*Dedicated to Marius Tucsnak on the occasion of his $60^{th}$ anniversary*

# CGO SOLUTIONS FOR COUPLED CONDUCTIVITY EQUATIONS

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This paper is devoted to study of *complex geometrical optics* (CGO) solutions to the coupled conductivity equations written in a matrix form $\operatorname{div}(Q\cdot\nabla U) = 0$ in $\mathbb{R}^2$ for symmetric, positive definite matrix functions $Q$. The CGO solutions were introduced by Faddeev in 1966 [8] to prove the uniqueness in the inverse potential scattering problem for Schödinger equation, later Sylvester and Uhlmann in 1987 [26] use the CGO functions to study the uniqueness of the Calderón's inverse problem.

Following the ideas of Astala and Päivärinta [3], we compute CGO solutions considering the vectorial solutions of an associated Beltrami system. In this work, we first prove the existence of CGO solution and then use a numerical strategy based on the method introduced by Huhtanem and Perämäki in [12] for the Beltrami equation. Numerical experiments are considered to show the influence of coupled equations.

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## 1. INTRODUCTION

*Electrical Impedance Tomography* (EIT) is an imaging technique in which electrodes are placed on the surface of the body and low-frequency current is applied on the electrodes which can then be measured. The measurement is repeated for a specified set of current patterns, or choices of current amplitudes at each electrode. The resulting current-to-voltage map serves as data for the inverse problem. The mathematical model of EIT is called the inverse conductivity problem: recover the conductivity distribution inside the body given electric boundary measurements performed on the surface of the body. It is a nonlinear and severely ill-posed problem.

In 1980 Alberto Calderón published a paper entitled “On an inverse boundary value problem” [5]. Two questions were posed by Calderón in [5] which is often pointed to as the mathematical beginnings of the inverse conductivity problem. The first question is, is it possible to uniquely determine the conductivity of an unknown object from boundary measurements? The other question is, how can this conductivity be reconstructed? Calderón shows that the linearized problem has an affirmative answer to the uniqueness question, and he proposed a linearized reconstruction scheme.

The seminal Calderón's problem can be described as follows. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. The electrical conductivity of $\Omega$ is represented by a bounded and positive function $q(x)$. In the absence of sinks or sources of current, the equation for potential is given by

$$\operatorname{div}(q\nabla u) = 0, \text{ in } \Omega, \tag{1.1}$$

where $q\nabla u$ represents the current flux.

Given a potential $\phi \in H^{1/2}(\partial\Omega)$ on the boundary, the induced potential $u \in H^1(\Omega)$ solves (1.1) with the Dirichlet boundary condition

$$u = \phi, \quad \text{on } \partial\Omega.$$

Thus we define the Dirichlet to Neumann map, or voltage to current map,

$$\begin{aligned} \Lambda_q : H^{1/2}(\partial\Omega) &\to H^{-1/2}(\partial\Omega), \\ \phi &\mapsto q\, \frac{\partial u}{\partial n}\bigg|_{\partial\Omega}, \end{aligned}$$

where $n$ denotes the unit outer normal to $\partial\Omega$. The inverse problem is to determine $q$ knowing $\Lambda_q$.

In this work we are interested in studying an extension to the above problem to a coupled conductivity equation defined in matrix sense, that is, let $Q$ be a matrix and $U \in [H^1(\Omega)]^2$, such that,

$$\operatorname{div}(Q\nabla U) = 0, \quad \text{in} \quad \mathbb{R}^2, \tag{1.2}$$

with the Dirichlet boundary condition

$$U = \Phi, \quad \text{on } \partial\Omega.$$

If $Q$ and $\partial\Omega$ are smooth, we can define the Dirichlet-to-Neumann, or voltage-to-current map by

$$\begin{aligned} \Lambda_Q : [H^{1/2}(\partial\Omega)]^2 &\to [H^{-1/2}(\partial\Omega)]^2, \\ \Phi &\mapsto n \cdot Q\nabla U. \end{aligned}$$

*Remark* 1. When considering the matrix $Q = [q_{ij}]_{ij=1}^2$ diagonal in the equation (1.2), we recover two decoupled equations of the form (1.1) where the conductivities take the value $q_{11}$ and $q_{22}$.

The Calderón's work has motivated many developments in inverse problems, in particular in the construction of the called *Complex Geometrical Optics* (CGO) solutions of partial differential equations to solve several inverse problems. The problem that Calderón considered was whether one can determine the electrical conductivity of a medium by making voltage and current measurements at the boundary of the medium. His methods have inspired a multitude of research on the problem, including the use of CGO solutions for answering both of his questions and for designing a regularized inversion method for practical EIT. EIT also arises in medical imaging given that human organs and tissues have quite different conductivities [16, 30, 14]. This inverse problem has also been used to detect leaks from buried pipes [15].

It is difficult to find a systematic way of prescribing voltage measurements at the boundary to be able to find the conductivity. Calderón took instead a different route. Using the divergence theorem, we have

$$Q_q(\phi) := \int_\Omega q|\nabla u|^2 \mathrm{d}x = \int_{\partial\Omega} \Lambda_q(\phi)\phi \,\mathrm{d}S, \tag{1.3}$$

where $dS$ denotes surface measure and $u$ is the solution of (1.1). $Q_q(\phi)$ is the quadratic form associated to the linear map $\Lambda_q(\phi)$, and knowing $\Lambda_q(\phi)$ or $Q_q(\phi)$ for all $\phi \in H^{1/2}(\partial\Omega)$ is equivalent. $Q_q(\phi)$ measures the energy needed to maintain the potential $\phi$ at the boundary. Calderón's point of view is that if one looks at $Q_q(\phi)$ the problem is changed to finding enough solutions $u \in H^1(\Omega)$ of the equation (1.1) to find $q$ in the interior.

In [5], Calderón used complex exponential harmonic functions $u = e^{x\cdot\rho}$ and $v = e^{-x\cdot\rho}$, where $\rho \in \mathbb{C}^N$ and $\rho\cdot\rho = 0$, to prove that the linearization of (1.3) is injective at constant conductivities. He also gave an approximation formula to reconstruct a conductivity which is, a priori, close to a constant conductivity.

Several uniqueness results have been obtained for the inverse conductivity problem, for example, in dimension higher than two for smooth conductivities by Sylvester and Uhlmann in 1987 [26]. In dimension two, Nachman [20] produced in 1995 a uniqueness result for conductivities with two derivatives. Earlier, the problem was solved for piecewise analytic conductivities by Koh and Vogelius in [17, 18] and the generic uniqueness was established by Sun and Uhlmann [25]. In dimension two, Astala and Päivärinta in [3] proved that the Dirichlet-to-Neumann map, $\Lambda_\sigma$, uniquely determines the conductivity $\sigma \in L^\infty(\Omega)$, $0 < c \le \sigma$. In [24], Santacesaria proposes an idea to tackle Calderon's problem in $\mathbb{R}^n$, based in the Astala and Päivärinta method [3] and Clifford algebras.

The crucial technical tools for the uniqueness results are *Complex Geometrical Optics* (CGO) solutions, sometimes also called exponentially growing solutions. These solutions have their origin in optics, and the complex-valued CGO solutions have exponential growth in certain directions and exponential decay in others. Faddeev in 1966 [8] proposed the idea of using complex exponential solutions to demonstrate uniqueness in the inverse potential scattering problem for Schödinger equation and later used in the context of inverse problems. CGO solutions are a valuable tool both theoretically and computationally points of view since many proofs involving them are constructive and lend themselves well to computational algorithms. For a thorough survey see [27].

In dimension two, Astala and Päivärinta [3] use the CGO solutions to solve Calderón's problem. In this case, considering $L^\infty$-conductivities, the CGO solutions need to be constructed via the Beltrami equation

$$\overline{\partial} f_\mu = \mu \overline{\partial f_\mu}, \tag{1.4}$$

where $\mu$ is a compactly supported $L^\infty$ function, connected to $q$ by the identity

$$\mu = \frac{1-q}{1+q}.$$

Indeed, the respective complex CGO solutions are related by the equation

$$2u(z,k) = f_\mu(z,k) + f_{-\mu}(z,k) + \overline{f_\mu(z,k)} - \overline{f_{-\mu}(z,k)}.$$

The simple reason behind these identities is that the real part $u(z,k)$ of $f_\mu(x,k)$ solves the equation (1.1) while the imaginary part solves the same equation with $q$ replaced by $1/q$.

Then, an asymptotic condition is required as well

$$f(z,k) = e^{ikz}(1+\omega(z,k)),\ \omega(z,k) = \mathcal{O}\left(\frac{1}{z}\right), |z| \to \infty. \tag{1.5}$$

The numerical computation of the CGO solutions of the equation (1.4) was first time introduced in [2], the authors proposed a complicated method to compute $\omega(z,k)$ in (1.5) *via* the solution of a $\mathbb{R}$-linear integral equation based on periodization, truncation of a Neumann series, discretization, *Fast Fourier Transform* (FFT), and the GMRES method [23]. In [7], a simpler numerical method for solving the same $\mathbb{R}$-linear integral equation was proposed, which solves the $\mathbb{R}$-linear integral equation in the unit disc directly, based on the fast algorithm in [6]. In [12] Huhtanen and Perämäki introduced an efficient method for the computation of the CGO, where they considered a new way to discretize the $\mathbb{R}$-linear integral equation.

Compared to the identification of the conductivity $q$ in (1.1), the problem of identifying the matrix $Q$ not only has not been demonstrated but also it has received less attention than scalar problems. However, there are some contributions treating the following problem:

$$\begin{aligned} \operatorname{div}(Q\nabla u) = \sum_{i,j=1}^{2} \frac{\partial}{\partial z_i}(q_{ij}(z)) \frac{\partial}{\partial z_j} u &= 0, \quad \text{in} \quad \Omega, \\ u &= \phi, \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{1.6}$$

Hoffmann and Sprekels in [11] proposed a dynamical system approach to reconstruct the matrix $Q$ in equation (1.6). In [22], Rannacher and Vexler employed the finite element method and showed error estimates for a matrix identification problem from pointwise measurements of the state variable, provided that the sought matrix is constant, and the exact data is smooth enough. Astala *et al.* [4] showed that it is possible to determine a $L^\infty$ smooth anisotropic conductivity up to a $W^{1,2}$ diffeomorphism $\phi$. In [10], an alternative method for reconstructing a matrix coefficient is proposed, based on convex energy functional method with Tikhonov regularization.

This work gives a first step to extend the Calderon problem to coupled conductivity systems by Astala Päivärinta method for $L^\infty$ matrix coefficients. It can give the possibility to extend the problem to other vector equations, such as elasticity equation and Stokes equations, using the same method. The main idea of this work is to reconstruct numerically the CGO solutions for the matrix conductivity case where we consider diverse types of conductivities which represent distinct types of materials. Indeed, suppose that $\Omega \subset \mathbb{R}^2$ is the unit disc and $Q$ is a symmetric $2 \times 2$-matrix. Let $U \in [H^1(\Omega)]^2$ be the unique solution to

$$\begin{array}{rcccc} \operatorname{div}(Q\nabla U) & = & 0, & \text{in} & \Omega, \\ U & = & \Phi, & \text{on} & \partial\Omega \end{array}$$

where, these solutions are specified by their asymptotics

$$U(z,k) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = e^{ikz} \begin{pmatrix} 1 + \mathcal{W}_1(z) \\ 1 + \mathcal{W}_2(z) \end{pmatrix}, \tag{1.7}$$

with

$$\mathcal{W}_i(z) = \mathcal{O}\left(\frac{1}{z}\right), \text{ as } |z| \to \infty, \text{ for } i = 1,2.$$

Here $k$ is a complex parameter, $i$ is the imaginary unit, the conductivity $Q = [q_{ij}]_{ij=1}^2$ is a given symmetric, positive definite real matrix function, with $q_{ij}(z) = 1$, for $i = j$, and $q_{ij}(z) = 0$, for $i \neq j$, outside a compact set $\Omega$. For simplicity let us take $\Omega$ as the unit disc, this is not a significant loss of generality, as a large class of more general setting can be reduced to this case.

Other interesting works in the context to vectorial equations are the following. In [28], Uhlmann and Wang construct CGO solutions for the isotropic elasticity system concentrated near spheres, where the domain is modeled as an inhomogeneous, isotropic, elastic medium characterized by the Lamé parameters $\lambda(x) \in C^2(\overline{\Omega})$ and $\mu \in C^4(\overline{\Omega})$. In [9], Heck *et al.*, transform the Stokes equations into a decoupled system which is a matrix-valued Schrödinger equation. In [13], Imanuvilov and Yamamoto considered an inverse source problem for the Stokes equation, when the main result is the Lipschitz stability in the inverse problem and the proof is based on a Carleman estimate for the Stokes system.

This paper is organized as follows. In Section 2, we present the principal definitions and properties used in our problem. Section 3 concerns the proof of the existence of the CGO solutions for the Beltrami system associated to conductivities in a matrix form. In Section 4, we introduce the algorithm of reconstruction for the CGO of Beltrami Systems. In Section 5, we reduce the problem to a periodic integral equation and define the discretization. Finally in Section 6 we show several examples for this reconstruction.

## 2. VECTORIAL CONDUCTIVITY EQUATION

We consider the coupled conductivity equation in the open unit disc $\Omega \subseteq \mathbb{R}^2$

$$\operatorname{div}(Q\nabla U) = 0, \quad \text{in } \Omega, \tag{2.1}$$

where $Q = [q_{jk}]_{j,k=1}^2 \in \mathbb{M}_{2\times 2}(\mathbb{R})$ is such that

$$[q_{jk}] \in L^\infty(\Omega;\mathbb{R}^{2\times 2}), \quad [q_{jk}]^t = [q_{jk}], \quad C^{-1}I \le [q_{jk}] \le CI,$$

with $j = 1,2$ and $j \neq k$. Here, and in what follows, we consider that $Q$ belongs to the set of symmetric $2\times 2$-matrices, equipped with the inner product $M \cdot N = trace(M^\top N)$ and the norm

$$\|M\| = (M\cdot M)^{1/2} = \left(\sum_{i,j=1}^{2} m_{ij}^2\right)^{1/2}.$$

For two symmetric matrices $M, N \in \mathbb{M}_{2\times 2}(\mathbb{R})$, let us consider the order relation $M \le N$ by

$$M\xi\cdot\xi \le N\xi\cdot\xi, \text{ for all } \xi \in \mathbb{R}^2.$$

Finally, in the space $L^\infty(\Omega;\mathbb{R}^{2\times 2})$ we use the norm

$$\|H\|_{L^\infty(\Omega;\mathbb{R}^{2\times 2})} := \max_{1\le i,j\le 2} \|h_{ij}\|_{L^\infty(\Omega)},$$

where $H = [h_{ij}]_{i,j=1}^2 \in L^\infty(\Omega;\mathbb{R}^{2\times 2})$.

In this work we identify $\mathbb{R}^2$ and $\mathbb{C}$ by the map $(x_1,x_2) \mapsto x_1 + ix_2$ and denote $z = x_1 + ix_2$, with $i$ satisfying $i^2 = -1$. We use the standard notations for complex derivatives:

$$\partial = \partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$$

$$\overline{\partial} = \partial_{\overline{z}} = \frac{1}{2}(\partial_1 + i\partial_2)$$

where $\partial_j = \dfrac{\partial}{\partial x_j}$, $j = 1, 2$. We let $D$ and $\overline{D}$ be the operators

$$D = \begin{pmatrix} \partial & 0 \\ 0 & \partial \end{pmatrix}, \qquad \overline{D} = \begin{pmatrix} \overline{\partial} & 0 \\ 0 & \overline{\partial} \end{pmatrix},$$

and let us consider

$$D_i = \begin{pmatrix} \partial_i & 0 \\ 0 & \partial_i \end{pmatrix}, \qquad \text{for } i = 1, 2.$$

Following [3], the aim of this work is, for $k \in \mathbb{C}$, to compute the unique solutions $U_1$ and $U_2$ of the following equations

$$\begin{aligned} \operatorname{div}\left(Q(z)\nabla U_1(z,k)\right) &= 0, \\ \operatorname{div}\left(Q(z)^{-1}\nabla U_2(z,k)\right) &= 0, \end{aligned}$$

where $U_1$ and $U_2$ have asymptotic behavior in sense of (1.7). To get the solutions $U_1$ and $U_2$, let us consider the following Beltrami system

$$\overline{D}F = M\overline{DF}, \tag{2.2}$$

where

$$M(z) = \left(I - Q(z)\right)\left(I + Q(z)\right)^{-1}. \tag{2.3}$$

Here, the connection between the equations (2.1) and (2.2) is given by the following Lemma.

LEMMA 2.1. *Suppose $U \in [H^1(\Omega)]^2$ satisfies the equation (2.1). Then there exists a function $V \in [H^1(\Omega)]^2$, such that, $F = U + iV$ satisfies the Beltrami system (2.2), where $M$ is defined by (2.3).*

*Conversely, if $F \in [H^1(\Omega)]^2$ satisfies (2.2) with $M \in L^\infty(\Omega; \mathbb{R}^{2\times 2})$, then $U = Re(F)$ and $V = Im(F)$ satisfy*

$$\begin{aligned} \operatorname{div}(Q\nabla U) &= 0 \\ \operatorname{div}(Q^{-1}\nabla V) &= 0, \end{aligned}$$

*respectively, where*

$$Q = (I + M)^{-1}(I - M).$$

*Proof.* Let $W$ be the vector $W = (-QD_2U, QD_1U)$. By (2.1),

$$\begin{aligned} 0 &= \operatorname{div}(Q\nabla U), \\ &= \operatorname{div}\left(Q\left(D_1U \middle| D_2U\right)\right), \\ &= D_1(QD_1U) + D_2(QD_2U), \end{aligned}$$

then $D_2W_1 = D_1W_2$. Therefore, there exists $V \in [H^1(\Omega)]^2$, unique up to a constant such that

$$\left\{\begin{array}{lcl} D_1V &=& -QD_2U, \\ D_2V &=& QD_1U. \end{array}\right. \tag{2.4}$$

Then, using (2.4)

$$\overline{D}F = \frac{1}{2}(I-Q)(I+Q)^{-1}(I+Q)\left(D_1U + iD_2U\right).$$

Considering $M = (I-Q)\,(I+Q)^{-1}$ and (2.4), the equation above can be written as

$$\overline{D}F = \frac{1}{2}M(I+Q)\left(D_1U + iD_2U\right) = \frac{1}{2}M\overline{(D_1 - iD_2)\,(U+iV)} = M\overline{DF}.$$

Hence $F$ satisfies the Beltrami system (2.2).

Conversely, let $F$ satisfying the Beltrami system (2.2) with $U = Re(F)$ and $V = Im(F)$. Then

$$\begin{aligned}\overline{D}F = M\overline{DF} \Rightarrow D_1U - D_2V + i\left(D_1V - D_2U\right) \\ = M\left(D_1U + D_2V\right) + iM\left(D_1V - D_2U\right).\end{aligned}$$

According to the above, it follows that

$$\left\{\begin{array}{lcl} D_1V &=& -(I+M)^{-1}(I-M)D_2U, \\ D_2V &=& (I+M)^{-1}(I-M)D_1U. \end{array}\right.$$

Then taking $Q = (I+M)^{-1}(I-M)$, we conclude that

$$\operatorname{div}(Q\nabla U) = 0 \text{ and } \operatorname{div}(Q^{-1}\nabla V) = 0.$$

□

*Remark* 2. We note that the condition for $Q$ and $M$ implies the existence of a constant $0 \leq \kappa < 1$ such that

$$\|M\|_{L^\infty(\Omega;\mathbb{R}^{2\times 2})} \leq \kappa,$$

holds for almost every $z \in \mathbb{C}$ and for $Q \in L^\infty(\Omega;\mathbb{R}^{2\times 2})$.

## 3. EXISTENCE OF CGO SOLUTION

Following the ideas of the scalar case [3, 2], we consider that the Beltrami system (2.2) and its solutions are governed and controlled by the extension of the two basic operators, the Cauchy and the Beurling transforms.

The Cauchy transform is extended for the vectorial case by

$$\mathcal{P}G(z) = \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}\begin{pmatrix} G_1(z) \\ G_2(z) \end{pmatrix} = \begin{pmatrix} PG_1(z) \\ PG_2(z) \end{pmatrix},$$

where the scalar Cauchy transform is defined by

$$Pg(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{g(\omega)}{\omega - z}\,\mathrm{d}\omega.$$

*Remark* 3. The vectorial Cauchy transform $\mathcal{P}$ acts as the inverse operator to $\overline{D}$, *i.e.*, $\mathcal{P}\overline{D}G = \overline{D}\mathcal{P}G = G$ for $G \in [C_0^\infty(\mathbb{C})]^2$.

The operator $P$ has some properties in an appropriate Lebesgue, Sobolev and Lipschitz space as it is shown in [3, 29]. The extension of these properties can be easily extended to the operator $\mathcal{P}$, where we consider

$$\|(f,g)\| = \max\left(\|f\|_{L^p\to L^p}, \|f\|_{L^p\to L^p}\right), \quad \forall (f,g) \in [L^p(\Omega)]^2.$$

PROPOSITION 1. *Let $\Omega \subset \mathbb{C}$ be a bounded domain and let $1 < q < 2$ and $2 < p < \infty$. Then*

1. $\mathcal{P} : [L^p(\mathbb{C})]^2 \to [Lip_\alpha(\mathbb{C})]^2$, *where* $\alpha = 1 - \frac{2}{p}$.
2. $\mathcal{P} : [L^p(\Omega)]^2 \to [W^{1,p}(\mathbb{C})]^2$ *is bounded.*
3. $\mathcal{P} : [L^p(\Omega)]^2 \to [L^p(\mathbb{C})]^2$ *is compact.*
4. $\mathcal{P} : [L^p(\mathbb{C}) \cap L^q(\mathbb{C})]^2 \to [C_0(\mathbb{C})]^2$ *is bounded, where $C_0$ is the closure of $C_0^\infty$ in $L^\infty$.*

*Here, we denote*

$$[L^p(\Omega)]^2 = \left\{ G \in [L^p(\mathbb{C})]^2 \,:\, G_i\big|_{\mathbb{C}\setminus\Omega} \equiv 0,\ i = 1,2 \right\}.$$

On the other hand, the Beurling transform is also extended for the vectorial case by

$$\mathcal{S}G(z) = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}\begin{pmatrix} G_1(z) \\ G_2(z) \end{pmatrix} = \begin{pmatrix} SG_1(z) \\ SG_2(z) \end{pmatrix} = \begin{pmatrix} DPG_1(z) \\ DPG_2(z) \end{pmatrix} = D\mathcal{P}G(z),$$

where $S$ is determined as a principal-value integral

$$Sg(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{g(\omega)}{(\omega - z)^2}\mathrm{d}\omega.$$

*Remark* 4. In particular, the operator $\mathcal{S}$ transforms $\overline{D}$ derivatives into $D$ derivatives, that is, $\mathcal{S}\left(\overline{D}G\right) = DG$.

The following proposition is an extension for the vectorial case of the useful result shown in [1], where we consider that $\overline{\mathcal{S}}$ denotes the operator $\overline{\mathcal{S}}(G) = \overline{\mathcal{S}(G)}$.

Proposition 2. *Let $M, N \in L^\infty(\mathbb{C};\mathbb{C}^{2\times 2})$ such that*

$$2\left(\|M\|_{L^\infty(\mathbb{C};\mathbb{C}^{2\times 2})} + \|N\|_{L^\infty(\mathbb{C};\mathbb{C}^{2\times 2})}\right) \leq \kappa,$$

*holds with a constant $0 \leq \kappa < 1$. Suppose that $1+\kappa < p < 1 + 1/\kappa$. Then the operator*

$$\mathcal{B} = \mathcal{I} - M\mathcal{S} - N\overline{\mathcal{S}} \tag{3.1}$$

*is invertible in $[L^p(\mathbb{C})]^2$, where $\mathcal{I}$ is the extension of the identity operator for the vectorial case, that is*

$$\mathcal{I} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

*Proof.* Let us consider

$$\|M\mathcal{S} + N\mathcal{S}\| \leq \|M\mathcal{S}\| + \|N\mathcal{S}\| \leq \left(\|M\|_{L^\infty(\mathbb{C};\mathbb{C}^{2\times 2})} + \|N\|_{L^\infty(\mathbb{C};\mathbb{C}^{2\times 2})}\right)\|\mathcal{S}\|.$$

Following the hypothesis, the first part of the right hand of the inequality can be bounded by

$$\left(\|M\|_{L^\infty(\mathbb{C};\mathbb{C}^{2\times 2})} + \|N\|_{L^\infty(\mathbb{C};\mathbb{C}^{2\times 2})}\right) \leq \frac{\kappa}{2}.$$

For the second term of the right hand of the inequality, it can be seen that

$$\|\mathcal{S}\| = \max\left(\|S\|_{L^p\to L^p}, \|S\|_{L^p\to L^p}\right).$$

Following [21], we consider the upper bound for $\|S\|_{L^p\to L^p}$ of the form $\|S\|_{L^p\to L^p} \leq 2(p^*-1)$ for

$$p^* = \max\left(p, \frac{p}{p-1}\right)$$

and considering the condition for $p$ defined in the hypothesis, the norm of the operator $\mathcal{S}$ is bounded as follows

$$\|\mathcal{S}\| < \frac{2}{\kappa}.$$

Thus,

$$\|M\mathcal{S} + N\mathcal{S}\| < 1$$

and therefore, the operator $\mathcal{B}$ is invertible. □

We need the following useful proposition as well, which is a natural extension of the result shown in [3] and [29].

Proposition 3. *Let $\mathcal{F} = (F_1, F_2)^T \in [W^{1,p}_{loc}]^2$ and $\Gamma \in L^p_{loc}(\mathbb{C})$ for some $p > 2$. Suppose that for some constant $0 \leq \kappa < 1$,*

$$|\overline{\partial} F_i(z)| \leq \kappa|\partial F_i(z)| + \Gamma(z)|F_i(z)|, \quad i = 1, 2,$$

*holds for almost every $z \in \mathbb{C}$. Then, if $\mathcal{F}(z) \to 0$ as $|z| \to \infty$ and $\Gamma$ has a compact support then*

$$\mathcal{F}(z) \equiv 0.$$

Now, we can establish the existence of the CGO solutions to (2.2) of the form

$$F_M(z,k) = e^{ikz} \begin{pmatrix} 1 + \mathcal{N}_1(z) \\ 1 + \mathcal{N}_2(z) \end{pmatrix} \tag{3.2}$$

with

$$\mathcal{N}_i(z,k) = \mathcal{O}\left(\frac{1}{z}\right), \text{ as } |z| \to \infty, \text{ for } i = 1, 2. \tag{3.3}$$

In order to establish the existence of CGO solutions and the expression (3.2), we begin with the following proposition.

PROPOSITION 4. *Suppose that* $2 < p < 1 + 1/\kappa$, $\alpha \in L^\infty(\mathbb{C}; \mathbb{R}^{2\times 2})$ *with* $\operatorname{supp}(\alpha) \subset \Omega$ *and* $\|N\|_{L^\infty(\mathbb{C};\mathbb{R}^{2\times 2})} \le \kappa \chi_\Omega(z)$ *for almost every* $z \in \Omega$. *Define the operator* $\mathcal{K} : [L^p(\mathbb{C})]^2 \to [L^p(\mathbb{C})]^2$ *by*

$$\mathcal{K}G = \mathcal{P}\left(I - N\overline{\mathcal{S}}\right)^{-1}\left(\alpha\overline{G}\right).$$

*Then* $\mathcal{K} : [L^p(\mathbb{C})]^2 \to [W^{1,p}(\mathbb{C})]^2$ *and* $I - \mathcal{K}$ *is invertible in* $[L^p(\mathbb{C})]^2$.

*Proof.* First, since $\|N\|_{L^\infty(\mathbb{C};\mathbb{R}^{2\times 2})} \le \kappa\chi_\Omega(z)$, by Proposition 2, we have that $I - N\overline{\mathcal{S}}$ is invertible in $L^p$ and, by Proposition 1, the operator $\mathcal{K} : [L^p(\mathbb{C})]^2 \to [L^p(\mathbb{C})]^2$ is well-defined and compact. We also have

$$\operatorname{supp}\left(I - N\overline{\mathcal{S}}\right)^{-1}\left(\alpha\overline{G}\right) \subset \Omega.$$

Finally, to prove the invertibility of $I - \mathcal{K}$ in $[L^p(\mathbb{C})]^2$, we use the Fredholm's alternative. For this, let us prove that $I - \mathcal{K}$ is injective in $[L^p(\mathbb{C})]^2$.

Let us suppose that $G = (G_1, G_2)^T \in [L^p(\mathbb{C})]^2$ satisfying

$$G = \mathcal{P}\left(\left(I - N\overline{\mathcal{S}}\right)^{-1}\left(\alpha\overline{G}\right)\right),$$

by Proposition 1, we have that $G \in [W^{1,p}(\mathbb{C})]^2$ and thus

$$\overline{D}G = \left(I - N\overline{\mathcal{S}}\right)^{-1}\left(\alpha\overline{G}\right),$$

which is equivalent to

$$\overline{D}G - N\overline{DG} = \alpha\overline{G}. \tag{3.4}$$

Finally, from (3.4) we can conclude that $\overline{D}G = 0$ outside $\Omega$, and therefore $G$ is analytic. Then this combined with $G \in [L^p(\mathbb{C})]^2$ implies that

$$G_i(z) = \mathcal{O}\left(\frac{1}{z}\right), \text{ for } |z| \to \infty, \text{ for } i = 1, 2.$$

Thus, the assumptions of Proposition 3 are fulfilled and we must have $G \equiv 0$. $\square$

Finally, the following theorem establishes the existence of the Complex Geometric Optics solutions to the Beltrami system (2.2).

THEOREM 3.1. *For each $k \in \mathbb{C}$ and for each $2 < p < 1 + 1/\kappa$ the system* (2.2) *admits a unique solution $F \in [W^{1,p}_{loc}(\mathbb{C})]^2$ of the form* (3.2) *such that the asymptotic formula* (3.3) *holds true.*

*Proof.* If we write

$$F_M(z,k) = e^{ikz}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \mathcal{N}(z)\right), \tag{3.5}$$

and plug this into the Beltrami system (2.2) we obtain

$$\overline{D}\mathcal{N} - e_{-k}M\overline{D\mathcal{N}} \;=\; \alpha\overline{\mathcal{N}} + \alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where

$$\begin{aligned} e_{-k}(z) &= e^{-i\left(kz+\overline{k}\overline{z}\right)}. \\ \alpha(z) &= -i\overline{k}e_{-k}(z)M(z). \end{aligned} \tag{3.6}$$

Since $\overline{\mathcal{S}}\left(\overline{D}G\right) = \overline{DG}$, we obtain

$$\overline{D}\mathcal{N} = \left(I - e_{-k}M\overline{\mathcal{S}}\right)^{-1}\left(\alpha\overline{\mathcal{N}} + \alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right).$$

If now, $\mathcal{K}$ is defined, as in the Proposition 4, with $N = e_{-k}M$ we get

$$\mathcal{N} - \mathcal{K}\mathcal{N} = \mathcal{K}\left(\chi_\Omega\right) \in [L^p(\mathbb{C})]^2. \tag{3.7}$$

Since $I - \mathcal{K}$ is invertible in $[L^p(\mathbb{C})]^2$, and $\mathcal{N}$ is analytic in $\mathbb{C}\setminus\Omega$ the result holds. $\square$

*Remark* 5. By the Theorem 3.1, the Complex Geometrical Optics solutions $F_M$ are given by substituting the unique solution of equation (3.7) by the formula (3.5).

## 4. **COMPUTE THE CGO SOLUTIONS**

Once the existence of the CGO solution of (2.2) has been proved, we want to calculate this solution $F_M$ numerically. For the scalar case, the numerical computation of CGO was introduced in [2], based on the original construction in [3]. In [2], the difficulty of the numerical computation of CGO solutions is the lack of complex-linearity in the equation that was compensated by keeping the real and imaginary parts of the solution separately in a real-linear solution process. To amend this Huhtanen and Perämäki introduced in [12] an efficient method for the computation of the CGO solutions. Let us describe this method below:

First, by Theorem 3.1 we know that the function $\mathcal{N}$ satisfies the equation

$$\overline{D}\mathcal{N} - N\overline{D\mathcal{N}} - \alpha\overline{\mathcal{N}} - \alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0. \tag{4.1}$$

Define $U \in [L^p(\Omega)]^2$ by $\overline{U} = -\overline{D}\mathcal{N}$. Then $\mathcal{N} = -\mathcal{P}\overline{U}$ and $D\mathcal{N} = -\mathcal{S}U$. Substituting $U$ into (4.1) leads to the real-linear integral equation

$$-\overline{U} - N\overline{\left(-\mathcal{S}\overline{U}\right)} - \alpha\overline{\left(-\mathcal{P}\overline{U}\right)} = \alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Then

$$U + \left(-\overline{N}\mathcal{S} - \overline{\alpha}\mathcal{P}\right)\overline{U} = -\overline{\alpha}\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{4.2}$$

Let us denote the complex conjugate $\overline{G}$ of an operator $G$ as $\overline{G} = \rho(G)$, then (4.2) takes the form

$$(I + A\rho)\,U = -\overline{\alpha}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{4.3}$$

where $A := \left(-\overline{N}\mathcal{S} - \overline{\alpha}\mathcal{P}\right)$.

The operator $I + A$ is invertible in $[L^p(\Omega)]^2$. Indeed, note that, by Proposition 2, $I - \overline{N}\mathcal{S}\rho$ is invertible. Hence the equation

$$\left(I - \left(I - \overline{N}\mathcal{S}\rho\right)^{-1}(\overline{\alpha}\mathcal{P}\rho)\right)U = \left(I - \overline{N}\mathcal{S}\rho\right)^{-1}\left(-\overline{\alpha}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

is equivalent to (4.3). The operator on the left-hand side is invertible by the fact that its null space is trivial and $\left(I - \overline{N}\mathcal{S}\rho\right)^{-1}(\overline{\alpha}\mathcal{P}\rho)$ is compact. Therefore, $I + A$ is invertible as well.

Following the ideas proposed in [12], a special preconditioning step is introduced, it consists of the transformation of the real-linear equation (4.3) into a complex-linear equation allowing standard iterative solution by GMRES. Consider the following equation in the space $[L^p(\Omega)]^2$:

$$\left(I - A\overline{A}\right)V = -\overline{\alpha}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{4.4}$$

where $\overline{A}V = \overline{A\overline{V}}$. Now (4.4) is complex-linear, and the solution $U$ of (4.3) can be written as $U = (I - A\rho)\,V$.

Summarizing, the computation of the function $\mathcal{N}(z,k)$ defined in (3.7) for a given $k \in \mathbb{C}$ proceeds as follows:

**Algorithm 1** Solution of $\mathcal{N}_M$

1: **return** Solution of the function $\mathcal{N}(z, k)$.
2: Given $k, \alpha, M$ ,where $k \in \mathbb{C}$, $\alpha$ is computed from (3.6), $M$ is computed from (2.3).
3: Solve for $V$ from (4.4). Note that $V$ is supported in $\Omega$.
4: Calculate $U = (I - A\rho)V$. Note that $U$ is supported in $\Omega$.
5: Compute $\mathcal{N} = -\mathcal{P}\overline{U}$.
6: $\mathcal{N}_M = \mathcal{N}$.

## 5. REDUCTION TO A PERIODIC INTEGRAL EQUATION AND DISCRETIZATION

As shown in [12] and discussed in the previous section, the computation of CGO solution to the real-linear Beltrami equation can be reduced to the solution of the complex-linear equation (4.4). Furthermore, one can use the iterative GMRES method for the solution of periodic and discrete version of (4.4). To that end, following [2], we need to introduce a periodic version of the operator

$$A := \left(-\overline{N}\mathcal{S} - \overline{\alpha}\mathcal{P}\right).$$

Take $s > 2$ and define a square $Q \subset \mathbb{R}^2$ by

$$Q := \left\{(x, y) \in \mathbb{R}^2 : \ -s \leq x < s, -s \leq y < s\right\}.$$

We consider tiling of the plane by translated copies of $Q$ and work with 2s-periodic functions $f : \mathbb{R}^2 \to \mathbb{C}$ satisfying

$$\tilde{f}(x + 2j_1 s, y + 2j_2 s) = \tilde{f}(x, y), \text{ for } j_1, j_2 \in \mathbb{Z},$$

where we indicate 2s-periodic functions adding $\tilde{\cdot}$ on top of symbols.

Choose a smooth cutoff function $\eta$ satisfying

$$\eta(z) = \begin{cases} 1, & \text{for} \quad |z| \leq 2, \\ 0, & \text{for} \quad |z| \geq 2 + (s-2)/2, \end{cases}$$

and $0 \leq \eta(z) \leq 1$ for all $z \in \mathbb{C}$. Define a $2s$-periodic approximate Green's function $\tilde{g}$ for the D-bar operator by setting it to $\eta(z)/(\pi z)$ inside $Q$ and extending periodically by

$$\tilde{g}(z + 2j_1 s + i2j_2 s) = \frac{\eta(z)}{\pi z}$$

for $z \in Q\backslash\{0\}$ and $j_1, j_2 \in \mathbb{Z}$. Define a periodic approximate Cauchy transform by

$$\tilde{P}f(z) := (\tilde{g}\tilde{*}f)(z) = \int_Q \tilde{g}(z - w) f(w) \, \mathrm{d}w_1 \, \mathrm{d}w_2, \tag{5.1}$$

where $\tilde{*}$ denotes convolution on the torus.

The Beurling transform is approximated in the periodic context by writing

$$\tilde{\beta}(z + 2j_1 s + i2j_2 s) = \frac{\eta(z)}{\pi z^2}$$

for $z \in Q \setminus 0$ and $j_1, j_2 \in \mathbb{Z}$, and defining

$$\tilde{S}g(z) := \left(\tilde{\beta}\tilde{*}g\right)(z) = \int_Q \tilde{\beta}(z-w)g(w)\,\mathrm{d}w_1\,\mathrm{d}w_2. \tag{5.2}$$

Then the extension of the periodic Cauchy transform $\tilde{P}$ and periodic Beurling transform $\tilde{S}$ are defined by

$$\tilde{\mathcal{P}} = \begin{pmatrix} \tilde{P} \\ \tilde{P} \end{pmatrix}, \quad \tilde{\mathcal{S}} = \begin{pmatrix} \tilde{S} \\ \tilde{S} \end{pmatrix}.$$

Set $\tilde{A} := \left(-\overline{\tilde{N}}\tilde{\mathcal{S}} - \overline{\tilde{\alpha}}\tilde{\mathcal{P}}\right)$ with the functions $\tilde{\alpha}$ and $\tilde{N}$ being trivial periodic extensions of functions defined in (3.6), $\alpha$ and $N$, which are both supported in the unit disc. The periodic version of (4.4) takes the form

$$\left(I - \tilde{A}\overline{\tilde{A}}\right)\tilde{V} = \overline{\tilde{\alpha}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{5.3}$$

Now, for discretization, choose a positive integer $m$, denote $M = 2^m$, and set $h = 2s/M$. Define a grid $\mathcal{G}_m \subset Q$ by

$$\mathcal{G}_m = \left\{jh : j \in \mathbb{Z}_m^2\right\},$$
$$\mathbb{Z}_m^2 = \left\{j = (j_1, j_2) \in \mathbb{Z}^2 : \ -2^{m-1} \le j_l < 2^{m-1}, l = 1, 2\right\}.$$

Note that the number of points in $\mathcal{G}_m$ is $M^2$. Define the grid approximation $\varphi_h : \mathbb{Z}_m^2 \to \mathbb{C}$ of a function $\varphi : Q \to \mathbb{C}$ by

$$\varphi_h(j) = \varphi(jh).$$

Our strategy is to use the iterative GMRES method for the solution of the discrete version of the periodic equation (5.3). To that end, we need to discretize the periodic Cauchy and Beurling transforms defined in (5.1) and (5.2), respectively.

Set

$$\tilde{g}_h(j) = \begin{cases} \tilde{g}(jh) & \text{for} \quad j \in \mathbb{Z}_m^2 \setminus 0, \\ 0 & \text{for} \quad j = 0, \end{cases}$$

and

$$\tilde{\beta}_h(j) = \begin{cases} \tilde{\beta}(jh) & \text{for} \quad j \in \mathbb{Z}_m^2 \setminus 0, \\ 0 & \text{for} \quad j = 0, \end{cases}$$

where the point $jh \in \mathbb{R}^2$ is interpreted as the complex number $hj_1 + ihj_2$. Now $\tilde{g}_h$ and $\tilde{\beta}_h$ are $M \times M$ matrices with complex entries. Given a periodic function $\varphi$, the discrete transforms $\tilde{P}\varphi$ are defined by

$$\left(\tilde{P}\varphi_h\right)_h = h^2\,\mathtt{IFFT}\left(\mathtt{FFT}(\tilde{g}_h)\cdot\mathtt{FFT}(\varphi_h)\right), \tag{5.4}$$

$$\left(\tilde{S}\varphi_h\right)_h = h^2 \texttt{IFFT}\left(\texttt{FFT}(\tilde{\beta}_h)\cdot\texttt{FFT}(\varphi_h)\right), \tag{5.5}$$

where FFT and IFFT are the *Fast Fourier Transform* and *Inverse Fast Fourier Transform*, respectively. Thus, all the ingredients for the numerical solution are in place.

## 6. NUMERICAL RESULTS

In this section we assess the ability of the proposed approach for obtaining numerical approximation on several contexts. We will take strictly positive conductivities $\sigma_{lm} : \Omega \to \mathbb{R}$ that models an idealized cross-section of human chest. Since the Astala-Päivärinta theory is developed for nonsmooth conductivities $\sigma \in L^\infty(\Omega)$ we will consider the background conductivity has the value one and the conductive heart and resistive lungs are separated from the background by a discontinuity.

For the first test, based on the example in [2, 19], let us consider $\sigma_{lm}(z) = \sigma(z)$, when the conductivity of the heart is 2 and the conductivity of the lungs is 0.7, see Figure 1 for a graphical representation of the conductivity. In this example we consider that the two equations act separately, that is, they are not coupled, the equation is the following

$$\operatorname{div}\left(\sigma(z)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\nabla U\right) = 0, \quad \text{in} \quad \Omega.$$

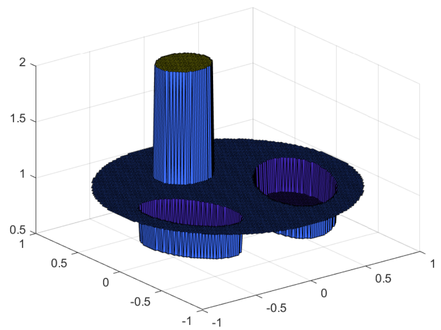
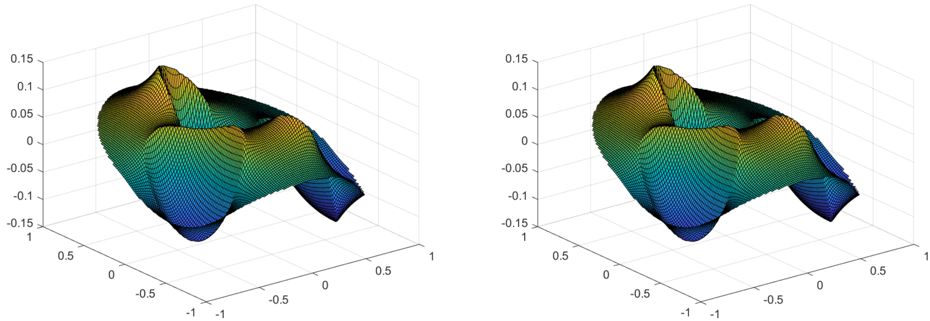



Fig. 1 – Three-dimensional mesh plot of the discontinuous conductivity $\sigma$.

In the Figure 2 the two equations act independently with the same conductivity, obtaining the same result for the CGO solution for $\mathcal{N}_1$ and $\mathcal{N}_2$. It is possible to observe that the solutions in the Figure 2 are equivalent to the solutions shown in [19] for the case of scalar case. It's possible to observe that our procedure solves the fully vectorial expression and recover the scalar case for the diagonal matrices, then our procedure is an extension of the scalar algorithm.




(a) Real part of $\mathcal{N}_1(z, 2)$. (b) Real part of $\mathcal{N}_2(z, 2)$.

Fig. 2 – Three-dimensional mesh plot of the CGO solutions corresponding to the non-smooth conductivity $\sigma$. Here $k = 2$ and $m = 8$.

For the second example we consider, as in the previous example, the conductivity $\sigma_{lm}(z) = \sigma(z)$ and as a diagonal matrix as well, but in this case the conductivity is different in the diagonal, the equation that represent this is the following

$$\operatorname{div}\left(\sigma(z)\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\nabla U\right) \quad = \quad 0, \quad \text{in} \quad \Omega.$$

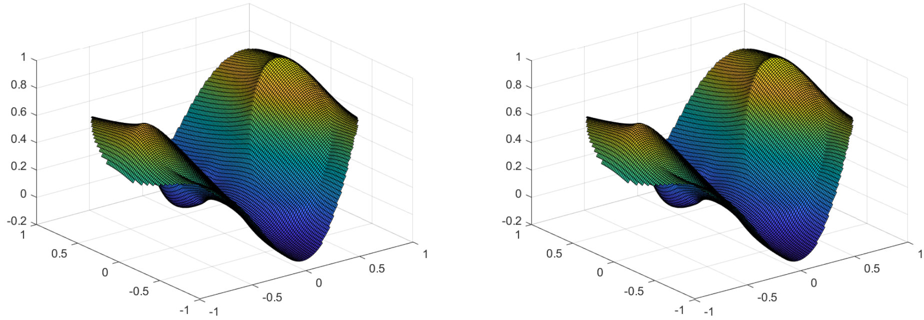
For this example, we will compare the results obtained by the vectorial algorithm and the results obtained by the scalar algorithm presented in [12]. For these cases, the scalar algorithm solves the following problems:

$$\operatorname{div}\left(\lambda\sigma(z)\nabla u\right) \quad = \quad 0, \quad \text{in} \quad \Omega,$$

where $\lambda = 2, 3$ and $\overline{\mathcal{N}}_1$, $\overline{\mathcal{N}}_2$ represent the scalar solution for $\lambda = 2$ and $\lambda = 3$, respectively. In the Figures 3-4 the two equations act independently, but since the factor that multiplies $\sigma$ is different in each equation, a different solution is observer for $\mathcal{N}_1$ and $\mathcal{N}_2$. In the Figures 3-4, it is possible to see that, the solutions have the same form as well.

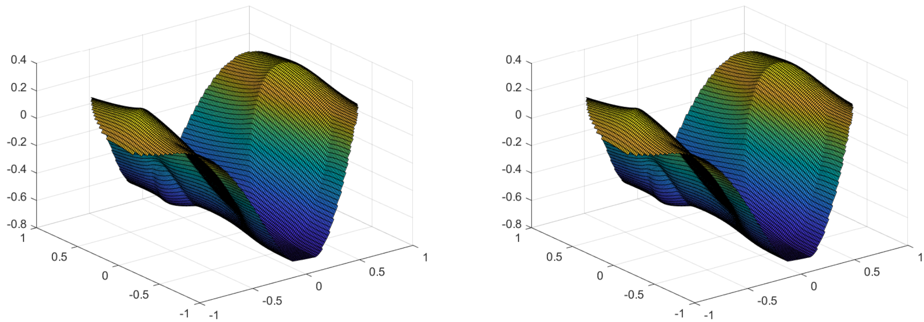
For the third example, we also take $\sigma_{lm}(z) = \sigma(z)$, but in this case, we consider a non-diagonal matrix, with this choice we have that the equations are coupled. This will be represented by the following equation

$$\operatorname{div}\left(\sigma(z)\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\nabla U\right) \quad = \quad 0, \quad \text{in} \quad \Omega.$$




(a) Real part of $\mathcal{N}_1(z, 2)$. (b) Real part of $\overline{\mathcal{N}}_1(z, 2)$.

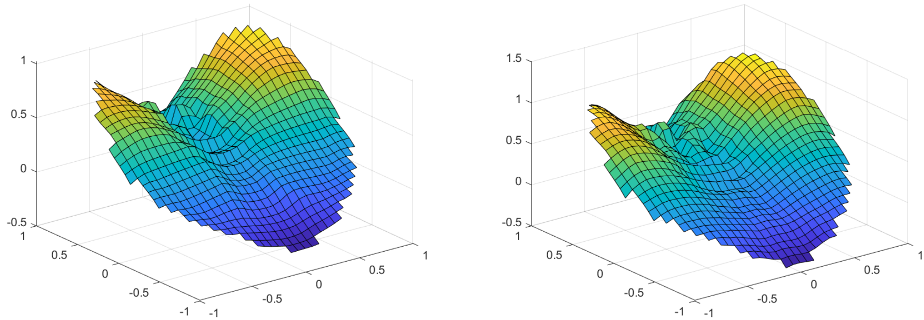
Fig. 3 – Three-dimensional mesh plot of the CGO solutions. Here $k = 2$ and $m = 8$.




(a) Real part of $\mathcal{N}_2(z, 2)$. (b) Real part of $\overline{\mathcal{N}}_2(z, 2)$.

Fig. 4 – Three-dimensional mesh plot of the CGO solutions. Here $k = 2$ and $m = 8$.

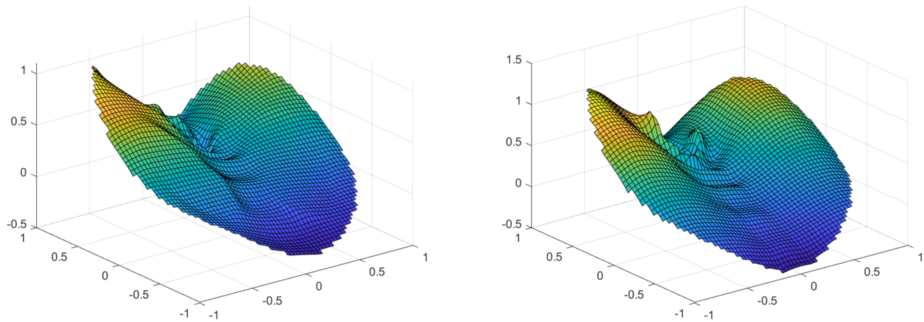
To assess the consistency and mesh independence of the method in this example, we have three different mesh refinements, $m = 6, 7, 8$, and compare their different results. In the Figures 5-7 can be seen that the coupled equation gives us solutions that differ both in their magnitude and their shape, so there is an interaction between the conductivities of each of the solutions. From Figures 5-7 it is possible to see that, for different refinements, the solution is the same. In the Tables 1-2 it can be seen the error between the discretization $m$ and $m - 1$, where it can be observed that as the discretization increases, the error compared to the previous discretization is lower. To compare both solutions, we consider only the common nodes that contain both discretization.




(a) Real part of $\mathcal{N}_1(z, 2)$.

(b) Real part of $\mathcal{N}_2(z, 2)$.

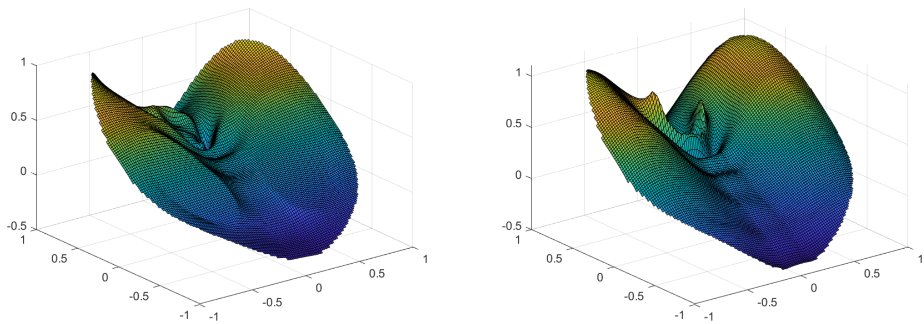
Fig. 5 – Three-dimensional mesh plot of the CGO solutions. Here $k = 2$ and $m = 6$.




(a) Real part of $\mathcal{N}_1(z, 2)$.

(b) Real part of $\mathcal{N}_2(z, 2)$.

Fig. 6 – Three-dimensional mesh plot of the CGO solutions. Here $k = 2$ and $m = 7$.




(a) Real part of $\mathcal{N}_1(z, 2)$.

(b) Real part of $\mathcal{N}_2(z, 2)$.

Fig. 7 – Three-dimensional mesh plot of the CGO solutions. Here $k = 2$ and $m = 8$.

| $m$ | $\\|\mathcal{N}_1^m\\|_\infty$ | $\\|\mathcal{N}_1^m - \mathcal{N}_1^{m-1}\\|_\infty$ |
|---|---|---|
| 7 | 1.0990 | 0.2955 |
| 8 | 0.9603 | 0.1619 |
| 9 | 0.9837 | 0.0809 |
| 10 | 0.9833 | 0.0394 |

Table 1 – Comparison of $\mathcal{N}_1$ for different refinements $m$.

| $m$ | $\\|\mathcal{N}_2^m\\|_\infty$ | $\\|\mathcal{N}_2^m - \mathcal{N}_2^{m-1}\\|_\infty$ |
|---|---|---|
| 7 | 1.2320 | 0.3385 |
| 8 | 1.1026 | 0.2319 |
| 9 | 1.1264 | 0.1241 |
| 10 | 1.1264 | 0.0480 |

Table 2 – Comparison of $\mathcal{N}_2$ for different refinements $m$.

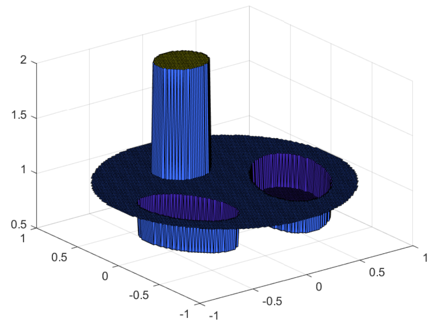
Finally, for the last example, we will consider different conductivities for the matrix but with the condition that the matrix remains symmetric, see Figure 8 for plot of the conductivities. The equation that represents this is the following

$$\operatorname{div}\left(\begin{pmatrix} \sigma_{11}(z) & \sigma_{12}(z) \\ \sigma_{21}(z) & \sigma_{22}(z) \end{pmatrix} \nabla U\right) = 0, \quad \text{in} \quad \Omega.$$

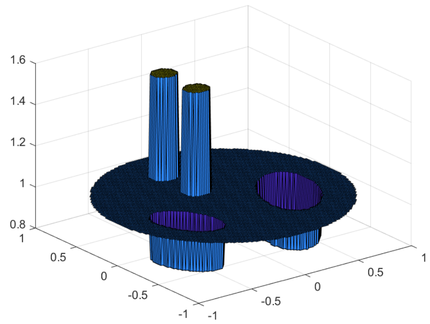
In the Figure 9 the coupled equation gives us solutions that differ both in their magnitude and their shape, so there is an interaction between the conductivities of each of the solutions. In this example it is possible to consider different conductivities that represent deformities of organs and tumors that may have these organs.

## 7. CONCLUSIONS AND COMMENTS

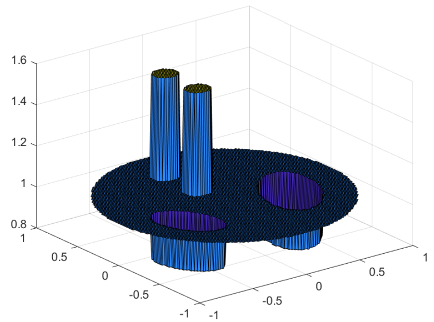
In this paper, we have presented a numerical reconstruction of the CGO solutions for the conductivity systems, this reconstruction is based on solving the problem for a Beltrami system equivalent to the conductivity system. For this, we prove the existence of the CGO solutions to the Beltrami system and then reconstruct the solution of this system. Several examples were presented to represent all the possible cases of conductivity systems, where we consider a simple case of a diagonal matrix to a more complex example of anisotropic conductivity and non-symmetric matrix.



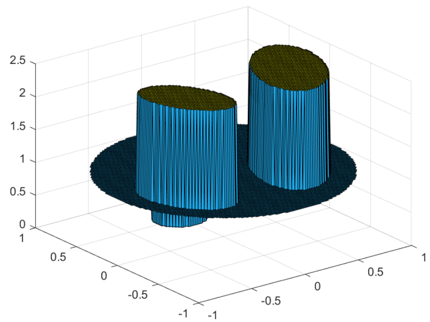
(a) $\sigma_{11}(z)$.



(b) $\sigma_{12}(z)$.

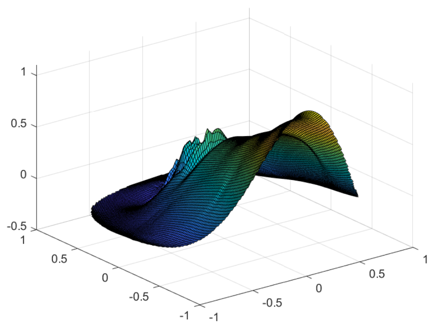


(c) $\sigma_{21}(z)$.

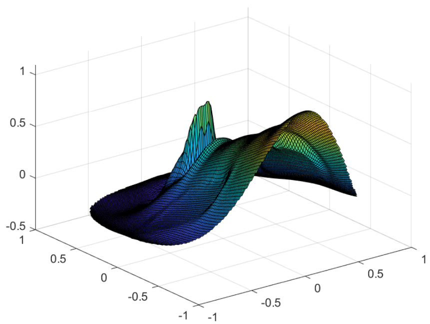


(d) $\sigma_{22}(z)$.

Fig. 8 – Three-dimensional mesh plot of the discontinuous conductivity $\sigma_{lm}$.



(a) Real part of $\mathcal{N}_1(z, 2)$.



(b) Real part of $\mathcal{N}_2(z, 2)$.

Fig. 9 – Three-dimensional mesh plot of the CGO solutions corresponding to the non-smooth conductivity $\sigma_{lm}$. Here $k = 2$ and $m = 8$.

In the examples we have considered diverse ways to validate the method where we can see that we can replicate the solution obtained for the scalar case, the method converges under mesh refinements.

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