Dedicated to Marius Tucsnak, a friend and a great mathematician, on the occasion of his 60th birthday

LACK OF COMPLETE STABILIZABILITY OF SOME COUPLED ODE-PARABOLIC SYSTEMS

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In this article, we consider several coupled ode-parabolic systems. These systems are known to be not null controllable at any time by localized interior controls. We show that, these systems are not exponentially stabilizable with arbitrary decay rate. And consequently, we recover the known results that they are not null controllable at any time.

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1. INTRODUCTION

The purpose of this work is to discuss exponential stabilizability properties of some coupled ode-parabolic linear systems such as linearized compressible Navier-Stokes, linear viscoelastic flows etc. For the convenience of the reader, let us first describe some basic concepts of controllability and stabilizability in an abstract framework.

Let $\mathcal{H}$ and $\mathcal{U}$ be two Hilbert spaces. Let us consider the linear control system

$$z'(t) = Az(t) + Bu(t), \quad t \geqslant 0, \quad z(0) = z_0 \in \mathcal{H}, \tag{1.1}$$

where $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ is the infinitesimal generator of a strongly continuous semigroup $\mathbb{T}$ on $\mathcal{H}$, and $B \in \mathcal{L}(\mathcal{U}, \mathcal{H})$ is a bounded control operator. It is well known that, given initial data $z_0 \in \mathcal{H}$ and a control $u \in L^2_{\text{loc}}(0, \infty; \mathcal{U})$, the system (1.1) admits a unique solution $y \in C([0, \infty); \mathcal{H}) \cap H^1_{\text{loc}}(0, \infty; \mathcal{D}(A^*)')$, where $\mathcal{D}(A^*)'$ is the dual of $\mathcal{D}(A)$ with respect to the pivot space $\mathcal{H}$.

Let us now introduce several notions of stabilizability.

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Definition 1.1 (Open loop stabilizable). The system (1.1) or the pair (A, B) is said to be open loop stabilizable if for any initial data $z_0 \in \mathcal{H}$ there exists a control $u \in L^2(0, \infty; \mathcal{U})$ such that the solution z of (1.1) belongs to $L^2(0, \infty; \mathcal{H})$.

Definition 1.2 (Exponentially stabilizable). The system (1.1) or the pair (A, B) is said to be exponentially stabilizable if for any initial data $z_0 \in \mathcal{H}$ there exists a control $u \in L^2(0, \infty; \mathcal{U})$ such that the solution z of (1.1) satisfy

$$\|z(t)\|_{\mathcal{H}} \leqslant Ce^{-\nu t} \|z_0\|_{\mathcal{H}}, \qquad t \geqslant 0,$$

for some constant $C > 0$ and $\nu > 0$.

Definition 1.3 (Feedback stabilizable). The system (1.1) or the pair (A, B) is said to be feedback stabilizable if there exists an operator $K \in \mathcal{L}(\mathcal{H}; \mathcal{U})$ such that the operator $A + BK$ with the domain $\mathcal{D}(A + BK) = \mathcal{D}(A)$ generates an exponentially stable semigroup $\mathbb{T}^K$ on $\mathcal{H}$, *i.e.*, there exist constants $M \geqslant 1$ and $\nu > 0$, such that

$$\left\|\mathbb{T}_t^K\right\|_{\mathcal{L}(\mathcal{H})} \leqslant Me^{-\nu t}, \quad t \geqslant 0.$$

In particular, feedback stabilizability is a special case of exponential stabilizability. Obviously feedback stabilizability or exponential stabilizability implies open loop stabilizability. The converse is also true; see for instance [3, Part V, Theorem 3.1]. For these reasons, these notions will be used in an exchangeable manner.

Definition 1.4 (Complete stabilizability). The system (1.1) or the pair (A, B) is said to be feedback stabilizable with a decay rate $\omega > 0$ if the pair $(A + \omega I, B)$ is feedback stabilizable. The system (1.1) or the pair (A, B) is said to be completely stabilizable if it is feedback stabilizable with any decay rate $\omega > 0$.

Finally, we introduce the notion of null controllability.

Definition 1.5 (Null controllability). The system (1.1) or the pair (A, B) is said to be null controllable in $\mathcal{H}$ at time $\tau > 0$ if for any $z_0 \in \mathcal{H}$, there exists a control $u \in L^2(0, T; \mathcal{U})$, such that the solution z of (1.1) satisfy $z(\tau) = 0$.

If the system (1.1) is null controllable at time $\tau > 0$ then it is completely stabilizable. The converse is not true in general. However, if A generates a group of operators, then they are equivalent ([18, Part IV, Theorem 3.4]).

In this article, we discuss complete stabilizability or their lack of, for some coupled ode-parabolic linear system. More precisely, we will consider the following linear systems

1. Compressible Navier-Stokes system linearized around zero velocity (cf. [8, 10])
2. One dimensional blood flow models linearized around constant steady state (cf. [12])
3. Linear viscoelastic Jeffreys system (cf. [11])

These systems are known to be not null controllable at any time by localized interior controls or by boundary controls. We refer to articles [8, 10, 11, 9, 1] for detailed discussions regarding the lack of null controllability of such systems. In this article, we prove a stronger result. More precisely, we have the following.

THEOREM. *The systems mentioned above are not exponentially stabilizable with an arbitrary decay rate. In other words, they are not completely stabilizable.*

As a consequence of the above result, we recover the fact that, these systems are not null controllable at any time. The proof of these results is based on duality arguments. Recently, a dual characterization of exponential stabilizability has been proved in [16, Theorem 1]. More precisely,

THEOREM 1.6 ([16, Theorem 1]). *The system* (1.1) *or the pair* (A, B) *is exponentially stabilizable if and only if there exist* $\beta \in (0,1)$, $\tau > 0$, $C \geqslant 0$ *such that*

$$\|\mathbb{T}^*_\tau z_0\|_{\mathcal{H}} \leqslant C \|B^*\mathbb{T}^*_t z_0\|_{L^2(0,\tau;\mathcal{U})} + \beta \|z_0\|_{\mathcal{H}} . \tag{1.2}$$

Note that, if $\beta = 0$, the observability inequality (1.2) coincides with the so-called final state observability of the pair (A^*, B^*), which is equivalent to null controllability of (A, B) (see for instance [17, Section 11.2]). Thus, the observability inequality equivalent to exponential stabilizability is *weaker* than the one which is equivalent to null controllability. In order to prove our results, we shall construct special solutions known as *Gaussian beam solutions*, such that corresponding observability inequalities do not hold. There are some results available in the literature where it has been shown that similar coupled ode-parabolic systems are not exponentially stabilizable with any decay rate; see for instance [8, Corollary 7.12], [5, Theorem 2.7]. However, the proofs are based on explicit computation of the eigenvalues and eigenfunctions of the linear operator, and thus are restricted to certain boundary conditions and dimensions. The method we present here is quite robust, and it does not depend on explicit computation of the eigenvalues and eigenfunctions of the linear operator. Thus, this method can be applied to many such similar models. In this article we only consider a few of the examples mentioned earlier.

The remaining part of this work is organized as follows. In Section 2, we study the lack of complete stabilizability of compressible Navier-Stokes linearized around constant state in any space dimension. In Section 3, we discuss similar results for two other coupled ODE-parabolic systems in one dimension. Section 4 is devoted towards proving lack of complete stabilization for viscoelastic flows. Finally, in Section 5, we discuss some open questions.

2. COMPRESSIBLE NAVIER-STOKES LINEARIZED AROUND ZERO VELOCITY

Let Ω be smooth bounded domain in $\mathbb{R}^n$, $n \geqslant 1$. We consider the compressible Navier-Stokes system linearised around $(\bar{\rho}, 0)$, $\bar{\rho} > 0$ (see [10, Eq. 1.10]). More precisely, we consider the following linear control system

(2.1)
$$\begin{cases} \partial_t \rho + \bar{\rho}\mathrm{div} u = \mathbb{1}_{\mathcal{O}_1} f_1 & (t,x) \in (0,\infty)\times\Omega, \\ \partial_t u - \dfrac{\mu}{\bar{\rho}}\Delta u - \dfrac{\alpha+\mu}{\bar{\rho}}\nabla(\mathrm{div}\ u) + a\gamma\bar{\rho}^{\gamma-2}\nabla\rho = \mathbb{1}_{\mathcal{O}_2} f_2 & (t,x) \in (0,\infty)\times\Omega, \\ u = 0 & (t,x) \in (0,\infty)\times\partial\Omega, \\ \rho(0) = \rho_0, \qquad u(0) = u_0 & x \in \Omega, \end{cases}$$

where $\mu > 0$ and $\alpha + 2\mu > 0$. In the above system $\rho(t,\cdot) : \Omega \to \mathbb{R}$ and $u(t,\cdot) : \Omega \to \mathbb{R}^n$ represent the fluid density and velocity respectively. Let us take

$$\mathcal{H} = \left[L^2(\Omega)\right]^{1+n},$$

endowed with the inner product

$$\left\langle \begin{bmatrix}\rho\\ u\end{bmatrix}, \begin{bmatrix}\sigma\\ v\end{bmatrix}\right\rangle_{\mathcal{H}} = a\gamma\bar{\rho}^{\gamma-2}\int_\Omega \rho\sigma\ \mathrm{d}x + \bar{\rho}\int_\Omega u\cdot v\ \mathrm{d}x.$$

We also define

$$L^2_{\mathrm{m}}(\Omega) = \left\{ f \in L^2(\Omega) \mid \int_\Omega f\ \mathrm{d}x = 0 \right\}, \qquad \mathcal{H}_{\mathrm{m}} = L^2_{\mathrm{m}}(\Omega) \times \left[L^2(\Omega)\right]^n.$$

We consider the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{H}$ defined by

$$\mathcal{D}(\mathcal{A}) = \left\{ (\rho, u) \in L^2(\Omega)\times H^1_0(\Omega)^n \mid \frac{\mu}{\bar{\rho}}\Delta u + \frac{\alpha+\mu}{\bar{\rho}}\nabla(\mathrm{div}\ u) - a\gamma\bar{\rho}^{\gamma-2}\nabla\rho \in L^2(\Omega)^n \right\},$$

$$\mathcal{A}\begin{bmatrix}\rho\\ u\end{bmatrix} = \begin{bmatrix} -\bar{\rho}\mathrm{div} u \\ \dfrac{\mu}{\bar{\rho}}\Delta u + \dfrac{\alpha+\mu}{\bar{\rho}}\nabla(\mathrm{div}\ u) - a\gamma\bar{\rho}^{\gamma-2}\nabla\rho \end{bmatrix}. \tag{2.2}$$

We introduce the input space $\mathcal{U} = \mathcal{H}$ and the control operator $\mathcal{B} \in \mathcal{L}(\mathcal{U};\mathcal{H})$ defined by

$$\mathcal{B}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \mathbb{1}_{\mathcal{O}_1} f_1 \\ \mathbb{1}_{\mathcal{O}_2} f_2 \end{bmatrix}. \tag{2.3}$$

With the above notations, the system (2.1) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \rho \\ u \end{bmatrix} = \mathcal{A}\begin{bmatrix} \rho \\ u \end{bmatrix} + \mathcal{B}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \qquad \begin{bmatrix} \rho \\ u \end{bmatrix}(0) = \begin{bmatrix} \rho_0 \\ u_0 \end{bmatrix}. \tag{2.4}$$

The fact that the system we consider is well-posed follows from the following result:

LEMMA 2.1. *The operator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ is the infinitesimal generator of a strongly continuous semigroup $\mathbb{T}$ on $\mathcal{H}$. Moreover, $\mathcal{H}_{\mathrm{m}}$ is invariant under $\mathbb{T}$, and the operator $\mathcal{A}$ can be restricted to $\mathcal{H}_{\mathrm{m}}$. The part of $\mathcal{A}$ in $\mathcal{H}_{\mathrm{m}}$ is the restriction of $\mathcal{A}$ to the domain $\mathcal{D}(\mathcal{A}) \cap \mathcal{H}_{\mathrm{m}}$.*

Proof. For all $(\rho, u) \in \mathcal{D}(\mathcal{A})$, we have

$$\left\langle \mathcal{A}\begin{bmatrix} \rho \\ u \end{bmatrix}, \begin{bmatrix} \rho \\ u \end{bmatrix} \right\rangle_{\mathcal{H}} = -\left(\mu \int_\Omega |\nabla u|^2 \mathrm{d}x + (\alpha+\mu) \int_\Omega (\operatorname{div} u)^2 \mathrm{d}x \right).$$

Thus, if $\alpha + \mu \geqslant 0$ the operator $\mathcal{A}$ is dissipative. If $\alpha + \mu < 0$, we rewrite the above formula as

$$\left\langle \mathcal{A}\begin{bmatrix} \rho \\ u \end{bmatrix}, \begin{bmatrix} \rho \\ u \end{bmatrix} \right\rangle_{\mathcal{H}} = -\Big((\alpha+2\mu)\, \|\nabla u\|^2_{L^2(\Omega)^{n\times n}} + (\alpha+\mu)(\|\operatorname{div} u\|^2_{L^2(\Omega)} - \|\nabla u\|^2_{L^2(\Omega)^{n\times n}}) \Big).$$

Now, using the fact that $\|\operatorname{div} u\|^2_{L^2(\Omega)} \leqslant \|\nabla u\|^2_{L^2(\Omega)^{n\times n}}$, we infer that $\mathcal{A}$ is dissipative. We now show that is also maximal. For any $(f, g) \in \mathcal{H}$, let us consider the system

$$\begin{cases} \rho + \bar\rho \operatorname{div} u = f & \text{in } \Omega, \\ u - \dfrac{\mu}{\bar\rho}\Delta u - \dfrac{\alpha+\mu}{\bar\rho}\nabla(\operatorname{div} u) + a\gamma\bar\rho^{\gamma-2}\nabla\rho = g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

By setting $\rho = f - \bar\rho \operatorname{div} u$, the above system can be rewritten as

$$\begin{cases} u - \dfrac{\mu}{\bar\rho}\Delta u - \dfrac{\alpha+\mu+a\gamma\bar\rho^{\gamma}}{\bar\rho}\nabla(\operatorname{div} u) = g - a\gamma\bar\rho^{\gamma-2}\nabla f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that, the right hand side of the above system $g - a\gamma\bar{\rho}^{\gamma-2}\nabla f$ belongs to $[H^{-1}(\Omega)]^n$. We set the continuous bilinear form $a(\cdot,\cdot)$ over $H_0^1(\Omega)^n \times H_0^1(\Omega)^n$ as follows: for all $u \in H_0^1(\Omega)^n, v \in H_0^1(\Omega)^n$,

$$a(u,v) = \int_\Omega u \cdot v \ \mathrm{d}x + \frac{\mu}{\bar{\rho}} \int_\Omega \nabla u : \nabla v \ \mathrm{d}x + \frac{\alpha + \mu + a\gamma\bar{\rho}^\gamma}{\bar{\rho}} \int_\Omega (\text{div } u)(\text{div } v)\mathrm{d}x.$$

By the similar calculation above, it follows that the bilinear form $a(\cdot,\cdot)$ is coercive over $H_0^1(\Omega)^n$, *i.e.*, it satisfies

$$a(u,u) \geqslant C\|u\|^2_{H_0^1(\Omega)}, \qquad \text{for all } u \in H_0^1(\Omega)^n,$$

for some positive constant C. Using Lax-Milgram theorem, we have $u \in H_0^1(\Omega)^n$.

Finally, the fact that $\mathcal{H}_m$ is invariant under $\mathbb{T}$, follows easily by integrating $(2.1)_1$, with $f_1 = 0$. $\square$

Next, we determine the adjoint of the operator $\mathcal{A}$.

Proposition 2.2. *The adjoint of $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ in $\mathcal{H}$ is defined by*

$$\mathcal{D}(\mathcal{A}^*) = \Big\{(\sigma, v) \in L^2(\Omega) \times H_0^1(\Omega)^n \mid \frac{\mu}{\bar{\rho}}\Delta v + \frac{\alpha+\mu}{\bar{\rho}}\nabla(\text{div } v) + a\gamma\bar{\rho}^{\gamma-2}\nabla\sigma \in L^2(\Omega)^n\Big\},$$

$$\mathcal{A}^* \begin{bmatrix} \sigma \\ v \end{bmatrix} = \begin{bmatrix} \bar{\rho}\,\text{div}\, v \\ \frac{\mu}{\bar{\rho}}\Delta v + \frac{\alpha+\mu}{\bar{\rho}}\nabla(\text{div } v) + a\gamma\bar{\rho}^{\gamma-2}\nabla\sigma \end{bmatrix}. \tag{2.5}$$

The adjoint operator $(\mathcal{A}^*, \mathcal{D}(\mathcal{A}^*))$ is the infinitesimal generator of a strongly continuous semigroup on $\mathcal{H}$.

Proof. Let us consider an unbounded operator $\mathcal{A}_0$ on $\mathcal{H}$ defined by

$$\mathcal{D}(\mathcal{A}_0) = \Big\{(\sigma, v) \in L^2(\Omega) \times H_0^1(\Omega)^n \mid \frac{\mu}{\bar{\rho}}\Delta v + \frac{\alpha+\mu}{\bar{\rho}}\nabla(\text{div } v) + a\gamma\bar{\rho}^{\gamma-2}\nabla\sigma \in L^2(\Omega)^n\Big\},$$

$$\mathcal{A}_0 \begin{bmatrix} \sigma \\ v \end{bmatrix} = \begin{bmatrix} \bar{\rho}\,\text{div}\, v \\ \frac{\mu}{\bar{\rho}}\Delta v + \frac{\alpha+\mu}{\bar{\rho}}\nabla(\text{div } v) + a\gamma\bar{\rho}^{\gamma-2}\nabla\sigma \end{bmatrix}.$$

Integrating by parts, it is easy to verify that

$$\left\langle \mathcal{A}\begin{bmatrix} \rho \\ u \end{bmatrix}, \begin{bmatrix} \sigma \\ v \end{bmatrix} \right\rangle_{\mathcal{H}} = \left\langle \begin{bmatrix} \rho \\ u \end{bmatrix}, \mathcal{A}_0\begin{bmatrix} \sigma \\ v \end{bmatrix} \right\rangle_{\mathcal{H}} \qquad \text{for all } (\rho, u) \in \mathcal{D}(\mathcal{A}), (\sigma, v) \in \mathcal{D}(\mathcal{A}_0),$$

and so $\mathcal{D}(\mathcal{A}_0) \subset \mathcal{D}(\mathcal{A}^*)$. To prove the reverse inclusion, let us first note that, by similar arguments as the proof of Lemma 2.1, we can show that $I - \mathcal{A}_0$ is an invertible operator, and $(I - \mathcal{A}_0)^{-1}(f, g) \in \mathcal{D}(\mathcal{A}_0)$ for any $(f, g) \in \mathcal{H}$. We take $(\sigma, v) \in \mathcal{D}((I - \mathcal{A})^*)$. Then there exists $(f, g) \in \mathcal{H}$ such that

$$\left\langle \begin{bmatrix} \sigma \\ v \end{bmatrix}, (I - \mathcal{A}) \begin{bmatrix} \rho \\ u \end{bmatrix} \right\rangle_{\mathcal{H}} = \left\langle \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} \rho \\ u \end{bmatrix} \right\rangle_{\mathcal{H}} \qquad \text{for all } (\rho, u) \in \mathcal{D}(\mathcal{A}).$$

Let $(\varsigma, w) = (I - \mathcal{A}_0)^{-1}(f, g) \in \mathcal{D}(\mathcal{A}_0)$. Then using the above identities, we infer that

$$\left\langle \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} \rho \\ u \end{bmatrix} \right\rangle_{\mathcal{H}} = \left\langle (I - \mathcal{A}_0) \begin{bmatrix} \varsigma \\ w \end{bmatrix}, \begin{bmatrix} \rho \\ u \end{bmatrix} \right\rangle_{\mathcal{H}} = \left\langle \begin{bmatrix} \varsigma \\ w \end{bmatrix}, (I - \mathcal{A}) \begin{bmatrix} \rho \\ u \end{bmatrix} \right\rangle_{\mathcal{H}}$$

for all $(\rho, u) \in \mathcal{D}(\mathcal{A})$. Therefore,

$$\left\langle \begin{bmatrix} \sigma \\ v \end{bmatrix} - \begin{bmatrix} \varsigma \\ w \end{bmatrix}, (I - \mathcal{A}) \begin{bmatrix} \rho \\ u \end{bmatrix} \right\rangle_{\mathcal{H}} = 0 \qquad \text{for all } (\rho, u) \in \mathcal{D}(\mathcal{A}),$$

and $(\sigma, v) = (\varsigma, w) \in \mathcal{D}(\mathcal{A}_0)$. This completes the proof. □

We now state our main results of this section. Our goal is to show that the system (2.1) is not exponentially stabilizable with arbitrary decay rate. More precisely, we shall prove the following result

THEOREM 2.3. *Let*

$$\mathcal{O}_1 \subset \Omega, \quad \mathcal{O}_2 \subseteq \Omega, \qquad \omega_0 := \frac{a\gamma\bar{\rho}^\gamma}{2\mu + \alpha}. \tag{2.6}$$

The system (2.1) *is not exponentially stabilizable in* $\mathcal{H}$ *with an exponential decay rate* $\omega \geqslant \omega_0$, *by interior controls* $f_1 \in L^2(0, \infty; L^2(\mathcal{O}_1))$ *and* $f_2 \in L^2(0, \infty; L^2(\mathcal{O}_2))^n$. *In other words, for* $\omega \geqslant \omega_0$, *the pair* $(\mathcal{A} + \omega I, \mathcal{B})$ *is not exponentially stabilizable on* $\mathcal{H}$.

Let us now give special attention to the case where the control is not active in the first component, *i.e.*, $f_1 \equiv 0$. In this case, it turns out that, $\mathcal{H}_{\mathrm{m}}$ is the appropriate space for stabilization than $\mathcal{H}$. For instance, let us consider the system (2.1) with $f_1 \equiv 0$. Integrating $(2.1)_1$ we have

$$\int_\Omega \rho(t, x)\mathrm{d}x = \int_\Omega \rho_0(x)\ \mathrm{d}x \text{ for all } t \geqslant 0.$$

And, therefore

$$\int_\Omega |\rho(t, x)|^2\ \mathrm{d}x \geqslant \frac{1}{|\Omega|} \left| \int_\Omega \rho_0(x)\ \mathrm{d}x \right|^2 \qquad (t > 0).$$

Thus, if $\displaystyle\int_\Omega \rho_0(x)\ \mathrm{d}x \neq 0$, the system can not be stabilized with the control f_2 only. With this observation in mind, when $f_1 \equiv 0$, Theorem 2.3 can be reformulated as

THEOREM 2.4. *Assume $f_1 \equiv 0$, and $\mathcal{O}_2 \subseteq \Omega$. The system (2.1) is not exponentially stabilizable in $\mathcal{H}_{\mathrm{m}}$ with an exponential decay rate $\omega \geqslant \omega_0$, by interior controls $f_2 \in L^2(0,\infty;L^2(\mathcal{O}_2))^n$.*

Remark 2.5. From [8, Lemma 2.5], we observe that in one-dimension, ω_0 in Theorem 2.3 coincides with the accumulation point of the spectrum of the operator $\mathcal{A}$.

Let us now mention some related works from the literature. Controllability of the system (2.1) in dimensions one and two has been studied in [8, 10]. In these articles, it was shown that the system is not null controllable at any time by localized interior controls. Regarding exponential stabilizability, in [2, 7], it was proved that the system (2.1) in dimension one is exponentially stabilizable with a decay $\omega < \omega_0$, where ω_0 is the same as (2.6), either by boundary control or by localized interior control on the velocity component only. Still, in dimension one, in [8, Corollary 7.12], it was established that the system is not exponentially stabilizable with decay rate $\omega > \omega_0$. Recently, in [6] via backstepping method, it was shown that the decay rate ω_0 is achievable by boundary control, provided initial data belongs to suitable regular space and the system is considered with Neumann type boundary conditions. In fact, in [6, Proposition 3.3], the authors pointed out that, with Dirichlet boundary condition at one end and Dirichlet boundary control on the other end, decay of $\|\rho(t)\|_{L^2}$ is not possible with the feedback control constructed in that paper. In Theorem 2.3 and Theorem 2.4, we show that, albeit for interior control, there is no feedback control that stabilizes the system (2.1) with a decay $\omega \geqslant \omega_0$.

The proof of the above results relies on a suitable observability inequality of the adjoint system, which is equivalent to the exponential stabilizability. More precisely, let $\omega \geqslant 0$, and we consider the following adjoint system

(2.7)
$$\begin{cases} \partial_t \sigma - \omega\sigma - \bar{\rho}\operatorname{div} v = 0 & (t,x) \in (0,\tau)\times\Omega, \\ \partial_t v - \omega v - \dfrac{\mu}{\bar{\rho}}\Delta v - \dfrac{\mu+\alpha}{\bar{\rho}}\nabla(\operatorname{div} v) - a\gamma\bar{\rho}^{\gamma-2}\nabla\sigma = 0 & (t,x) \in (0,\tau)\times\Omega, \\ v = 0 & (t,x) \in (0,\tau)\times\partial\Omega, \\ \sigma(0) = \sigma_0, \quad v(0) = v_0 & x \in \Omega. \end{cases}$$

Note that the well-posedness of the above problem follows from Proposition 2.2. According to [16, Theorem 1], the exponential stabilizability of the pair $(\mathcal{A} + \omega I, \mathcal{B})$ is equivalent to the following observability inequality:

PROPOSITION 2.6. *Let $\omega > 0$.*

1. *The system (2.1) is exponentially stabilizable with a decay rate ω, by interior controls $f_1 \in L^2(0,\infty;L^2(\mathcal{O}_1))$ and $f_2 \in L^2(0,\infty;L^2(\mathcal{O}_2))^n$, if and*

only if, there exist $\beta \in (0,1)$, $\tau > 0$ *and* $C \geqslant 0$ *such that, for any* $(\sigma_0, v_0) \in \mathcal{H}$, *the solution of* (2.7) *satisfies the following observability estimate*

(2.8)
$$\|\sigma(\tau,\cdot)\|_{L^2(\Omega)} + \|v(\tau,\cdot)\|_{L^2(\Omega)^n} \leqslant C\left(\|\sigma\|_{L^2(0,\tau;L^2(\mathcal{O}_1))} + \|v\|_{L^2(0,\tau;L^2(\mathcal{O}_2))^n}\right) + \beta\left(\|\sigma_0\|_{L^2(\Omega)} + \|v_0\|_{L^2(\Omega)^n}\right).$$

2. *Assume further that* $f_1 \equiv 0$. *The system* (2.1) *is exponentially stabilizable with a decay rate* ω, *by interior controls* $f_2 \in L^2(0,\infty;L^2(\mathcal{O}_2))^n$, *if and only if, there exist* $\beta \in (0,1)$, $\tau > 0$ *and* $C \geqslant 0$ *such that, for any* $(\sigma_0, v_0) \in \mathcal{H}_{\mathrm{m}}$, *the solution of* (2.7) *satisfies the following observability estimate*

$$\|\sigma(\tau,\cdot)\|_{L^2(\Omega)} + \|v(\tau,\cdot)\|_{L^2(\Omega)^n} \leqslant C\,\|v\|_{L^2(0,\tau;L^2(\mathcal{O}_2))^n} + \beta\left(\|\sigma_0\|_{L^2(\Omega)} + \|v_0\|_{L^2(\Omega)^n}\right). \tag{2.9}$$

To prove Theorem 2.3, we shall construct special initial data (σ_0, v_0) belonging to $\mathcal{H}$ (or $\mathcal{H}_{\mathrm{m}}$), so that the corresponding solution to system (2.7) violates the observability inequality (2.8) (or (2.9)) for $\omega \geqslant \omega_0$. To this aim, we first construct highly localised solutions known as "Gaussian beam solutions" to the adjoint of (2.1) in $(0,T)\times\mathbb{R}^n$. More precisely, let $\omega > 0$, and we consider the following system :

(2.10)
$$\begin{cases} \partial_t\sigma - \omega\sigma - \bar{\rho}\,\mathrm{div}\, v = 0 & (t,x)\in(0,\tau)\times\mathbb{R}^n,\\ \partial_t v - \omega v - \dfrac{\mu}{\bar{\rho}}\Delta v - \dfrac{\mu+\alpha}{\bar{\rho}}\nabla(\mathrm{div}\, v) - a\gamma\bar{\rho}^{\gamma-2}\nabla\sigma = 0 & (t,x)\in(0,\tau)\times\mathbb{R}^n,\\ \sigma(0) = \sigma_0 := \mathrm{div}\ w_0, \quad v(0) = v_0 & x\in\mathbb{R}^n. \end{cases}$$

We shall construct a special solution to the above system. Note that in (2.10), the initial data σ_0 is taken in the form $\sigma_0 := \mathrm{div}\ w_0$ to ensure that the initial condition in (2.7) belongs to $\mathcal{H}_{\mathrm{m}}$. This is needed to prove Theorem 2.4. In dimension 1 and 2, similar solutions were constructed in [10]. We follow a similar approach here. We set

$$\widehat{\sigma}(t,\xi) := \mathcal{F}_x\sigma(t,x), \quad \widehat{v}(t,\xi) := \mathcal{F}_x v(t,x), \tag{2.11}$$
$$\widehat{\sigma}_0(\xi) := \mathcal{F}_x\sigma_0(x), \quad \widehat{w}_0(\xi) := \mathcal{F}_x w_0(x), \quad \widehat{v}_0(\xi) := \mathcal{F}_x v_0(x), \tag{2.12}$$

where $\mathcal{F}_x(f)$ denotes the Fourier transformation of f with respect to the spatial variable x. Applying Fourier transform to (2.10), we obtain

$$\frac{d}{dt}\begin{bmatrix}\widehat{\sigma}\\ \widehat{v}\end{bmatrix} = (\widehat{\mathcal{A}}(\xi) + \omega\mathbb{I}_n)\begin{bmatrix}\widehat{\sigma}\\ \widehat{v}\end{bmatrix}, \quad \begin{bmatrix}\widehat{\sigma}\\ \widehat{v}\end{bmatrix}(0) = \begin{bmatrix}\widehat{\sigma}_0\\ \widehat{v}_0\end{bmatrix} = \begin{bmatrix}i\xi\cdot\widehat{w}_0\\ \widehat{v}_0\end{bmatrix} \quad t\in(0,\tau), \tag{2.13}$$

where

$$\widehat{\mathcal{A}}(\xi) = \begin{bmatrix} 0 & i\bar{\rho}\xi \\ ia\gamma\bar{\rho}^{\gamma-2}\xi^\top & -\frac{\mu}{\bar{\rho}}|\xi|^2\mathbb{I}_n - \frac{\mu+\alpha}{\bar{\rho}}\xi\otimes\xi \end{bmatrix}, \quad \text{for all } \xi \in \mathbb{R}^n, \tag{2.14}$$

where $\mathbb{I}_n$ is the $n \times n$ identity matrix and $\xi \otimes \xi$ is the $n \times n$ matrix with $(\xi \otimes \xi)_{ij} = \xi_i \xi_j$ for all $i = 1, \cdots, n$ and $j = 1, \cdots, n$.

We first study eigenvalues and eigenvectors of $\widehat{\mathcal{A}}(\xi)$. We have the following result:

LEMMA 2.7. *The following holds*

1. *The characteristic equation of* $\widehat{\mathcal{A}}(\xi)$ *is*

$$\left(\lambda + \frac{\mu}{\bar{\rho}}|\xi|^2\right)^{n-1} \left(\lambda^2 + \frac{\alpha + 2\mu}{\bar{\rho}}|\xi|^2\lambda + a\gamma\bar{\rho}^{\gamma-1}|\xi|^2\right) = 0. \tag{2.15}$$

2. *The eigenvalues of* $\widehat{\mathcal{A}}(\xi)$ *are*

$$\lambda_1(\xi) = \cdots = \lambda_{n-1}(\xi) = -\frac{\mu}{\bar{\rho}}|\xi|^2,$$

$$\lambda_n(\xi) = -\frac{(2\mu+\alpha)|\xi|^2}{2\bar{\rho}}\left(1 + \sqrt{1 - \frac{4a\gamma\bar{\rho}^{\gamma+1}}{(2\mu+\alpha)^2|\xi|^2}}\right),$$
$$\delta(\xi) = -\frac{(2\mu+\alpha)|\xi|^2}{2\bar{\rho}}\left(1 - \sqrt{1 - \frac{4a\gamma\bar{\rho}^{\gamma+1}}{(2\mu+\alpha)^2|\xi|^2}}\right).$$

3. *There exists* $\xi_0 > 0$*, such that for all* $|\xi| \geqslant \xi_0$*, all the eigenvalues are real. They satisfy*

$$\lim_{|\xi|\to\infty} \frac{\lambda_k(\xi)}{|\xi|^2} = -\frac{\mu}{\bar{\rho}}, k = 1, 2, \ldots, n-1 \qquad \lim_{|\xi|\to\infty} \frac{\lambda_n(\xi)}{|\xi|^2} = -\frac{2\mu+\alpha}{\bar{\rho}},$$
$$\lim_{|\xi|\to\infty} \delta(\xi) = -\frac{a\gamma\bar{\rho}^\gamma}{2\mu+\alpha} := -\omega_0.$$

4. *For* $|\xi| \geqslant \xi_0$*, we have*

$$-2\frac{a\gamma\bar{\rho}^\gamma}{2\mu+\alpha} < \delta(\xi) < -\frac{a\gamma\bar{\rho}^\gamma}{2\mu+\alpha}, \ \textit{i.e.} -2\omega_0 < \delta(\xi) < -\omega_0,$$

and

$$\delta(\xi_2) > \delta(\xi_1) \ \textit{for} \ |\xi_2| > |\xi_1| > \xi_0,$$

and

$$\left|\nabla_\xi^k \delta(\xi)\right| \leqslant \frac{C}{|\xi|^k}, \qquad (k \in \mathbb{N} \cup \{0\}), \tag{2.16}$$

for some positive constant C *independent of* ξ.

5. *The eigenfunction of* $\widehat{\mathcal{A}}(\xi)$ *corresponding to* $\delta(\xi)$ *is* $\left(1, \dfrac{\delta(\xi)\xi}{i\bar{\rho}|\xi|^2}\right)^\top$.

Proof. The proof follows easily from the expression of $\delta(\xi)$. □

We now construct *Gaussian beam* solutions for the system (2.10).

THEOREM 2.8. *Let* $\beta \in (0,1)$, $\tau > 0$, *and* $\omega \geqslant \omega_0$. *Let* $\bar{\xi} = (\frac{1}{c_0}, 0, \cdots, 0) \in \mathbb{R}^n$ *with* $c_0 > 1$, *and* $x_0 \in \mathbb{R}^n$. *Let* ψ *be a smooth function compactly supported in the unit ball and of unit* $L^2(\mathbb{R}^n)$ *norm. For any* $\varepsilon > 0$, *we define*

$$\psi_\varepsilon(\xi) = \varepsilon^{\frac{n}{4}} \psi\left(\sqrt{\varepsilon}\left(\xi - \frac{\bar{\xi}}{\varepsilon}\right)\right) e^{-ix_0\cdot\xi},$$

and

$$\widehat{w}_0(\xi) = \varepsilon\psi_\varepsilon(\xi)e_1, \quad \widehat{\sigma}_0(\xi) = i\xi\cdot\widehat{w}_0(\xi), \quad \widehat{v}_0(\xi) = \frac{\delta(\xi)\xi}{i\bar{\rho}|\xi|^2}\widehat{\sigma}_0(\xi), \tag{2.17}$$

where $e_1 = (1, 0, \ldots, 0)^\top$. *Then* (σ, v) *defined by*

$$\sigma(t,x) = \mathcal{F}_\xi^{-1}\left(\widehat{\sigma}_0(\xi)e^{(\delta(\xi)+\omega)t}\right), \quad v(t,x) = \mathcal{F}_\xi^{-1}\left(\widehat{v}_0(\xi)e^{(\delta(\xi)+\omega)t}\right), \tag{2.18}$$

satisfies the system (2.10), *with*

$$w_0(x) = \mathcal{F}_\xi^{-1}\left(\widehat{w}_0(\xi)\right), \ \sigma_0(x) = \operatorname{div}\ w_0 = \mathcal{F}_\xi^{-1}\left(\widehat{\sigma}_0(\xi)\right), \ v_0(x) = \mathcal{F}_\xi^{-1}\left(\widehat{v}_0(\xi)\right). \tag{2.19}$$

Moreover, there exists $\varepsilon_0 > 0$ *such that, for all* $0 < \varepsilon < \varepsilon_0$, *we have*

(i) *There exists a positive constant* C *depending on* ε_0, *but independent of* ε *such that*

$$\|w_0\|_{L^2(\mathbb{R}^n)^n} = \frac{\varepsilon}{(2\pi)^n}, \quad \|\sigma_0\|_{L^2(\mathbb{R}^n)} \leqslant \frac{1}{(2\pi)^n}, \tag{2.20}$$

$$\frac{\beta^{1/m}}{c_0^2(2\pi)^n}e^{\frac{(m-1)(\omega-\omega_0)\tau}{m}} \leqslant \|\sigma(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \leqslant \frac{1}{(2\pi)^n}e^{(\omega-\omega_0)\tau}, \quad m\in\mathbb{N}, \tag{2.21}$$

$$\|v(t,\cdot)\|_{H^s(\mathbb{R}^n)^n} \leqslant C\varepsilon^{(1-s)}, \qquad s\in[0,1),\ t\in[0,\tau]. \tag{2.22}$$

(ii) *For any* $\eta > 0$, *there exists a constant* C, *depending on* ω, τ, η *and* ε_0, *but independent of* ε *such that*

$$\|\sigma(t,\cdot)\|_{L^2(|x-x_0|\geqslant\eta)} \leqslant C\varepsilon^{k-n/4}, \qquad k\in\mathbb{N}, t\in[0,\tau], \tag{2.23}$$

$$\|w_0\|_{H^1(|x-x_0|\geqslant\eta))^n} \leqslant C\varepsilon^{k-1-n/4}, \qquad k\in\mathbb{N}, \tag{2.24}$$

$$\|v\|_{H^1(0,\tau;H^2(|x-x_0|\geqslant\eta))^n} \leqslant C\varepsilon^{k-1-n/4}, \qquad k\in\mathbb{N}. \tag{2.25}$$

Proof. Let us recall

$$\sigma(t,x)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\widehat{\sigma}_0(\xi)e^{ix\cdot\xi}e^{(\delta(\xi)+\omega)t}\,\mathrm{d}\xi. \tag{2.26}$$

By Parseval's relation, for all $t\in[0,\tau]$ we have

$$\begin{aligned}
\|\sigma(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} &= \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^n}|\widehat{\sigma}_0(\xi)|^2e^{2(\mathrm{Re}\delta(\xi)+\omega)t}\,\mathrm{d}\xi\\
&= \frac{\varepsilon^{2+\frac{n}{2}}}{(2\pi)^{2n}}\int_{\mathbb{R}^n}\xi_1^2\left|\psi\left(\sqrt{\varepsilon}\left(\xi-\frac{\bar{\xi}}{\varepsilon}\right)\right)\right|^2e^{2(\mathrm{Re}\delta(\xi)+\omega)t}\,\mathrm{d}\xi\\
&= \frac{1}{(2\pi)^{2n}}\int_{|\zeta|\leqslant 1}|\zeta_1\sqrt{\varepsilon}+\bar{\xi}_1|^2|\psi(\zeta)|^2\exp\left(2\mathrm{Re}\delta\left(\frac{\zeta}{\sqrt{\varepsilon}}+\frac{\bar{\xi}}{\varepsilon}\right)t\right)e^{2\omega t}\,\mathrm{d}\zeta.
\end{aligned}$$

From the above relation, by choosing ε sufficiently small, we get (2.20). Next, we define

$$\kappa=(\omega-\omega_0)-\frac{\ln\beta}{\tau}.$$

Note that, by the hypothesis of the theorem we have $\kappa>0$. Let $m\in\mathbb{N}$. We choose ε sufficiently small so that for any $|\zeta|\leqslant 1$, $\left|\frac{\zeta}{\sqrt{\varepsilon}}+\frac{\bar{\xi}}{\varepsilon}\right|>\xi_0$, and then from Lemma 2.7 we have

$$\mathrm{Re}\delta\left(\frac{\zeta}{\sqrt{\varepsilon}}+\frac{\bar{\xi}}{\varepsilon}\right)=\delta\left(\frac{\zeta}{\sqrt{\varepsilon}}+\frac{\bar{\xi}}{\varepsilon}\right)\in\left(-\omega_0-\frac{\kappa}{m},-\omega_0\right),$$
$$\frac{1}{c_0^4}\leqslant|\zeta_1\sqrt{\varepsilon}+\bar{\xi}_1|^2\leqslant 1 \text{ for } |\zeta|\leqslant 1.$$

With the above choice of ε and using the above estimates, it is easy to see that

$$\frac{1}{c_0^2(2\pi)^n}e^{(\omega-\omega_0-\kappa/m)\tau}\leqslant\|\sigma(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\leqslant\frac{1}{(2\pi)^n}e^{(\omega-\omega_0)\tau}.$$

Using the definition of κ the above expression can be simplified as

$$\frac{\beta^{1/m}}{c_0^2(2\pi)^n}e^{\frac{(m-1)(\omega-\omega_0)\tau}{m}}\leqslant\|\sigma(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\leqslant\frac{1}{(2\pi)^n}e^{(\omega-\omega_0)\tau}.$$

This completes the proof of (2.21). Recalling the expression of σ from (2.26), we have

$$\begin{aligned}
\sigma(t,x) &= \frac{\varepsilon^{1+\frac{n}{4}}}{(2\pi)^n}\int_{\mathbb{R}^n}i\xi_1\psi\left(\sqrt{\varepsilon}\left(\xi-\frac{\bar{\xi}}{\varepsilon}\right)\right)e^{i(x-x_0)\cdot\xi}e^{(\delta(\xi)+\omega)t}\,d\xi\\
&= \frac{\varepsilon^{-n/4}}{(2\pi)^n}\int_{|\zeta|\leqslant 1}i(\zeta_1\sqrt{\varepsilon}+\bar{\xi}_1)\psi(\zeta)e^{i(x-x_0)\cdot(\frac{\zeta}{\sqrt{\epsilon}}+\frac{\bar{\xi}}{\varepsilon})}e^{\delta(\frac{\zeta}{\sqrt{\varepsilon}}+\frac{\bar{\xi}}{\varepsilon})t}e^{\omega t}\,d\zeta.
\end{aligned}$$

Note that

$$\Delta_\zeta^k e^{i(x-x_0)\cdot(\frac{\zeta}{\sqrt{\varepsilon}}+\frac{\bar{\xi}}{\varepsilon})}=(-1)^k\left(\frac{|x-x_0|^2}{\varepsilon}\right)^k e^{i(x-x_0)\cdot(\frac{\zeta}{\sqrt{\varepsilon}}+\frac{\bar{\xi}}{\varepsilon})}\qquad k\in\mathbb{N}.$$

Thus for $|x-x_0|\geqslant \eta>0$ we have

$$\begin{aligned}\sigma(t,x) &= \frac{(-1)^k i\varepsilon^{k-n/4}}{(2\pi)^n|x-x_0|^{2k}}\int_{|\zeta|\leqslant 1}\Delta_\zeta^k\left(e^{i(x-x_0)\cdot(\frac{\zeta}{\sqrt{\varepsilon}}+\frac{\bar\xi}{\epsilon})}\right)\\ &\qquad\times(\zeta_1\sqrt{\varepsilon}+\bar\xi_1)\psi(\zeta)e^{\delta(\frac{\zeta}{\sqrt{\varepsilon}}+\frac{\bar\xi}{\varepsilon})t}e^{\omega t}\,d\zeta\\ &= \frac{(-1)^k i\varepsilon^{k-n/4}}{(2\pi)^n|x-x_0|^{2k}}\int_{|\zeta|\leqslant 1}e^{i(x-x_0)\cdot(\frac{\zeta}{\sqrt{\varepsilon}}+\frac{\bar\xi}{\varepsilon})}\\ &\qquad\times\Delta_\zeta^k\left((\zeta_1\sqrt{\varepsilon}+\bar\xi_1)\psi(\zeta)e^{\delta(\frac{\zeta}{\sqrt{\varepsilon}}+\frac{\bar\xi}{\varepsilon})t}\right)e^{\omega t}\,d\zeta.\end{aligned}\tag{2.27}$$

A simple calculation yields

$$\begin{aligned}&\Delta_\zeta\left((\zeta_1\sqrt{\varepsilon}+\bar\xi_1)\psi(\zeta)e^{\delta(\frac{\zeta}{\sqrt{\varepsilon}}+\frac{\bar\xi}{\varepsilon})t}\right)\\ &= e^{\delta(\frac{\zeta}{\sqrt{\varepsilon}}+\frac{\bar\xi}{\varepsilon})t}\Big[2\sqrt{\varepsilon}\partial_{\zeta_1}\psi(\zeta)+2t\psi(\zeta)\partial_1\delta(\frac{\zeta}{\sqrt{\varepsilon}}+\frac{\bar\xi}{\varepsilon})\\ &\qquad+\frac{2t}{\sqrt{\varepsilon}}(\zeta_1\sqrt{\varepsilon}+\bar\xi_1)\nabla\delta(\frac{\zeta}{\sqrt{\varepsilon}}+\frac{\bar\xi}{\varepsilon})\cdot\nabla\psi(\zeta)+(\zeta_1\sqrt{\varepsilon}+\bar\xi_1)\Delta_\zeta\psi\\ &\qquad+\frac{\zeta_1\sqrt{\varepsilon}+\bar\xi_1}{\varepsilon}\Big(t\Delta\delta(\frac{\zeta}{\sqrt{\varepsilon}}+\frac{\bar\xi}{\varepsilon})+t^2\Big|\nabla\delta(\frac{\zeta}{\sqrt{\varepsilon}}+\frac{\bar\xi}{\varepsilon})\Big|^2\Big)\Big].\end{aligned}$$

Using (2.16) and the fact that ψ is compactly supported, we infer that, there exists $\varepsilon_0>0$ such that, for $\varepsilon<\varepsilon_0$

$$\begin{aligned}&\left|\Delta_\zeta\left((\zeta_1\sqrt{\varepsilon}+\bar\xi_1)\psi(\zeta)e^{\delta(\frac{\zeta}{\sqrt{\epsilon}}+\frac{\bar\xi}{\epsilon})t}\right)\right|\\ &\leqslant C\Big(1+\sqrt{\varepsilon}+\frac{\varepsilon+\sqrt{\varepsilon}}{|\zeta\sqrt{\varepsilon}+\bar\xi|}+\frac{\varepsilon}{|\zeta\sqrt{\varepsilon}+\bar\xi|^2}\Big)\leqslant C \qquad (|\zeta|\leqslant 1),\end{aligned}$$

for some constant C independent of ε. Moreover, proceeding similarly, we can show that, there exists $\varepsilon_0>0$ such that, for $\varepsilon<\varepsilon_0$

$$\left|\Delta_\zeta^k\left((\zeta_1\sqrt{\varepsilon}+\bar\xi_1)\psi(\zeta)e^{\delta(\frac{\zeta}{\sqrt{\epsilon}}+\frac{\bar\xi}{\epsilon})t}\right)\right|\leqslant C \qquad (|\zeta|\leqslant 1),$$

for some constant C independent of ε.

Thus for $0<\varepsilon<\varepsilon_0$ and for $|x-x_0|\geqslant\eta>0$ we get

$$|\sigma(t,x)|\leqslant Ce^{\omega t}\frac{\varepsilon^{k-n/4}}{(2\pi)^n|x-x_0|^{2k}}.\tag{2.28}$$

This proves (2.23). (2.24) can be proved in a similar manner. To prove estimate (2.22), we note that, for $t\in[0,\tau]$ and $s\in[0,1)$

$$\begin{aligned}\|v(t,\cdot)\|^2_{H^s(\mathbb{R}^n)} &= \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^n}\frac{(1+|\xi|^2)^s}{|\xi|^2}|\widehat{\sigma}_0(\xi)|^2\left|\frac{\delta(\xi)}{\bar{\rho}i}\right|^2 e^{2(\mathrm{Re}\delta(\xi)+\omega)t}\,\mathrm{d}\xi\\ &\leqslant Ce^{\omega t}\frac{\varepsilon^{2+\frac{n}{2}}}{(2\pi)^{2n}}\int_{\mathbb{R}^n}\frac{(1+|\xi|^2)^s}{|\xi|^2}\xi_1^2\left|\psi\left(\sqrt{\varepsilon}\left(\xi-\frac{\bar{\xi}}{\varepsilon}\right)\right)\right|^2\,\mathrm{d}\xi\\ &= Ce^{\omega t}\frac{\varepsilon^{2(1-s)}}{(2\pi)^{2n}}\int_{|\zeta|\leqslant 1}\frac{(|\zeta\sqrt{\varepsilon}+\bar{\xi}|^2+\varepsilon^2)^s}{|\zeta\sqrt{\varepsilon}+\bar{\xi}|^2}|\zeta_1\sqrt{\varepsilon}+\bar{\xi}_1|^2|\psi(\zeta)|^2\,\mathrm{d}\zeta\\ &\leqslant Ce^{\omega t}\frac{\varepsilon^{2(1-s)}}{(2\pi)^{2n}}.\end{aligned}$$

From the above estimate we deduce (2.22). To prove (2.25), we may proceed as in the proof of (2.23) above. □

We are now in a position to prove Theorem 2.3 and Theorem 2.4.

Proof of Theorem 2.3. The proof is by contradiction. Let $\omega \geqslant \omega_0$. We assume that the system (2.1) is exponentially stabilizable in $\mathcal{H}$ with an exponential decay rate $\omega \geqslant \omega_0$. Then according to Proposition 2.6, there exist $\beta \in (0,1)$ and $\tau > 0$, such that the observability inequality (2.8) holds for any $(\sigma_0, v_0) \in \mathcal{H}$. In particular, there exist $\beta \in (0,1)$, $\tau > 0$ and $C \geqslant 0$ such that, for any $(\sigma_0, v_0) \in \mathcal{H}$, the solution of (2.7) satisfies the following observability estimate

$$\begin{aligned}\|\sigma(\tau,\cdot)\|_{L^2(\Omega)} - \beta\,\|\sigma_0\|_{L^2(\Omega)} + \|v(\tau,\cdot)\|_{L^2(\Omega)^n}\\ \leqslant C\left(\|\sigma\|_{L^2(0,\tau;L^2(\mathcal{O}_1))} + \|v\|_{L^2(0,\tau;L^2(\mathcal{O}_2))^n}\right) + \beta\,\|v_0\|_{L^2(\Omega)^n}.\end{aligned}\tag{2.29}$$

Let us fix such β and τ, and we take $k \in \mathbb{N}$ be such that $k > 1+\frac{n}{4}$. We are going to construct (σ_0, v_0) such that the corresponding solution of (2.7) does not satisfy the above observability estimate. We choose x_0 and $\eta > 0$ such that

$$B(x_0;\eta) \subset \Omega \text{ and } B(x_0;\eta) \cap \mathcal{O}_1 = \emptyset.$$

We also fix $m \in \mathbb{N}, m \geqslant 1$ and $c_0 > 1$ in Theorem 2.8 such that

$$c_0^4\beta^{1-1/m} < 1.\tag{2.30}$$

With the above choice of parameters, let $(\sigma^\sharp, v^\sharp)$ be constructed as in Theorem 2.8. We define

$$h(t,x) := v^\sharp(t,x) \qquad t \in [0,\tau], x \in \partial\Omega.$$

By Theorem 2.8, there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon < \varepsilon_0$, we have

$$\left\|w_0^\sharp\right\|_{[H^1(\Omega)\setminus B(x_0,\eta)]^n} \leqslant C\varepsilon^{k-1-n/4},\tag{2.31}$$

$$\sigma_0^\sharp = \operatorname{div} w_0^\sharp \text{ in } \Omega, \qquad \left\|\sigma_0^\sharp\right\|_{L^2(\Omega)} \leqslant \frac{1}{(2\pi)^n}, \tag{2.32}$$

$$\begin{aligned}\left\|\sigma^\sharp(\tau,\cdot)\right\|_{L^2(\Omega)} \geqslant \left\|\sigma^\sharp(\tau,\cdot)\right\|_{L^2(B(x_0;\eta))} &\geqslant \frac{\beta^{1/m}}{c_0^2(2\pi)^n} e^{\frac{(m-1)(\omega-\omega_0)\tau}{m}} - C\varepsilon^{k-n/4} \\ &\geqslant \frac{\beta^{1/m}}{c_0^3(2\pi)^n} e^{\frac{(m-1)(\omega-\omega_0)\tau}{m}} \geqslant \frac{\beta^{1/m}}{c_0^3(2\pi)^n},\end{aligned} \tag{2.33}$$

$$\left\|v_0^\sharp\right\|_{L^2(\Omega)^n} \leqslant C\varepsilon, \quad \left\|v^\sharp(\tau,\cdot)\right\|_{L^2(\Omega)^n} \leqslant C\varepsilon, \tag{2.34}$$

$$\left\|\sigma^\sharp\right\|_{L^2(0,\tau;L^2(\mathcal{O}_1))} \leqslant C\varepsilon^{k-n/4}, \quad \left\|v^\sharp\right\|_{L^2(0,\tau;L^2(\mathcal{O}_2))^n} \leqslant C\varepsilon, \tag{2.35}$$

$$\|h\|_{H^1(0,\tau;H^{3/2}(\partial\Omega))^n} \leqslant C\varepsilon^{k-1-n/4}, \tag{2.36}$$

for some constant C independent of ε. Let ζ be a smooth function in $\overline{\Omega}$ such that $0 \leqslant \zeta \leqslant 1$, $\zeta = 0$ in $\overline{B(x_0,\eta)}$ and $\zeta = 1$ on $\partial\Omega$. Next, we consider the following system:

$$\begin{cases} \partial_t\sigma^\dagger - \omega\sigma^\dagger - \bar\rho \operatorname{div} v^\dagger = 0 & \text{in } (0,\tau)\times\Omega, \\ \partial_t v^\dagger - \omega v^\dagger - \dfrac{\mu}{\bar\rho}\Delta v^\dagger - \dfrac{\mu+\alpha}{\bar\rho}\nabla(\operatorname{div} v^\dagger) - a\gamma\bar\rho^{\gamma-2}\nabla\sigma^\dagger = 0 & \text{in } (0,\tau)\times\Omega, \\ v^\dagger = h & \text{on } (0,\tau)\times\partial\Omega, \\ \sigma^\dagger(0) = \operatorname{div}(\zeta w_0^\sharp), \quad v^\dagger(0) = 0 & \text{in } \Omega. \end{cases} \tag{2.37}$$

The uniqueness of the solution to the above system is obvious. Let h_e be an interior lifting of h such that

$$\begin{aligned} &h_e \in H^1(0,\tau;H^2(\Omega))^n, \quad h_e = h \text{ on } [0,\tau]\times\partial\Omega, \\ &\|h_e\|_{H^1(0,\tau;H^2(\Omega))^n} \leqslant C\,\|h\|_{H^1(0,\tau;H^{3/2}(\partial\Omega))^n}. \end{aligned} \tag{2.38}$$

We set $\widetilde{v} = v^\dagger - h_e$, so that $(\sigma^\dagger, \widetilde{v})$ solves

$$\begin{cases} \partial_t\sigma^\dagger - \omega\sigma^\dagger - \bar\rho \operatorname{div}\widetilde{v} = f_1 & \text{in } (0,\tau)\times\Omega, \\ \partial_t\widetilde{v} - \omega\widetilde{v} - \dfrac{\mu}{\bar\rho}\Delta\widetilde{v} - \dfrac{\mu+\alpha}{\bar\rho}\nabla(\operatorname{div}\widetilde{v}) - a\gamma\bar\rho^{\gamma-2}\nabla\sigma^\dagger = f_2 & \text{in } (0,\tau)\times\Omega, \\ \widetilde{v} = 0 & \text{on } (0,\tau)\times\partial\Omega, \\ \sigma^\dagger(0) = \operatorname{div}(\zeta w_0^\sharp), \quad v^\dagger(0) = h_e(0) & \text{in } \Omega. \end{cases} \tag{2.39}$$

where $f_1 = \bar\rho \operatorname{div} h_e$ and $f_2 = -\partial_t h_e + \omega h_e + \frac{\mu}{\bar\rho}\Delta h_e + \frac{\mu+\alpha}{\bar\rho}\nabla(\operatorname{div} h_e)$. Note that, $(f_1, f_2) \in L^2(0,\tau;\mathcal{H})$. Then using the fact that $\mathcal{A}^*$ generates a C^0-semigroup on $\mathcal{H}$ as mentioned in Proposition 2.2 and (2.38), we get

$$\left\|(\sigma^\dagger, v^\dagger)\right\|_{C([0,\tau];\mathcal{H})} \leqslant \left\|(\sigma^\dagger, \widetilde{v})\right\|_{C([0,\tau];\mathcal{H})} + \|h_e\|_{C([0,\tau];L^2(\Omega))^n}$$

$$\leqslant C\left(\left\|\mathrm{div}\,(\zeta w_0^\sharp)\right\|_{L^2(\Omega)} + \|h_e(0)\|_{L^2(\Omega)^n} + \|(f_1,f_2)\|_{L^2(0,\tau;\mathcal{H})}\right) + \|h_e\|_{C([0,\tau];L^2(\Omega))^n}$$
$$\leqslant C\left(\left\|\mathrm{div}\,(\zeta w_0^\sharp)\right\|_{L^2(\Omega)} + \|h_e\|_{H^1(0,\tau;H^2(\Omega))^n}\right)$$
$$\leqslant C\left(\left\|\mathrm{div}\,(\zeta w_0^\sharp)\right\|_{L^2(\Omega)} + \|h\|_{H^1(0,\tau;H^{3/2}(\partial\Omega))^n}\right).$$

Moreover, using (2.31) and (2.36), for $\varepsilon < \varepsilon_0$, we have
(2.40)
$$\left\|(\sigma^\dagger, v^\dagger)\right\|_{C([0,\tau];\mathcal{H})} \leqslant C\left(\|h\|_{H^1(0,\tau;H^{3/2}(\partial\Omega))} + \left\|\mathrm{div}\,(\zeta w_0^\sharp)\right\|_{L^2(\Omega)}\right) \leqslant C\varepsilon^{k-1-n/4},$$
for some constant C independent of ε. We set
$$\sigma(t,x) = \sigma^\sharp(t,x) - \sigma^\dagger(t,x), \qquad v(t,x) = v^\sharp(t,x) - v^\dagger(t,x) \quad t\in[0,\tau], x\in\Omega.$$
Then (σ, v) satisfies the system (2.7) with the initial data
$$\sigma_0(x) = \sigma_0^\sharp(x) - \sigma^\dagger(0,x) = \mathrm{div}(w_0^\sharp(x) - \zeta w_0^\sharp(x)), \quad v_0(x) = v_0^\sharp(x). \tag{2.41}$$
Combining (2.40) together with (2.33) - (2.36), it is easy to verify that, for ε sufficiently small
$$\|\sigma(\tau,\cdot)\|_{L^2(\Omega)} \geqslant \frac{\beta^{1/m}}{c_0^3(2\pi)^n} - C\varepsilon^{k-1-n/4},$$
$$\|\sigma_0\|_{L^2(\Omega)} \leqslant \frac{1}{(2\pi)^n} + C\varepsilon^{k-1-n/4},$$

where the constant C is independent of ε. Thus, using (2.30) along with the above estimates, for ε sufficiently small, we obtain
$$\text{L.H.S of (2.29)} > \|\sigma(\tau,\cdot)\|_{L^2(\Omega)} - \beta\,\|\sigma_0\|_{L^2(\Omega)}$$
$$> \frac{\beta^{1/m}}{c_0^3(2\pi)^n} - C\varepsilon^{k-1-n/4} - \frac{\beta}{(2\pi)^n} - \beta C\varepsilon^{k-1-n/4}$$
$$> \frac{1}{(2\pi)^n}\left(\frac{\beta^{1/m}}{c_0^4} - \beta\right) > 0,$$

whereas
$$\text{R.H.S of (2.29)} \leqslant C\varepsilon.$$
This is a contradiction to (2.29). Therefore, the system (2.1) is not exponentially stabilizable in $\mathcal{H}$ with an exponential decay rate $\omega \geqslant \omega_0$. □

Proof of Theorem 2.4. The proof of Theorem 2.4 is similar to the proof of Theorem 2.3 given above. The main difference is that we now have to construct initial data $(\sigma_0, v_0) \in \mathcal{H}_{\mathrm{m}}$ instead of $\mathcal{H}$. However, this follows from (2.41). In fact, integrating σ^0 over Ω and using the fact that $\zeta = 1$ on $\partial\Omega$ we have

$$\int_{\Omega} \sigma^0(x)\, \mathrm{d}x = \int_{\partial\Omega} (w_0^\sharp(x) - \zeta w_0^\sharp(x)) \cdot \nu\, \mathrm{d}\gamma = 0.$$

Thus the initial data (σ_0, v_0) constructed in (2.41) belongs to $\mathcal{H}_{\mathrm{m}}$. The rest of the proof is similar to that of Theorem 2.3. □

3. OTHER ONE-DIMENSIONAL ODE-PARABOLIC COUPLED MODELS

In this section, we consider two other linear compressible type models in one space dimension. The first one is compressible Navier-Stokes-Fourier system linearized around constant trajectories. This is a linear system coupled with two parabolic equations and an ODE. The second one is a linearized blood flow model. This system is similar to (2.1) in one space dimension with some lower order terms in the parabolic component.

3.1. Linearized Navier-Stokes-Fourier system

We consider the following linear control problem

(3.1)

$$\begin{cases} \partial_t \rho + \bar{\rho}\partial_x u = \mathbb{1}_{\mathcal{O}_1} f_1 & (t,x) \in (0,\infty) \times (0,L), \\ \partial_t u - \dfrac{\alpha + 2\mu}{\bar{\rho}}\partial_{xx} u + \dfrac{R\bar{\theta}}{\bar{\rho}}\partial_x \rho + R\partial_x \vartheta = \mathbb{1}_{\mathcal{O}_2} f_2 & (t,x) \in (0,\infty) \times (0,L), \\ \partial_t \vartheta - \dfrac{\kappa}{\bar{\rho} c_v}\partial_{xx}\vartheta + \dfrac{R\bar{\theta}}{c_v}\partial_x u = \mathbb{1}_{\mathcal{O}_3} f_3 & (t,x) \in (0,\infty) \times (0,L), \\ u(t,0) = u(t,L) = \vartheta(t,0) = \vartheta(t,L) = 0 & t \in (0,\infty), \\ \rho(0,x) = \rho_0(x),\ u(0,x) = u_0(x),\ \vartheta(0,x) = \vartheta_0(x) & x \in (0,L), \end{cases}$$

where $(\alpha+2\mu), \bar{\rho}, R, \bar{\theta}, \kappa$ and c_v are positive constants, and f_1, f_2, f_3 are interior controls. Null controllability of the above system was studied in [10]. More precisely, if $\mathcal{O}_1 \subset (0,L)$ and $\mathcal{O}_2, \mathcal{O}_3 \subseteq (0,L)$ the system (3.1) is not null controllable in $L^2(0,L)^3$ at any time $\tau > 0$ (see for instance [10, Theorem 1.1]). Here we prove the following result

THEOREM 3.1. *Let*

$$\mathcal{O}_1 \subset (0,L), \quad \mathcal{O}_2, \mathcal{O}_3 \subseteq (0,L), \qquad \omega_0 := \frac{R\bar{\rho}\bar{\theta}}{2\mu + \alpha}. \tag{3.2}$$

The system (3.1) is not exponentially stabilizable in $L^2(0,L)^3$ with an exponential decay rate $\omega \geqslant \omega_0$, by interior controls $f_1 \in L^2(0,\infty;L^2(\mathcal{O}_1))$, $f_2 \in L^2(0,\infty;L^2(\mathcal{O}_2))$ and $f_3 \in L^2(0,\infty;L^2(\mathcal{O}_3))$.

This result can be proved in a similar manner as we proved Theorem 2.3. At first, we obtain a suitable observability inequality which is equivalent to the exponential stabilizability. Next, we construct suitable *Gaussian beam* solutions to show that observability inequality does not hold. In fact, the *Gaussian beam* solution of the adjoint of system (3.1) was already constructed in [10, Section 2.2].The proof is left to the reader.

3.2. **Linear blood flow type model**

Let us consider the following linear control system

(3.3)

$$\begin{cases} \partial_t a + \partial_x q = \mathbb{1}_{\mathcal{O}_1} f_1 & (t,x) \in (0,\infty)\times(0,L), \\ \partial_t q - \mu_0 \partial_{xx} q + \mu_1 \partial_x q + \mu_2 q + \alpha_1 \partial_x a + \alpha_2 a = \mathbb{1}_{\mathcal{O}_2} f_2 & (t,x) \in (0,\infty)\times(0,L), \\ q(t,0) = q(t,L) = 0 & t \in (0,\infty), \\ a(0,x) = a_0(x), \quad q(0,x) = q_0(x) & x \in (0,L), \end{cases}$$

where $\mu_0, \mu_1, \mu_2, \alpha_1, \alpha_2$ are constants and $\alpha_1, \mu_0 > 0$. This system can be obtained by linearising system (1.3) of [12], which models one dimensional blood flow in a vessel with viscoelastic walls, around a constant steady state. Note that, the system (2.1) in dimension one is a special case of the above system.

The system (3.3) is also not null controllable at any time $\tau > 0$ by the interior controls f_1 and f_2 (see for instance [1, Theorem 1.4]). We have the following result regarding the lack of stabilizability

Theorem 3.2. *Let*

$$\mathcal{O}_1 \subset (0,L), \quad \mathcal{O}_2 \subseteq (0,L), \qquad \omega_0 := \frac{\alpha_1}{\mu_0}. \tag{3.4}$$

The system (3.3) is not exponentially stabilizable in $L^2(0,L)^2$ with an exponential decay rate $\omega \geqslant \omega_0$, by interior controls $f_1 \in L^2(0,\infty;L^2(\mathcal{O}_1))$ and $f_2 \in L^2(0,\infty;L^2(\mathcal{O}_2))$.

The proof of this theorem can also be obtained by following the same philosophy as the proof of Theorem 2.3. The construction of *Gaussian beam* solutions of the adjoint of system (3.3) is similar to Theorem 2.8 above and [10, Theorem 2.9]. Thus the details are left to the reader.

4. LINEAR VISCOELASTIC JEFFREYS SYSTEM

In this section, we consider linear viscoelastic flows in a smooth bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$. More precisely, we consider linear single mode Jeffreys system (see for instance [11, Eq. 1.3]):

$$
\begin{cases}
\bar{\rho}\partial_t u - \mu_0 \Delta u + \nabla p - \operatorname{div} \mathbb{S} = \mathbb{1}_{\mathcal{O}} f & (t,x) \in (0,\infty) \times \Omega,\\
\operatorname{div} u = 0 & (t,x) \in (0,\infty) \times \Omega,\\
\partial_t \mathbb{S} + \mu_1 \mathbb{S} - 2\mu_2 \mathbb{D} u = 0 & (t,x) \in (0,\infty) \times \Omega,\\
u = 0 & (t,x) \in (0,\infty) \times \partial\Omega,\\
u(0) = u_0, \quad \mathbb{S}(0) = \mathbb{S}_0 & x \in \Omega,
\end{cases}
\tag{4.1}
$$

where

$$
\mathbb{D} u = \frac{1}{2}(\nabla u + \nabla u^\top),
$$

and μ_0, μ_1 and μ_2 are positive constants. In the above system $u(t,\cdot) : \Omega \to \mathbb{R}^n$, $p(t,\cdot) : \Omega \to \mathbb{R}$ and $\mathbb{S}(t,\cdot) : \Omega \to \mathbb{R}^{n\times n}$ represent the fluid velocity, pressure and stress tensor respectively. Let us consider the spaces

$$
L^2_\sigma(\Omega) = \left\{ u \in (L^2(\Omega))^n \mid \operatorname{div} u = 0 \text{ in } \Omega,\ u.\nu = 0 \text{ in } \partial\Omega \right\}
$$

and

$$
H^1_{\sigma,0}(\Omega) = L^2_\sigma(\Omega) \cap H^1_0(\Omega)^n,
$$

where ν is the outward unit normal to $\partial\Omega$. Let $\mathcal{P}$ be the Leray projector from $L^2(\Omega)^n$ onto $L^2_\sigma(\Omega)$. We denote by $A_0 = \mathcal{P}\Delta$, the Stokes operator in $L^2_\sigma(\Omega)$ with $\mathcal{D}(A_0) = H^2(\Omega)^n \cap H^1_{\sigma,0}(\Omega)$. Let $\mathcal{L}_S(\mathbb{R}^n)$ denote the space of all symmetric real $n \times n$ matrices, and we take

$$
\mathcal{H} = L^2_\sigma(\Omega) \times L^2(\Omega; \mathcal{L}_S(\mathbb{R}^n)),
$$

equipped with the inner product

$$
\left\langle \begin{pmatrix} u \\ \mathbb{S} \end{pmatrix}, \begin{pmatrix} w \\ \mathbb{Q} \end{pmatrix} \right\rangle_{\mathcal{H}} = \bar{\rho} \int_\Omega u \cdot w \,\mathrm{d}x + \frac{1}{2\mu_2} \int_\Omega \mathbb{S} : \mathbb{Q} \,\mathrm{d}x.
$$

We consider the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{H}$ defined by

$$
\mathcal{D}(\mathcal{A}) = \Big\{ (u, \mathbb{S}) \in H^1_{\sigma,0}(\Omega) \times L^2(\Omega; \mathcal{L}_S(\mathbb{R}^n)) \\
\mid \exists p \in L^2_{\mathrm{m}}(\Omega) \text{ such that } -\mu_0 \Delta u + \nabla p - \operatorname{div} \mathbb{S} \in L^2_\sigma(\Omega) \Big\},
$$

$$
\mathcal{A} \begin{bmatrix} u \\ \mathbb{S} \end{bmatrix} = \begin{bmatrix} \frac{1}{\bar{\rho}}(\mu_0 \Delta u - \nabla p + \operatorname{div} \mathbb{S}) \\ -\mu_1 \mathbb{S} + 2\mu_2 \mathbb{D} u \end{bmatrix}.
\tag{4.2}
$$

We have the following result regarding well-posedness of the system (4.1).

Lemma 4.1. *The operator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ is the infinitesimal generator of a strongly continuous semigroup on $\mathcal{H}$.*

Proof. Note that for all $(u, \mathbb{S}) \in \mathcal{D}(\mathcal{A})$, we have

$$\left\langle \mathcal{A}\begin{bmatrix} u \\ \mathbb{S} \end{bmatrix}, \begin{bmatrix} u \\ \mathbb{S} \end{bmatrix} \right\rangle_{\mathcal{H}} = -\mu_0 \int_\Omega |\nabla u|^2 \,\mathrm{d}x - \frac{\mu_1}{2\mu_2} \int_\Omega \mathbb{S} : \mathbb{S} \,\mathrm{d}x \leqslant 0,$$

and hence $\mathcal{A}$ is dissipative. In order to show $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ is maximal in $\mathcal{H}$, for any $(f, \mathbb{G}) \in \mathcal{H}$, we consider the system

$$\text{(4.3)} \qquad \begin{cases} u - \dfrac{1}{\bar{\rho}}(\mu_0 \Delta u - \nabla p + \operatorname{div} \mathbb{S}) = f, \qquad \operatorname{div} u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ (1 + \mu_1)\mathbb{S} - 2\mu_2 \mathbb{D}u = \mathbb{G} & \text{in } \Omega. \end{cases}$$

Note that, solving the second equation of $(4.3)_2$ we get $\mathbb{S} = \frac{2\mu_2}{1+\mu_1}\mathbb{D}u + \frac{1}{1+\mu_1}\mathbb{G}$, and therefore u solves

$$\begin{cases} u - \dfrac{1}{\bar{\rho}}[(\mu_0 + \dfrac{2\mu_2}{1+\mu_1})\Delta u - \nabla p] = f + \dfrac{1}{1+\mu_1}\operatorname{div} \mathbb{G} & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The right hand side of the above equation belongs to $[H^{-1}(\Omega)]^n$, the dual of $[H_0^1(\Omega)]^n$ with respect to the pivot space $L^2(\Omega)^n$. Therefore by [4, Theorem IV.5.1] the above system admits a unique solution $(u, p) \in H^1_{0,\sigma}(\Omega) \times L^2_{\mathrm{m}}(\Omega)$. Next, from $(4.3)_3$ it follows that $\mathbb{S} \in L^2(\Omega; \mathcal{L}_S(\mathbb{R}^n))$ and from $(4.3)_1$ it follows $\mu_0\Delta u - \nabla p + \operatorname{div} \mathbb{S} \in L^2_\sigma(\Omega)$. $\square$

Our main result of this section is as follows.

Theorem 4.2. *Let*

$$\text{(4.4)} \qquad \mathcal{O} \subseteq \Omega, \qquad \omega_0 := \frac{\mu_2}{\mu_0} + \mu_1.$$

The system (4.1) is not exponentially stabilizable in $\mathcal{H}$ with an exponential decay rate $\omega \geqslant \omega_0$, by interior control $f \in L^2(0, \infty; L^2(\mathcal{O}))^n$.

In order to prove the above result, we assume that $\mathbb{S}$ is of the form $\mathbb{S}(t, \cdot) = 2\mathbb{D}v(t, \cdot)$, for some $v(t, \cdot) \in H^1_{\sigma,0}(\Omega)$, $t \geqslant 0$. Then (u, v) satisfies the following system

$$
(4.5)\qquad \begin{cases} \bar\rho\partial_t u-\mu_0\Delta u+\nabla p-\Delta v=\mathbb{1}_{\mathcal{O}}f & (t,x)\in(0,\infty)\times\Omega,\\ \partial_t v+\mu_1 v-\mu_2 u=0 & (t,x)\in(0,\infty)\times\Omega,\\ \operatorname{div}u=0=\operatorname{div}v & (t,x)\in(0,\infty)\times\Omega,\\ u=0=v & (t,x)\in(0,\infty)\times\partial\Omega,\\ u(0)=u_0,\quad v(0)=v_0 & x\in\Omega. \end{cases}
$$

Let us take $\mathcal{H}_0=L^2_\sigma(\Omega)\times H^1_{\sigma,0}(\Omega)$ equipped with the inner product

$$
\left\langle \begin{pmatrix}u\\v\end{pmatrix},\begin{pmatrix}w\\z\end{pmatrix}\right\rangle_{\mathcal{H}_0}=\int_\Omega u\cdot w\,\mathrm{d}x+\frac{1}{\mu_2\bar\rho}\int_\Omega \nabla v:\nabla z\,\mathrm{d}x.
$$

We define an unbounded operator $(\mathcal{A}_0,\mathcal{D}(\mathcal{A}_0))$ on $\mathcal{H}_0$:

$$
\mathcal{D}(\mathcal{A}_0)=\left\{(u,v)\in H^1_{\sigma,0}(\Omega)\times H^1_{\sigma,0}(\Omega)\mid \mu_0 u+v\in\mathcal{D}(A_0)\right\},
$$

$$
(4.6)\qquad \mathcal{A}_0\begin{bmatrix}u\\v\end{bmatrix}=\begin{bmatrix}\frac{1}{\bar\rho}A_0(\mu_0u+v)\\ \mu_2u-\mu_1v\end{bmatrix}.
$$

Lemma 4.3. *The operator $(\mathcal{A}_0,\mathcal{D}(\mathcal{A}_0))$ is the infinitesimal generator of a strongly continuous semigroup on $\mathcal{H}_0$.*

Proof. Note that for all $(u,v)\in\mathcal{D}(\mathcal{A}_0)$, we have

$$
\left\langle \mathcal{A}_0\begin{bmatrix}u\\v\end{bmatrix},\begin{bmatrix}u\\v\end{bmatrix}\right\rangle_{\mathcal{H}_0}=-\frac{\mu_0}{\bar\rho}\int_\Omega|\nabla u|^2\,\mathrm{d}x-\frac{\mu_1}{\mu_2\bar\rho}\int_\Omega|\nabla v|^2\,\mathrm{d}x\leqslant 0,
$$

and hence $\mathcal{A}_0$ is dissipative. Now to show that $(\mathcal{A}_0,\mathcal{D}(\mathcal{A}_0))$ is maximal in $\mathcal{H}_0$, for any $(f,g)\in\mathcal{H}_0$, we consider the system

$$
(4.7)\qquad \begin{cases} u-\dfrac{1}{\bar\rho}A_0(\mu_0u+v)=f,\\ (1+\mu_1)v-\mu_2u=g. \end{cases}
$$

Note that, solving the second equation of $(4.7)_2$ we get $v=\frac{\mu_2}{1+\mu_1}u+\frac{1}{1+\mu_1}g$, and therefore u solves

$$
u-\frac{1}{\bar\rho}\left(\mu_0+\frac{\mu_2}{1+\mu_1}\right)A_0u=f+\frac{1}{1+\mu_1}A_0g.
$$

Next, using the fact that the Stokes operator A_0 is an isomorphism from $H^1_{0,\sigma}$ to $H^{-1}_\sigma(\Omega)$, the dual of $H^1_{0,\sigma}$ with respect to the pivot space $L^2_\sigma(\Omega)$, we obtain u and v belongs to $H^1_{0,\sigma}(\Omega)$. Next, setting $w=\mu_0u+v$, it is easy to see that w solves

$$
(\mu_0+\frac{\mu_2}{1+\mu_1})^{-1}w-\frac{1}{\bar\rho}A_0w=f+\frac{g}{\mu_0(1+\mu_1)+\mu_2},
$$

where $f+\dfrac{g}{\mu_0(1+\mu_1)+\mu_2} \in L^2_\sigma(\Omega)$ and $(\mu_0+\dfrac{\mu_2}{1+\mu_1})^{-1}$ is a positive constant. Using the well-posedness of the Stokes operator, we obtain $w \in \mathcal{D}(A_0)$. □

We also determine the adjoint operator of $\mathcal{A}_0$.

PROPOSITION 4.4. *The adjoint of* $(\mathcal{A}_0, \mathcal{D}(\mathcal{A}_0))$ *in* $\mathcal{H}_0$ *is defined by*

$$\mathcal{D}(\mathcal{A}_0^*) = \left\{(w,z) \in H^1_{\sigma,0}(\Omega) \times H^1_{\sigma,0}(\Omega) \mid \mu_0 w - z \in \mathcal{D}(A_0)\right\},$$

$$\mathcal{A}_0^* \begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\bar\rho} A_0(\mu_0 w - z) \\ -\mu_2 w - \mu_1 z \end{bmatrix}. \tag{4.8}$$

Proof. The proof of this proposition is similar to that of Proposition 2.2. We just have to use the fact that the Stokes operator A_0 is self adjoint. □

In view of Poincaré and Korn's inequalities, to prove Theorem 4.2, it is enough to prove the following result

THEOREM 4.5. *Assume the hypothesis of Theorem 4.2. The system* (4.5) *is not exponentially stabilizable in* $\mathcal{H}$ *with an exponential decay rate* $\omega \geqslant \omega_0$, *by interior control* $f \in L^2(0,\infty; L^2(\mathcal{O}))^n$.

The controllability of linear viscoelastic flows has been studied only in a few articles. For example, the lack of null controllability of the one-dimensional model has been proved in [15], whereas the lack of null controllability of the higher dimensional model has been proved in [11]. As far as we know, the stabilizability of the system (4.1) or (4.5) has not been studied rigorously. In this context, we mention the work [13] where the exponential stabilizability with a decay $\omega < \omega_0$, of linear viscoelastic flows with a rigid structure was proved.

The rest of this section is devoted towards the proof of Theorem 4.5. The proof is similar to the proof of Theorem 2.3 given in Section 2. First, we derive an observability inequality equivalent to the exponential stabilizability of the system (4.5). Let $\omega \geqslant 0$. We consider the adjoint system

$$\begin{cases} \bar\rho\partial_t w - \bar\rho\omega w - \mu_0\Delta w + \nabla q + \mu_2\Delta z = 0 & (t,x) \in (0,\tau)\times\Omega, \\ \partial_t z - \omega z + \mu_1 z + w = 0 & (t,x) \in (0,\tau)\times\Omega, \\ \operatorname{div} w = 0 = \operatorname{div} z & (t,x) \in (0,\tau)\times\Omega, \\ w = 0 = z & (t,x) \in (0,\tau)\times\partial\Omega, \\ w(0) = w_0, \quad z(0) = z_0 & x \in \Omega. \end{cases} \tag{4.9}$$

We have following equivalence result

Proposition 4.6. *Let $\omega > 0$. The system (4.5) is exponentially stabilizable with a decay rate ω, by interior controls $f \in L^2(0,\infty;L^2(\mathcal{O}))^n$, if and only if, there exist $\beta \in (0,1)$, $\tau > 0$ and $C \geqslant 0$ such that, for any $(w_0, z_0) \in \mathcal{H}_0$, the solution of (4.9) satisfies the following observability estimate*

$$\begin{aligned} \|w(\tau,\cdot)\|_{L^2(\Omega)^n} + \|z(\tau,\cdot)\|_{H^1(\Omega)^n} \leqslant C\,\|w\|_{L^2(0,\tau;L^2(\mathcal{O}))^n} \\ + \beta\left(\|w_0\|_{L^2(\Omega)} + \|z_0\|_{H^1(\Omega)^n}\right). \end{aligned} \tag{4.10}$$

As before, we shall construct special solutions violating the observability inequality (4.10). The construction here is slightly different due to divergence free conditions, and different order coupling between ODE and parabolic equations. We just provide the *Gaussian beam* construction. The rest of the proof of Theorem 4.5 is similar to the proof of Theorem 2.3, and thus left to the reader. Consider the following system in $(0,\tau)\times\mathbb{R}^n$

$$\begin{cases} \bar\rho\partial_t w - \omega\bar\rho w - \mu_0\Delta w + \mu_2\Delta z = 0 & (t,x)\in(0,\tau)\times\mathbb{R}^n, \\ \partial_t z - \omega z + \mu_1 z + w = 0 & (t,x)\in(0,\tau)\times\mathbb{R}^n, \\ \operatorname{div} w = 0 = \operatorname{div} z & (t,x)\in(0,\tau)\times\mathbb{R}^n, \\ w(0) = w_0 := \operatorname{curl} a_0, \quad z(0) = z_0 := \operatorname{curl} b_0 & x\in\mathbb{R}^n. \end{cases} \tag{4.11}$$

In the above, we use the same notation of "curl" in both two and three dimensions. If $n = 2$, the functions a_0 and b_0 are scalars and $\operatorname{curl} a_0 = \nabla^\perp a_0$. If $n = 3$, they are vectors in $\mathbb{R}^3$ and $\operatorname{curl} a_0 = \nabla\times a_0$. The initial datas are chosen as "curl" of some function so that they are automatically divergence free. Applying Fourier transform to (4.11), we obtain

$$\frac{d}{dt}\begin{bmatrix}\widehat w\\ \widehat z\end{bmatrix} = (\widehat{\mathcal{A}_0}(\xi) + \omega\mathbb{I}_n)\begin{bmatrix}\widehat w\\ \widehat z\end{bmatrix}, \quad \begin{bmatrix}\widehat w\\ \widehat z\end{bmatrix}(0) = \begin{bmatrix}\widehat{w}_0\\ \widehat{z}_0\end{bmatrix} = \begin{bmatrix} i\xi\times\widehat a_0\\ i\xi\times\widehat b_0\end{bmatrix} \quad t\in(0,\tau),$$

where

$$\widehat{\mathcal{A}}_0(\xi) = \begin{bmatrix} -\frac{\mu_0}{\bar\rho}|\xi|^2\mathbb{I}_n & \frac{\mu_2}{\bar\rho}|\xi|^2\mathbb{I}_n \\ -\mathbb{I}_n & -\mu_1\mathbb{I}_n \end{bmatrix}.$$

If $n = 2$, the term $\xi\times\widehat a_0$ to be understood as $\xi^\perp\widehat a_0$. We have the following lemma

Lemma 4.7. *The following holds*

1. *The eigenvalues of $\widehat{\mathcal{A}}_0(\xi)$ are*

$$\begin{aligned}\lambda_1(\xi) &= \cdots = \lambda_n(\xi) = \lambda(\xi) \\ &= \frac{-(\mu_1\bar\rho + \mu_0|\xi|^2) - \sqrt{(\mu_1\bar\rho + \mu_0|\xi|^2)^2 - 4\bar\rho(\mu_1\mu_0 + \mu_2)|\xi|^2}}{2\bar\rho}\end{aligned}$$

$$\delta_1(\xi) = \cdots = \delta_n(\xi) = \delta(\xi)$$
$$= \frac{-(\mu_1\bar{\rho} + \mu_0|\xi|^2) + \sqrt{(\mu_1\bar{\rho} + \mu_0|\xi|^2)^2 - 4\bar{\rho}(\mu_1\mu_0 + \mu_2)|\xi|^2}}{2\bar{\rho}}.$$

2. *There exists $\xi_0 > 0$, such that for all $|\xi| \geqslant \xi_0$, all the eigenvalues are real. They satisfy*
$$\lim_{|\xi|\to\infty} \frac{\lambda_k(\xi)}{|\xi|^2} = -\frac{\mu_0}{\bar{\rho}}, \quad \lim_{|\xi|\to\infty} \delta_k(\xi) = -\left(\mu_1 + \frac{\mu_2}{\mu_0}\right) := -\omega_0, \ k = 1, 2, \ldots, n.$$

3. *For $|\xi| \geqslant \xi_0$, we have*
$$-2\omega_0 < \delta(\xi) < -\omega_0, \qquad \delta(\xi_2) > \delta(\xi_1) \text{ for } |\xi_2| > |\xi_1| > \xi_0,$$
and
$$\left|\nabla_\xi^k \delta(\xi)\right| \leqslant \frac{C}{|\xi|^k}, \qquad (k \in \mathbb{N} \cup \{0\}),$$
for some positive constant C independent of ξ.

4. *The eigenfunction of $\widehat{\mathcal{A}}_0(\xi)$ corresponding to $\delta(\xi)$ is $(-(\mu_1+\delta(\xi))\mathbf{1}_n, \mathbf{1}_n)^\top$, where $\mathbf{1}_n = (1, \cdots, 1)^\top$.*

Regarding the construction of the *Gaussian beam* solutions for (4.11), we have the following result:

THEOREM 4.8. *Let $\beta \in (0,1)$, $\tau > 0$, and $\omega \geqslant \omega_0$. Let $\bar{\xi} = (\frac{1}{c_0}, 0, \cdots, 0) \in \mathbb{R}^n$ with $c_0 > 1$, and $x_0 \in \mathbb{R}^n$. For any $\varepsilon > 0$, let ψ_ε be defined as in Theorem 2.8, and we define*
$$n = 2, \quad \widehat{a}_0 = -\varepsilon^2(\mu_1 + \delta(\xi))\psi_\varepsilon, \quad \widehat{b}_0 = \varepsilon^2\psi_\varepsilon, \quad \widehat{w}_0 = i\xi^\perp \widehat{a}_0, \quad \widehat{z}_0 = i\xi^\perp \widehat{b}_0,$$
$$n = 3, \quad \widehat{a}_0 = -\varepsilon^2(\mu_1+\delta(\xi))\psi_\varepsilon e_3, \quad \widehat{b}_0 = \varepsilon^2\psi_\varepsilon e_3, \quad \widehat{w}_0 = i\xi \times \widehat{a}_0, \quad \widehat{z}_0 = i\xi \times \widehat{b}_0,$$
where $e_3 = (0,0,1)^\top$. Then (w, z) defined by
$$w(t,x) = \mathcal{F}_\xi^{-1}\left(\widehat{w}_0(\xi)e^{(\delta(\xi)+\omega)t}\right), \quad z(t,x) = \mathcal{F}_\xi^{-1}\left(\widehat{z}_0(\xi)e^{(\delta(\xi)+\omega)t}\right),$$
satisfies the system (4.11), with
$$a_0(x) = \mathcal{F}_\xi^{-1}\left(\widehat{a}_0(\xi)\right), \quad b_0(x) = \mathcal{F}_\xi^{-1}\left(\widehat{b}_0(\xi)\right),$$
$$w_0(x) = \operatorname{curl} a_0 = \mathcal{F}_\xi^{-1}\left(\widehat{w}_0(\xi)\right), \quad z_0(x) = \operatorname{curl} b_0 = \mathcal{F}_\xi^{-1}\left(\widehat{z}_0(\xi)\right).$$
Moreover, there exists $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon < \varepsilon_0$, we have

(i) *There exists a positive constant C depending on ε_0, but independent of ε such that*
$$\|a_0\|_{H^s(\mathbb{R}^n)} \leqslant \frac{C\varepsilon^{2-s}}{(2\pi)^n}, \ s \in [0,1], \quad \|b_0\|_{H^1(\mathbb{R}^n)} \leqslant \frac{C\varepsilon}{(2\pi)^n}$$

$$\frac{\beta^{1/m}}{c_0^4(2\pi)^n} e^{\frac{(m-1)(\omega-\omega_0)\tau}{m}} \leqslant \|z(\tau,\cdot)\|_{H^1(\mathbb{R}^n)} \leqslant \frac{1}{(2\pi)^n} e^{(\omega-\omega_0)\tau}, \quad m \in \mathbb{N},$$
$$\|w(t,\cdot)\|_{L^2(\mathbb{R}^n)^n} \leqslant \frac{C\varepsilon}{(2\pi)^n}, \qquad t \in [0,\tau].$$

(ii) *For any $\eta > 0$, there exists a constant C, depending on ω, τ, η and ε_0, but independent of ε such that*

$$\|z(t,\cdot)\|_{H^1(|x-x_0|\geqslant\eta)} \leqslant C\varepsilon^{k-n/4}, \qquad k \in \mathbb{N}, t \in [0,\tau],$$
$$\|b_0\|_{H^2(|x-x_0|\geqslant\eta))^n} \leqslant C\varepsilon^{k-1-n/4}, \qquad k \in \mathbb{N},$$
$$\|(w,z)\|_{H^1(0,\tau;H^2(|x-x_0|\geqslant\eta))^n} \leqslant C\varepsilon^{k-1-n/4}, \qquad k \in \mathbb{N}.$$

Proof. The proof is similar to that of Theorem 2.8. □

Now we can proceed as the proof of Theorem 2.3 in Section 2 above to prove Theorem 4.5.

5. CONCLUSION AND OPEN PROBLEMS

The main results in this article concern the lack of exponential stabilizability of some coupled ODE-parabolic systems. We show these systems are not exponentially stabilizable with an arbitrary decay rate by localized interior controls. As a consequence, we recover previously known results that these systems are not null controllable at any time. In view of our results, several open questions seem natural.

Optimality of the decay rate: In all of the examples above, we are able to identify the decay rate ω_0 such that the system is not exponentially stabilizable with decay rate $\omega \geqslant \omega_0$. Thus, the most natural question is whether these decay rates are optimal or not. In particular, it would be interesting to know whether these systems are exponentially stabilizable with a decay $\omega < \omega_0$. As mentioned earlier, linearized compressible Navier-Stokes system *i.e.*, system (2.1) in dimension one, is exponentially stabilizable in $\mathcal{H}_m$ with a decay rate $\omega < \omega_0$, ω_0 defined in (2.6), by a localized parabolic control only (see for instance [7, Proposition 3.4]). In [13, Theorem 1.1], it was proved that linear viscoelastic fluid-structure model is exponentially stabilizable with a decay $\omega < \omega_0$, with ω_0 defined as in (4.4). In view of all these results, we think, all the systems considered here are exponentially stablizable with decay $\omega < \omega_0$. More precisely, we conjecture that, the decay rate ω_0 for the systems (2.1), (3.1), (3.3) and (4.5) is optimal, in the sense that, the corresponding systems are exponentially stabilizable with decay $\omega < \omega_0$. In [7, 13], the exponential stabilizability was proved by checking Hautus criteria for exponential stabilizability, which requires the knowledge of the spectrum of the operator. However, in view of Theorem 1.6, perhaps Carleman type estimates may be helpful to prove the optimality of the decay rate.

Coupled ODE-parabolic system with variable coefficients: In this article, we have only considered constant coefficient operators. Recently, lack of null controllability of coupled ODE-parabolic systems with coefficients depending on both space and time, was proved in [1]. Thus, it would be interesting to see whether similar results can be proved for the variable coefficient operator. The Fourier transform technique used here to construct *Gaussian beam* solutions would not be possible. However, the ideas used in [14] and [1] to construct *Gaussian beam* solutions for variable coefficient operators may be useful.

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