in L^2(0,T)$ be given. Then we have*

$$\int_0^T w(t)\,u^2(t)\,\mathrm{d}t=\sum_{j=0}^{k-1}\int_{j\,T_0}^{(j+1)\,T_0}\left(\int_{j\,T_0}^{t}u^2(s)\,\mathrm{d}s\right)\mathrm{d}t. \tag{2}$$

Proof. For all $j\in\{0,...,k-1\}$ and $t\in[j\,T_0,\,(j+1)\,T_0]$ we have

$$w(t)=T_0-(t-j\,T_0)=(j+1)\,T_0-t$$

and thus

$$\int_{j\,T_0}^{(j+1)\,T_0}w(t)\,u^2(t)\,\mathrm{d}t=\int_{j\,T_0}^{(j+1)\,T_0}((j+1)T_0-t)\,u^2(t)\,\mathrm{d}t.$$

Integration by parts yields

$$\int_{j\,T_0}^{(j+1)\,T_0}((j+1)T_0-t)\,u^2(t)\,\mathrm{d}t$$

$$=\left[((j+1)T_0-t)\int_{j\,T_0}^{t}u^2(s)\,\mathrm{d}s\right]\Big|_{t=j\,T_0}^{(j+1)\,T_0}+\int_{j\,T_0}^{(j+1)\,T_0}\int_{j\,T_0}^{t}u^2(s)\,\mathrm{d}s\,\mathrm{d}t.$$

Since $\int_0^T w(t)\,u^2(t)\,\mathrm{d}t=\sum_{j=0}^{k-1}\int_{j\,T_0}^{(j+1)\,T_0}w(t)\,u^2(t)\,\mathrm{d}t$, the assertion follows. □

Proof of Theorem 2.1. For $u\in L^2(0,T)$, let

$$J(u)=\int_0^T E(t)+\frac{\gamma}{2}\,w(t)\,u^2(t)\,\mathrm{d}t$$

denote the objective function of the optimal control problem $\mathbf{OCP}(T,\gamma)$.

Due to Lemma 2.2, we can rewrite the objective function in the form

$$J(u) = \sum_{j=0}^{k-1} \int_{j\,T_0}^{(j+1)\,T_0} E(t) + \frac{\gamma}{2}\left(\int_{j\,T_0}^{t} u^2(s)\,\mathrm{d}s\right)\mathrm{d}t.$$

For all $j \in \{0, ..., k-1\}$ and $t \in [j\,T_0,\,(j+1)\,T_0]$, define the integrand

$$H^{u,j}(t) = E(t) + \frac{\gamma}{2} \int_{j\,T_0}^{t} u^2(s)\,\mathrm{d}s.$$

Then

$$J(u) = \sum_{j=0}^{k-1} \int_{j\,T_0}^{(j+1)\,T_0} H^{u,j}(t)\,\mathrm{d}t. \tag{3}$$

For a given control $u \in L^2(0,\,T)$, the function $H^{u,j}$ is differentiable almost everywhere on $(j\,T_0,\,(j+1)\,T_0)$ with respect to t, and we have

$$H_t^{u,j}(t) = E_t(t) + \frac{\gamma}{2} u^2(t).$$

Suppose that we have a control $\tilde{u} \in L^2(0,\,T)$, such that for all $u \in L^2(0,\,T)$ we have the inequality

$$H_t^{\tilde{u},j}(t) \le H_t^{u,j}(t) \tag{4}$$

almost everywhere on $(j\,T_0,\,(j+1)\,T_0)$ for all $j \in \{0, ..., k-1\}$. Note that due to the definition of H_u and since the initial energy $E(0)$ is determined by the initial state we have

$$H^{\tilde{u},0}(0) = E(0) = H^{u,0}(0).$$

Hence for all $j \in \{0, ..., k-1\}$ and $t \in (j\,T_0,\,(j+1)\,T_0)$ almost everywhere we have the inequality

$$H^{\tilde{u},j}(t) = E(0) + \sum_{l=0}^{j-1} \int_{l\,T_0}^{(l+1)\,T_0} H_t^{\tilde{u},l}(s)\,\mathrm{d}s + \int_{jT_0}^{t} H_t^{\tilde{u},j}(s)\,\mathrm{d}s$$

$$\le E(0) + \sum_{l=0}^{j-1} \int_{l\,T_0}^{(l+1)\,T_0} H_t^{u,l}(s)\,\mathrm{d}s + \int_{jT_0}^{t} H_t^{u,j}(s)\,\mathrm{d}s = H^{u,j}(t).$$

Due to (3) this implies

$$J(\tilde{u}) \le J(u).$$

Hence if $\tilde{u}$ satisfies (4), it solves problem $\mathbf{OCP}(T,\,\gamma)$.

Now we construct a control $\tilde{u}$ that satisfies (4). For this purpose, we first compute the time-derivative $E_t(t)$ for a given control $u \in L^2(0,\,T)$. For $t \in (0,\,T)$ almost everywhere we have

$$E_t(t) = y_t(t,\,L)\,y_x(t,\,L) = y_t(t,\,L)\,u(t).$$

For $j \in \{0, ..., k-1\}$ and $t \in (j\,T_0,\ (j+1)\,T_0)$ almost everywhere we consider the parametric optimization problem

$$\min_{u\in\mathbb{R}} H_t(t)^{u,j}. \tag{5}$$

With a travelling waves solution of the form $y(t,\,x) = \alpha\left(t + \frac{x}{c}\right) - \alpha\left(t - \frac{x}{c}\right)$, for the energy we obtain the representation

$$E(t) = \frac{1}{c^2}\int_0^L \left(\alpha'\left(t + \frac{x}{c}\right)\right)^2 + \left(\alpha'\left(t - \frac{x}{c}\right)\right)^2 \,\mathrm{d}t.$$

This yields

$$E'(t) = \frac{1}{c}\left[\alpha'\left(t + \frac{L}{c}\right)^2 - \alpha'\left(t - \frac{L}{c}\right)^2\right].$$

For our parametric optimization problem (5) we obtain

$$\min_{\alpha'\left(t+\frac{L}{c}\right)\in\mathbb{R}} \frac{1}{c}\left[\alpha'\left(t + \frac{L}{c}\right)^2 - \alpha'\left(t - \frac{L}{c}\right)^2\right] + \frac{\gamma}{2\,c^2}\left[\alpha'\left(t + \frac{L}{c}\right) + \alpha'\left(t - \frac{L}{c}\right)\right]^2.$$

The necessary optimality conditions yield the solution

$$\alpha'\left(t + \frac{L}{c}\right) = -\frac{\gamma}{2c+\gamma}\,\alpha'\left(t - \frac{L}{c}\right).$$

This solution can be transformed to the form $u = -\frac{1}{c+\gamma}\,y_t(t,\,L)$.

For $t \in (0,\,T)$ almost everywhere, we define $\tilde{u}$ by the feedback law

$$\tilde{u}(t) = -\frac{1}{c+\gamma}\,y_t(t,\,L). \tag{6}$$

Then due to the construction, for $t \in (0,\,T)$ almost everywhere, $\tilde{u}(t)$ solves (5). Thus (4) holds. Hence $\tilde{u}$ solves problem $\mathbf{OCP}(T,\,\gamma)$.

The velocity feedback law (6) is well–known and has been studied for example already in [15], [1] and [12] (Theorem 5.3). The properties of the feedback law imply the assertion. Thus Theorem 2.1 is proved. □

Remark 1. Note that for $\gamma = 0$, due to the finite-time turnpike property for the optimal control $\tilde{u}$ we have $\tilde{u}(t) = 0$ for $t > T_0$ and for the optimal state we have $E(t) = 0$ for $t > T_0$. In this case, $\tilde{u}$ also solves the corresponding infinite time-horizon problem with T replaced by ∞, that is $\mathbf{OCP}(\infty,\,\gamma)$. In fact, also for $\gamma > 0$, the state that is generated with the feedback law (6) is the optimal state for the infinite time-horizon problem $\mathbf{OCP}(\infty,\,\gamma)$. For $\gamma > 0$, the optimal state does not have the finite-time turnpike property but due to the exponential decay of the energy shows an exponential turnpike structure where the decay rate is independent of T.

An example for the switching delay feedback stabilization of the vibrating string is presented in [13]. In [14] optimal boundary feedback stabilization of a string with moving boundary is considered.

For the general theory on optimal control problems for the wave equation see the monographs [16] and [17].

3. CONCLUSIONS

While the turnpike phenomenon with non-smooth objective functions has already been studied, in this paper we show that a suitable chosen time-dependent weight function can also enforce that the optimal state reaches the desired state in finite time. This finite time turnpike property is useful for the numerical treatment of the problem since it allows to concentrate the dynamic computations on the first part of the time interval, because after finite time a static situation is reached.

We expect that a generalization of the result to networked systems is possible. This is important, since many systems in engineering can be modelled as networked systems, for example gas transportation networks or electrical grids. On the other hand, the question whether it is possible to generalize the result to dimensions 2 or 3 is completely open. A possible step in this direction would be to consider a system governed by the wave equation with control concentrated in interior curves, similar to [18]. Let me emphasize that in this case, also a slower turnpike property where the the optimal state converges to the turnpike only slowly would be of great interest.

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