Dedicated to Marius Tucsnak on the occasion of his 60^{th} anniversary

TOPOLOGY OPTIMIZATION FOR THE STOKES SYSTEM

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We discuss shape optimization problems associated to the Stokes system. Our analysis includes both topology and boundary variations and it is based on the functional variations approach. It approximates the geometric optimization problem by an optimal control problem that can be solved using gradient algorithms. Numerical examples are also indicated.

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1. INTRODUCTION

Let $D \subset \mathbb{R}^d$, $d \in \{2,3\}$ be a bounded domain with Lipschitz boundary and $\Omega \subset\subset D$ be an unknown open set (the "obstacle", not necessarily connected), with Lipschitz boundary. We assume the decomposition $\partial D = \overline{\Gamma}_D \cup \overline{\Gamma}_N$, where Γ_D and Γ_N are relatively open subsets, mutually disjoint. Some results mentioned below are valid in arbitrary dimension.

The domain $\omega = D \setminus \overline{\Omega}$ is filled with a fluid governed by the Stokes equations:

$$
\begin{aligned}
-\mu\Delta\mathbf{y}_\Omega + \nabla p_\Omega &= \mathbf{f}, \text{ in } \omega, && (1.1)\\
\nabla\cdot\mathbf{y}_\Omega &= 0, \text{ in } \omega, && (1.2)\\
\mathbf{y}_\Omega &= \boldsymbol{\phi}, \text{ on } \Gamma_D, && (1.3)\\
\mu\frac{\partial\mathbf{y}_\Omega}{\partial\mathbf{n}} - p_\Omega\mathbf{n} &= \boldsymbol{\psi}, \text{ on } \Gamma_N, && (1.4)\\
\mathbf{y}_\Omega &= 0, \text{ on } \partial\Omega, && (1.5)
\end{aligned}
$$

where $\mu > 0$ is the constant viscosity, $\mathbf{f} \in L^2(D)^d$ is given, $\mathbf{y}_\Omega : \overline{\omega} \to \mathbb{R}^d$ denotes the velocity and $p_\Omega : \overline{\omega} \to \mathbb{R}$ is the pressure of the fluid, while $\boldsymbol{\phi} : \Gamma_D \to \mathbb{R}^d$ is the imposed velocity, $\boldsymbol{\phi} \in H^{3/2}(\Gamma_D)^d$ and $\boldsymbol{\psi} : \Gamma_N \to \mathbb{R}^d$ is the imposed traction, $\boldsymbol{\psi} \in H^{1/2}(\Gamma_N)^d$, $\mathbf{n}$ is the unit outward normal to ∂D. On $\partial\Omega$ we have no-slip boundary condition. According to [6], Lemma 2.2, p. 24, there exists $y_{\phi} \in H^1(\omega)^d$, $\nabla \cdot y_{\phi} = 0$ in ω, $y_{\phi} = 0$ on $\partial\Omega$, $y_{\phi} = \phi$ on Γ_D and

$$\|y_{\phi}\|_{H^1(\omega)^d} \leq C(\omega) \|\phi\|_{H^{1/2}(\Gamma_D)^d}$$

with $C(\omega) > 0$ independent of ϕ and y_{ϕ}. We can take y_{ϕ} such that its support is in a neighborhood of ∂D, then y_{ϕ} is independent of Ω, if $\hat{d}(\overline{\Omega}, \partial D) = min\{||\mathbf{x}_1 - \mathbf{x}_2||; \mathbf{x}_1 \in \overline{\Omega}, \mathbf{x}_2 \in \partial D\} > \delta > 0$, where δ is some constant.

For $\mathbf{y}_{\Omega} \in H^2(\omega)^d$ and $p_{\Omega} \in H^1(\omega)$, the equalities (1.1)-(1.2) have sense in $L^2(\omega)^d$ and (1.3), (1.4) and (1.5) have sense in $H^{3/2}(\Gamma_D)^d$, $H^{1/2}(\Gamma_N)^d$ and $H^{3/2}(\partial\Omega)^d$, respectively.

We denote by

$$V_{\Omega} = \left\{ \mathbf{v} \in H^1(\omega)^d;\ \nabla \cdot \mathbf{v} = 0 \text{ in } \omega,\ \mathbf{v} = 0 \text{ on } \Gamma_D \cup \partial\Omega \right\}$$

the Hilbert space of test functions. Multiplying (1.1) by $\mathbf{v}_{\Omega} \in V_{\Omega}$, integrating over ω, using Green formula and $\mathbf{v}_{\Omega} = 0$ on $\Gamma_D \cup \partial\Omega$, we get

$$\mu \int_{\omega} \nabla \mathbf{y}_{\Omega} : \nabla \mathbf{v}_{\Omega} d\mathbf{x} - \int_{\omega} (\nabla \cdot \mathbf{v}_{\Omega}) p_{\Omega} d\mathbf{x} = \int_{\omega} \mathbf{f} \cdot \mathbf{v}_{\Omega} d\mathbf{x} + \int_{\Gamma_N} \left(\mu \frac{\partial \mathbf{y}_{\Omega}}{\partial \mathbf{n}} - p_{\Omega} \mathbf{n} \right) \cdot \mathbf{v}_{\Omega} ds.$$

In (1.2) and above, we have denoted by "$\cdot$" the scalar product in $\mathbb{R}^d$ ("$\nabla\cdot$" is the divergence operator) and by ":" the Frobenius matrix product $(a_{ij}) : (b_{ij}) = \sum_{i,j=1}^{d} a_{ij} b_{ij}$ with the corresponding norm $\|(a_{ij})\|_F = \sqrt{(a_{ij}) : (a_{ij})}$.

Using $\nabla \cdot \mathbf{v}_{\Omega} = 0$ in ω and (1.4), the weak solution of (1.1)-(1.5) is defined by:

$$\mu \int_{\omega} \nabla \mathbf{y}_{\Omega} : \nabla \mathbf{v}_{\Omega}\, d\mathbf{x} = \int_{\omega} \mathbf{f} \cdot \mathbf{v}_{\Omega}\, d\mathbf{x} + \int_{\Gamma_N} \boldsymbol{\psi} \cdot \mathbf{v}_{\Omega} ds,\ \forall \mathbf{v}_{\Omega} \in V_{\Omega}, \tag{1.6}$$

with $\mathbf{y}_{\Omega} \in y_{\phi} + V_{\Omega}$, $p_{\Omega} \in L^2(\omega)$ being defined in the distributional sense by (1.1), see Temam [23], Lemma 2.1, p. 16, or [24]. The condition (1.4) has a sense in the dual space of traces on Γ_N of functions from V_{Ω}, see [3], Chapter IV, Section 7. The bilinear application $(\mathbf{u}, \mathbf{v}_{\Omega}) \in V_{\Omega} \times V_{\Omega} \to \mu \int_{\omega} \nabla \mathbf{u} : \nabla \mathbf{v}_{\Omega}\, d\mathbf{x}$ is continuous and using the generalized Poincaré inequality, see [3], Prop. III.2.38, p. 179, it is elliptic, too. From [12], the application $\mathbf{v}_{\Omega} \in V_{\Omega} \to \nabla \cdot \mathbf{v}_{\Omega} \in L^2(\omega)$ is onto. Applying the result concerning the mixed variational problem, see [6], Chap. 1, Sect. 4.1, we obtain the existence and uniqueness of $\mathbf{y}_{\Omega} \in y_{\phi} + V_{\Omega}$, $p_{\Omega} \in L^2(\omega)$ and

$$\|\mathbf{y}_{\Omega}\|_{H^1(\omega)^d} + \|p_{\Omega}\|_{L^2(\omega)} \leq C(\omega) \left(\|\mathbf{f}\|_{L^2(\omega)^d} + \|\phi\|_{H^{1/2}(\Gamma_D)^d} + \|\boldsymbol{\psi}\|_{L^2(\Gamma_N)^d} \right).$$

We point out that the pressure is uniquely determined if $meas(\Gamma_N) > 0$.

In the case when ω is a three dimensional polyhedron and on ∂D we impose Dirichlet boundary condition $\mathbf{y}_{\Omega} = \phi^i$ on some faces Γ_i^D of ∂D and

Neumann boundary condition $\mu \frac{\partial \mathbf{y}_\Omega}{\partial \mathbf{n}} - p_\Omega \mathbf{n} = \boldsymbol{\psi}^i$ on the other faces Γ_i^N of ∂D, keeping (1.5) on $\partial\Omega$, then we have the regularity $\mathbf{y}_\Omega \in W^{2,s}(\omega)^3$, $p_\Omega \in W^{1,s}(\omega)$, if $\mathbf{f} \in L^s(\omega)^3$, $\boldsymbol{\phi}^i \in W^{2-1/s,s}(\Gamma_i^D)^3$, $\boldsymbol{\psi}^i \in W^{1-1/s,s}(\Gamma_i^N)^3$, where $1 < s \leq 8/7$ depends on the domain ω, see [12], where the authors work in weighted Sobolev spaces. When ω is a two dimensional angle, regularity results using weighted Sobolev spaces for Stokes equations with mixed boundary conditions can be found in [22], Section 3.2.

For Stokes equations with Dirichlet boundary conditions on the whole boundary of ω, see [23], and for Stokes equations with Neumann boundary conditions on the whole boundary of ω, see [3], the existence, uniqueness and regularity are obtained in arbitrary dimension $\omega \subset \mathbb{R}^d$, $d \geq 2$.

To the state system (1.1)-(1.5), some cost functional $J(\Omega)$ may be associated. For instance, the minimization of the dissipated energy:

$$\inf_{\Omega \in \mathcal{O}} \left\{ J(\Omega) = \int_\omega \|\mathbf{e}(\mathbf{y}_\Omega)\|_F^2 \, \mathrm{d}\mathbf{x} \right\}, \tag{1.7}$$

where $\mathbf{e}(\mathbf{y}_\Omega) = \frac{1}{2}\left(\nabla \mathbf{y}_\Omega + (\nabla \mathbf{y}_\Omega)^T\right)$ is the symmetrized gradient and $\mathcal{O}$ is some given family of open sets (obstacles), not necessarily connected.

Regularity conditions on the geometry, on the given mappings $\mathbf{f}$, $\boldsymbol{\phi}$, $\boldsymbol{\psi}$ will be imposed in the sequel, as necessity appears. More constraints on the geometry (for instance, prescribed volume for $\Omega \in \mathcal{O}$, see [5]) or on the state may be considered as well.

The literature on shape optimization problems associated to compressible or incompressible fluid systems is very rich and we quote the papers by Amstutz [2], Masmoudi and his co-authors [9], the monographs of Plotnikov and Sokolowski [20], Novotny and Sokolowski [18] and their references. Topological optimization questions, including numerical experiments are, in general, discussed via the well known topological asymptotic expansion approach (for the cost functional), also called topological sensitivity or topological gradient.

In this work, we use a different method, based on functional variations as introduced in [16], [17], on Hamiltonian systems in dimension two and on implicit parametrizations, [26]. These concepts have already been applied in other shape optimization problems in [13] and allow the introduction of general topological derivatives and the use of standard gradient methods. Applications to optimal control or mathematical programming can be found in [28].

We also underline that functional variations combine in a natural way boundary and topological variations. They are based on the use of level functions as in the well known method of Osher and Sethian [19], but there are essential differences: no Hamilton-Jacobi equation is necessary, no “evolution” of the geometry is taken into account. Instead, simple ordinary differential

Hamiltonian systems may be employed for the description of the unknown geometry.

In Section 2 we collect some preliminary information on the functional variations approach. Section 3 is devoted to the approximation method that we apply here and its differentiability properties, while Section 4 discusses some numerical examples.

2. FUNCTIONAL VARIATIONS

We consider that the family of admissible domains $\omega \subset D \subset \mathbb{R}^2$ and, implicitly, the family $\mathcal{O}$ of obstacles that appears in (1.7) are obtained as follows, starting from an admissible family of level functions $\mathcal{F} \subset \mathcal{C}(\overline{D})$, via the relations: $\Omega = D \setminus \overline{\omega}$ and

$$\omega = \omega_g = int\left\{\mathbf{x} \in D;\ g(\mathbf{x}) \leq 0\right\}, \quad g \in \mathcal{F}. \tag{2.1}$$

Clearly, (2.1) defines open subsets of D, not necessarily connected. The family of admissible functions $\mathcal{F}$ is a cone since we also impose the condition

$$g(\mathbf{x}) < 0, \quad \mathbf{x} \in \partial D,\ \forall g \in \mathcal{F} \tag{2.2}$$

that ensures that $\partial D \subset \overline{\omega_g}$ for any admissible ω_g. We shall denote as the fluid domain the component of ω_g that includes ∂D in its boundary. This makes sense due to (2.2). Its complement, $D \setminus \overline{\omega_g} = \Omega_g$, is not necessarily connected and defines the obstacles.

An important role is played by the set:

$$G = \left\{\mathbf{x} \in \Omega_g;\ g(\mathbf{x}) = 0\right\}. \tag{2.3}$$

In general, for $g \in \mathcal{C}(\overline{D})$, it may happen that $meas(G) > 0$ and the boundary of Ω_g may be very irregular (in principle, it satisfies just the segment property [1], [25]). However, the regularity properties appearing in (1.1)-(1.6) require at least Lipschitz properties for $\partial\Omega_g$ and we impose the hypotheses $\mathcal{F} \subset \mathcal{C}^1(\overline{D})$ and

$$|\nabla g(\mathbf{x})| > 0, \quad \mathbf{x} \in G,\ \forall g \in \mathcal{F}. \tag{2.4}$$

By the implicit functions theorem, the condition (2.4) gives $G = \partial\Omega_g$ is of class $\mathcal{C}^1$ (or even $\mathcal{C}^2$ if $\mathcal{F} \subset \mathcal{C}^2(\overline{D})$, etc.) and we have

$$\omega_g = \left\{\mathbf{x} \in D;\ g(\mathbf{x}) < 0\right\}. \tag{2.5}$$

One may impose the supplementary condition $g > 0$ in $D \setminus \omega_g$, for instance, by adding to g some multiple of the distance function to G, at some power to ensure smoothness. Let us assume that there is $\mathbf{x}_0 \in D$ such that

$$g(\mathbf{x}_0) = 0,\ \forall g \in \mathcal{F}. \tag{2.6}$$

Then, it is well known that $G = \partial\Omega_g$ is locally parametrized, around $\mathbf{x}_0$, by the solution of the Hamiltonian system:

$$(2.7)\qquad (z_g^1)'(t) = -\partial_2 g\left(z_g^1(t), z_g^2(t)\right),\ t \in I_g,$$

$$(2.8)\qquad (z_g^2)'(t) = \partial_1 g\left(z_g^1(t), z_g^2(t)\right),\ t \in I_g,$$

$$(2.9)\qquad \mathbf{z}_g(0) = \left(z_g^1(0), z_g^2(0)\right) = \mathbf{x}_0,$$

where I_g is the existence interval around the origin, ensured by the Peano theorem. The local representation (2.7)-(2.9) of the geometry $G = \partial\Omega_g$ can be extended to arbitrary dimension by using iterated Hamiltonian systems [26]. Moreover, an argument that again employs the implicit function theorem shows that the solution of (2.7)-(2.9) is unique, although the right-hand side is just continuous (and similarly in arbitrary dimension [26]). For these local results, the condition (2.4) may be imposed just in $\mathbf{x}_0 \in G$.

Under hypothesis (2.4), in dimension two, it yields, as a consequence of the Poincaré-Bendixson theorem [11], [21], that the solution of (2.7)-(2.9) is periodic and we obtain a global representation of $G = \partial\Omega_g$. This property of Hamiltonian systems in dimension two is based on the knowledge of the Hamiltonian functions $g \in \mathcal{F}$ and on their assumed properties, in order to show that the limit cycle situation is not possible here [27].

Functional variations of the (not necessarily simply connected) domain ω_g are obtained by (2.5) starting with the perturbation $g + \lambda h$, $\lambda \in \mathbb{R}$, $g, h \in \mathcal{F} \subset \mathcal{C}^1(\overline{D})$. It is clear that the perturbed boundary $G_\lambda = \{\mathbf{x} \in D;\ (g + \lambda h)(\mathbf{x}) = 0\}$ includes perturbations of the boundary G, but also the number of holes of $\omega_{g+\lambda h}$ (the connectivity type) may change, depending on λ. This allows the combined boundary and topological optimization with respect to ω_g.

PROPOSITION 2.1 ([27]). *Under the assumptions (2.2), (2.4), (2.6), G is a finite union of disjoint closed curves, without self intersections and not meeting ∂D, parametrized by the unique periodic solution of the Hamiltonian system (2.7)-(2.9), where $\mathbf{x}_0$ is some point chosen on each component of G.*

A similar statement is valid for G_λ, for $|\lambda|$ "small". For $\epsilon > 0$, we denote

$$V_\epsilon = \{\mathbf{x} \in D;\ d(\mathbf{x}, G) < \epsilon\},$$

a neighborhood of G. Then, there is $\lambda(\epsilon) > 0$ such that $G_\lambda \subset V_\epsilon$ for $|\lambda| < \lambda(\epsilon)$. In particular, this shows that $G_\lambda \to G$ in the sense of Hausdorff-Pompeiu, [15]. Moreover, we have

COROLLARY 2.2. *$\mathcal{F}$ is an open cone in $\mathcal{C}^1(\overline{D})$.*

$\mathcal{F}$ is clearly a cone. There is a constant $c_g > 0$ such that $|g(\mathbf{x})| \geq c_g$ on $\overline{D} \setminus V_\epsilon$ which is a finite union of compacts due to Prop. 2.1. Then, for any $h \in \mathcal{F}$, the perturbation $g + \lambda h$ satisfies $|(g + \lambda h)(\mathbf{x})| > 0$ for $|\lambda|$ small enough in $\overline{D} \setminus V_\epsilon$, that is $G_\lambda \subset V_\epsilon$ for $|\lambda| < \lambda(\epsilon)$. Similarly, (2.2) is also satisfied. By taking ϵ even smaller, hypothesis (2.4) gives $|\nabla g(\mathbf{x})| > C_g > 0$ on V_ϵ, due to the Weierstrass theorem. It follows that $g+\lambda h$ satisfies (2.4) as well, for $|\lambda|$ small. Due to this corollary, functional variations are possible in $\mathcal{F}$.

One of the most difficult points in investigating shape optimization problems, both from the theoretical and the computational points of view, is the variable/unknown character of the open set $\Omega \in \mathcal{O}$. For instance, in the standard numerical approaches, one has to remesh the domain and to recompute the mass matrix in each iteration. This increases considerably the computational effort and, as a consequence, fixed domain approximation methods have been developed and we quote [17] for a survey in this respect.

In the sequel, we take $j : D \times \mathbb{R}^2 \times \mathbb{R}^4 \to \mathbb{R}$ to be a Carathéodory function and

$$J(\Omega) = \int_\omega j\left(\mathbf{x}, \mathbf{y}_\Omega(\mathbf{x}), \nabla \mathbf{y}_\Omega(\mathbf{x})\right) \mathrm{d}\mathbf{x}, \tag{2.10}$$

where $\mathbf{y}_\Omega \in V_\Omega$ is the solution (1.1)-(1.5). Other constraints on $\Omega \in \mathcal{O}$, on the state $\mathbf{y}_\Omega$ may be added as well. Regularity conditions on $j(\cdot)$ will be imposed later.

We also notice that for any $g \in \mathcal{F}$, $H(g)$ (here $H(\cdot)$ is the Heaviside function) is the characteristic function of Ω_g (under the convention, already explained, that $g > 0$ in Ω_g).

The cost functional (2.10) may be written as an integral on D:

$$J(g) = \int_D (1 - H(g)) j\left(\mathbf{x}, \mathbf{y}_g(\mathbf{x}), \nabla \mathbf{y}_g(\mathbf{x})\right) \mathrm{d}\mathbf{x}. \tag{2.11}$$

The cost functional (2.11) is not differentiable with respect to $g \in \mathcal{F}$ since $H(\cdot)$ is not smooth. A penalization of the state system is discussed in the next section, again using the not differentiable Heaviside function. In order to overcome such difficulties, we use a smoothing $H^\epsilon(\cdot)$ of the Heaviside function and the study of the approximation properties for $\epsilon \to 0$ is necessary. Such a fixed domain approach and some variants are discussed in [13], [27], in a different context.

3. APPROXIMATION AND DIFFERENTIABILITY

We use a regularization of the Heaviside function adapted from [7]:

$$H^\epsilon(r) = \begin{cases} 1, & r \geq \epsilon, \\ \frac{(-2r+3\epsilon)r^2}{\epsilon^3}, & 0 < r < \epsilon, \\ 0, & r \leq 0, \end{cases} \tag{3.1}$$

with $\epsilon > 0$ and $1 - H^\epsilon(g)$ is a regularization of the characteristic function of ω_g.

The regularized optimization problem is

$$\inf_g J(g) = \int_D (1 - H^\epsilon(g)) j\left(\mathbf{x}, \mathbf{y}_g^\epsilon(\mathbf{x}), \nabla \mathbf{y}_g^\epsilon(\mathbf{x})\right) \mathrm{d}\mathbf{x} \tag{3.2}$$

where $\mathbf{y}_g^\epsilon \in H^1(D)^2$, $\nabla \cdot \mathbf{y}_g^\epsilon = 0$ in D, $\mathbf{y}_g^\epsilon = \boldsymbol{\phi}$ on Γ_D, such that

$$\mu \int_D \nabla \mathbf{y}_g^\epsilon : \nabla \mathbf{v}\, \mathrm{d}\mathbf{x} + \frac{1}{\epsilon} \int_D H^\epsilon(g) \mathbf{y}_g^\epsilon \cdot \mathbf{v}\, \mathrm{d}\mathbf{x} = \int_D \mathbf{f} \cdot \mathbf{v}\, \mathrm{d}\mathbf{x} + \int_{\Gamma_N} \boldsymbol{\psi} \cdot \mathbf{v} \mathrm{d}s, \tag{3.3}$$

for any $\mathbf{v} \in V$ where

$$V = \left\{\mathbf{v} \in H^1(D)^2;\ \nabla \cdot \mathbf{v} = 0 \text{ in } D,\ \mathbf{v} = 0 \text{ on } \Gamma_D\right\}. \tag{3.4}$$

PROPOSITION 3.1. *For fixed $g \in \mathcal{F}$, $\mathbf{f} \in L^2(D)^2$, $\boldsymbol{\phi} \in H^{1/2}(\Gamma_D)^2$, $\boldsymbol{\psi} \in L^2(\Gamma_N)^2$ and $\epsilon > 0$, the problem (3.3)-(3.4) has a unique solution $\mathbf{y}_g^\epsilon \in H^1(D)^2$, $\nabla \cdot \mathbf{y}_g^\epsilon = 0$ in D and $\mathbf{y}_g^\epsilon = \boldsymbol{\phi}$ on Γ_D, satisfying the inequality*

$$\|\mathbf{y}_g^\epsilon\|_{H^1(D)^2} \le C\left(\|\mathbf{f}\|_{L^2(D)^2} + \|\boldsymbol{\phi}\|_{H^{1/2}(\Gamma_D)^2} + \|\boldsymbol{\psi}\|_{L^2(\Gamma_N)^2}\right) \tag{3.5}$$

where $C > 0$ is independent of ϵ, g, $\mathbf{f}$, $\boldsymbol{\phi}$, $\boldsymbol{\psi}$.

Proof. Using generalized Poincaré inequality, see [3], Prop. III.2.38, p. 179, we obtain

$$\|\mathbf{v}\|^2_{H^1(D)^2} \le C_1(D) \int_D \nabla \mathbf{v} : \nabla \mathbf{v}\, \mathrm{d}\mathbf{x}$$

for any $\mathbf{v} \in H^1(D)^2$, $\mathbf{v} = 0$ on Γ_D. This yields the ellipticity over V of the bilinear functional from (3.3), since $0 \le H^\epsilon(g) \le 1$,

$$\begin{aligned} \alpha \|\mathbf{v}\|^2_{H^1(D)^2} &\le \mu \int_D \nabla \mathbf{v} : \nabla \mathbf{v}\, \mathrm{d}\mathbf{x} \\ &\le \mu \int_D \nabla \mathbf{v} : \nabla \mathbf{v}\, \mathrm{d}\mathbf{x} + \frac{1}{\epsilon} \int_D H^\epsilon(g) \mathbf{v} \cdot \mathbf{v}\, \mathrm{d}\mathbf{x},\ \forall \mathbf{v} \in V \end{aligned}$$

with $\alpha > 0$, independently of ϵ and g. We also have its boundedness.

The extension by zero in Ω_g of $\mathbf{y}_\phi \in H^1(\omega_g)^2$, also denoted by $\mathbf{y}_\phi$, verifies $\mathbf{y}_\phi \in H^1(D)^2$, $\nabla \cdot \mathbf{y}_\phi = 0$ in D, $\mathbf{y}_\phi = \boldsymbol{\phi}$ on Γ_D and $\|\mathbf{y}_\phi\|_{H^1(D)^2} \le C(D)\, \|\boldsymbol{\phi}\|_{H^{1/2}(\partial D)^2}$. Subtracting $\mu \int_D \nabla \mathbf{y}_\phi : \nabla \mathbf{v}\, \mathrm{d}\mathbf{x}$ from both sides of (3.3), using $\frac{1}{\epsilon} \int_D H^\epsilon(g) \mathbf{y}_\phi \cdot \mathbf{v}\, \mathrm{d}\mathbf{x} = 0$, since $supp(\mathbf{y}_\phi) \cap \Omega_g = \emptyset$ and $H^\epsilon(g) = 0$ in $\overline{D} \setminus \Omega_g$, we obtain

$$\begin{aligned} &\mu \int_D \nabla(\mathbf{y}_g^\epsilon - \mathbf{y}_\phi) : \nabla \mathbf{v}\, \mathrm{d}\mathbf{x} + \frac{1}{\epsilon} \int_D H^\epsilon(g)(\mathbf{y}_g^\epsilon - \mathbf{y}_\phi) \cdot \mathbf{v}\, \mathrm{d}\mathbf{x} \\ &= \int_D \mathbf{f} \cdot \mathbf{v}\, \mathrm{d}\mathbf{x} + \int_{\Gamma_N} \boldsymbol{\psi} \cdot \mathbf{v} \mathrm{d}s - \mu \int_D \nabla \mathbf{y}_\phi : \nabla \mathbf{v}\, \mathrm{d}\mathbf{x} \end{aligned} \tag{3.6}$$

and from the Lax-Milgram theorem, we obtain the existence and uniqueness of $\mathbf{y}_g^\epsilon \in \mathbf{y}_\phi + V$. Moreover, taking $\mathbf{v} = \mathbf{y}_g^\epsilon - \mathbf{y}_\phi$ and using ellipticity and Cauchy-Schwarz inequalities, we get

$$\begin{aligned} & \alpha \left\|\mathbf{y}_g^\epsilon - \mathbf{y}_\phi\right\|^2_{H^1(D)^2} \\ \leq & \left(\|\mathbf{f}\|_{L^2(D)^2} + \|\boldsymbol{\psi}\|_{L^2(\Gamma_N)^2} + \mu \|\mathbf{y}_\phi\|_{H^1(D)^2}\right) \left\|\mathbf{y}_g^\epsilon - \mathbf{y}_\phi\right\|_{H^1(D)^2}. \end{aligned}$$

After some computations, using the triangle inequality

$$\left\|\mathbf{y}_g^\epsilon\right\|_{H^1(D)^2} \leq \left\|\mathbf{y}_g^\epsilon - \mathbf{y}_\phi\right\|_{H^1(D)^2} + \|\mathbf{y}_\phi\|_{H^1(D)^2}$$

and $\|\mathbf{y}_\phi\|_{H^1(D)^2} \leq C(D) \|\boldsymbol{\phi}\|_{H^{1/2}(\partial D)^2}$, we obtain (3.5). $\square$

Remark 3.2. Let us introduce $W = \{\mathbf{w} \in H^1(D)^2;\ \mathbf{w} = 0 \text{ on } \Gamma_D\}$. The bilinear application $(\mathbf{u}, \mathbf{w}) \in W \times W \to \mu \int_D \nabla \mathbf{u} : \nabla \mathbf{w}\, \mathrm{d}\mathbf{x} + \frac{1}{\epsilon} \int_D H^\epsilon(g) \mathbf{u} \cdot \mathbf{w}\, \mathrm{d}\mathbf{x}$ is continuous and elliptic. The application $\mathbf{w} \in W \to \nabla \cdot \mathbf{w} \in L^2(D)$ is onto. Applying the results from [6], Chap. 1, Sect. 4.1, we obtain the existence and uniqueness of $p_g^\epsilon \in L^2(D)$ such that

$$\begin{aligned} & \mu \int_D \nabla \mathbf{y}_g^\epsilon : \nabla \mathbf{w}\, \mathrm{d}\mathbf{x} + \frac{1}{\epsilon} \int_D H^\epsilon(g) \mathbf{y}_g^\epsilon \cdot \mathbf{w}\, \mathrm{d}\mathbf{x} + \int_D (\nabla \cdot \mathbf{w}) p_g^\epsilon\, \mathrm{d}\mathbf{x} \\ = & \int_D \mathbf{f} \cdot \mathbf{w}\, \mathrm{d}\mathbf{x} + \int_{\Gamma_N} \boldsymbol{\psi} \cdot \mathbf{w} \mathrm{d}s, \quad \forall \mathbf{w} \in W, \end{aligned}$$

where $\mathbf{y}_g^\epsilon$ is the solution of (3.3). Moreover, we have

$$\left\|p_g^\epsilon\right\|_{L^2(D)} \leq C \left(\|\mathbf{f}\|_{L^2(D)^2} + \|\boldsymbol{\phi}\|_{H^{1/2}(\Gamma_D)^2} + \|\boldsymbol{\psi}\|_{L^2(\Gamma_N)^2} + \frac{1}{\epsilon} \left\|H^\epsilon(g) \mathbf{y}_g^\epsilon\right\|_{L^2(D)^2}\right)$$

depending on ϵ. In [8], using an H^1 penalization term and the Heaviside function without regularization, it is obtained that the pressure in the $L^2(D)$ norm is bounded independently of ϵ.

Proposition 3.3. *For g fixed, $\lim_{\epsilon \to 0} \mathbf{y}^\epsilon_{g|\omega_g} = \mathbf{y}_{\Omega_g}$ in $H^1(\omega_g)^2$ weakly, where $\mathbf{y}_{\Omega_g}$ is the solution of (1.6) for $\Omega = \Omega_g$.*

Proof. From (3.5) there exists $\widehat{\mathbf{y}}_g \in H^1(\omega_g)^2$ such that $\mathbf{y}_g^\epsilon \to \widehat{\mathbf{y}}_g$, on a subsequence, in $H^1(D)^2$, weakly, when $\epsilon \to 0$.

Putting $\mathbf{v} = \mathbf{y}_g^\epsilon - \mathbf{y}_\phi$ in (3.6), we obtain

$$\begin{aligned} & \frac{1}{\epsilon} \int_D H^\epsilon(g) (\mathbf{y}_g^\epsilon - \mathbf{y}_\phi) \cdot (\mathbf{y}_g^\epsilon - \mathbf{y}_\phi)\, \mathrm{d}\mathbf{x} \\ \leq & \left(\|\mathbf{f}\|_{L^2(D)^2} + \|\boldsymbol{\psi}\|_{L^2(\Gamma_N)^2} + \mu \|\mathbf{y}_\phi\|_{H^1(D)^2}\right) \left\|\mathbf{y}_g^\epsilon - \mathbf{y}_\phi\right\|_{H^1(D)^2} \end{aligned}$$

and using (3.5), we get

$$\int_{\Omega_g} H^\epsilon(g) \mathbf{y}_g^\epsilon \cdot \mathbf{y}_g^\epsilon\, \mathrm{d}\mathbf{x} = \int_D H^\epsilon(g) (\mathbf{y}_g^\epsilon - \mathbf{y}_\phi) \cdot (\mathbf{y}_g^\epsilon - \mathbf{y}_\phi)\, \mathrm{d}\mathbf{x} \leq \epsilon C$$

since $supp(\mathbf{y}_\phi) \cap \Omega_g = \emptyset$ and $H^\epsilon(g) = 0$ in $\overline{D} \backslash \Omega_g$, where $C > 0$ is independent of ϵ. Let $K \subset \Omega_g$ be a compact. There exists $\epsilon_K > 0$ such that $\epsilon_K \le \min_{\mathbf{x} \in K} g(\mathbf{x})$. Then $H^\epsilon(g(\mathbf{x})) = 1$ for $\mathbf{x} \in K$, $0 < \epsilon \le \epsilon_K$. We get

$$\int_K \mathbf{y}_g^\epsilon \cdot \mathbf{y}_g^\epsilon \,\mathrm{d}\mathbf{x} \le \epsilon C, \quad \forall \epsilon \in (0, \epsilon_K].$$

But $\mathbf{y}_g^\epsilon \to \widehat{\mathbf{y}}_g$ weakly in $H^1(D)^2$ and the inclusion $H^1(D) \subset L^2(D)$ is compact from the Sobolev theorem, then $\mathbf{y}_g^\epsilon \to \widehat{\mathbf{y}}_g$ strongly in $L^2(D)^2$. By passing to the limit in the above inequality, we obtain that $\|\widehat{\mathbf{y}}_g\|_{L^2(K)^2} = 0$, then $\widehat{\mathbf{y}}_g = 0$ in K, for all compact $K \subset \Omega_g$. From the trace theorem, we get that $\widehat{\mathbf{y}}_g = 0$ on $\partial\Omega_g$.

Let $\mathbf{v}_{\Omega_g}$ be in V_{Ω_g} and let $\widetilde{\mathbf{v}}$ be the extension of $\mathbf{v}_{\Omega_g}$ by zero in Ω_g. We have $\widetilde{\mathbf{v}} \in V$ and putting it in (3.3), it follows

$$\mu \int_D \nabla \mathbf{y}_g^\epsilon : \nabla \widetilde{\mathbf{v}} \,\mathrm{d}\mathbf{x} = \int_D \mathbf{f} \cdot \widetilde{\mathbf{v}} \,\mathrm{d}\mathbf{x} + \int_{\Gamma_N} \boldsymbol{\psi} \cdot \widetilde{\mathbf{v}} \mathrm{d}s,$$

since $H^\epsilon(g) = 0$ in $\overline{D} \setminus \Omega_g$. Passing to the limit, we get

$$\mu \int_{\omega_g} \nabla \widehat{\mathbf{y}}_g : \nabla \mathbf{v}_{\Omega_g} \,\mathrm{d}\mathbf{x} = \int_{\omega_g} \mathbf{f} \cdot \mathbf{v}_{\Omega_g} \,\mathrm{d}\mathbf{x} + \int_{\Gamma_N} \boldsymbol{\psi} \cdot \mathbf{v}_{\Omega_g} \mathrm{d}s.$$

Also, $\nabla \cdot \mathbf{y}_g^\epsilon = 0$ in D and $\mathbf{y}_g^\epsilon = \boldsymbol{\phi}$ on Γ_D, then $\nabla \cdot \widehat{\mathbf{y}}_g = 0$ in D and $\widehat{\mathbf{y}}_g = \boldsymbol{\phi}$ on Γ_D. Finally, we get that $\widehat{\mathbf{y}}_g = \mathbf{y}_{\Omega_g}$ the unique solution of (1.6), for $\Omega = \Omega_g$. □

PROPOSITION 3.4. *For $\epsilon > 0$ fixed, $g, r \in \mathcal{F}$ fixed, $\lim_{\lambda \to 0} \mathbf{y}_{(g+\lambda r)}^\epsilon = \mathbf{y}_g^\epsilon$ in $H^1(D)^2$ strongly.*

Proof. From (3.5), $\mathbf{y}_{(g+\lambda r)}^\epsilon$ is bounded in $H^1(D)^2$, independently of λ. Then there exists $\widetilde{\mathbf{y}}$ such that $\mathbf{y}_{(g+\lambda r)}^\epsilon \longrightarrow \widetilde{\mathbf{y}}$ weakly in $H^1(D)^2$ and strongly in $L^2(D)^2$, on a subsequence $\lambda_n \to 0$. From (3.3), we have

$$\mu \int_D \nabla \mathbf{y}_{(g+\lambda r)}^\epsilon : \nabla \mathbf{v} \,\mathrm{d}\mathbf{x} + \frac{1}{\epsilon} \int_D H^\epsilon(g+\lambda r) \mathbf{y}_{(g+\lambda r)}^\epsilon \cdot \mathbf{v} \,\mathrm{d}\mathbf{x} = \int_D \mathbf{f} \cdot \mathbf{v} \,\mathrm{d}\mathbf{x} + \int_{\Gamma_N} \boldsymbol{\psi} \cdot \mathbf{v} \mathrm{d}s,$$

for any $\mathbf{v} \in V$. As $H^\epsilon(g + \lambda r) \to H^\epsilon(g)$ uniformly, for $\lambda \to 0$, one can pass to the limit in the above equality. We obtain that $\widetilde{\mathbf{y}}$ verifies (3.3). We have $\nabla \cdot \mathbf{y}_{(g+\lambda r)}^\epsilon = 0$ in D and $\mathbf{y}_{(g+\lambda r)}^\epsilon = \boldsymbol{\phi}$ on Γ_D, then $\nabla \cdot \widetilde{\mathbf{y}} = 0$ in D and $\widetilde{\mathbf{y}} = \boldsymbol{\phi}$ on Γ_D. Finally, we get $\widetilde{\mathbf{y}} = \mathbf{y}_g^\epsilon$, consequently $\lim_{\lambda \to 0} \mathbf{y}_{(g+\lambda r)}^\epsilon = \mathbf{y}_g^\epsilon$ weakly in $H^1(D)^2$, without subsequence. Again, $\lim_{\lambda \to 0} \mathbf{y}_{(g+\lambda r)}^\epsilon = \mathbf{y}_g^\epsilon$ strongly in $L^2(D)^2$.

Subtracting (3.3) written for g from (3.3) written for $g + \lambda r$, we get

$$\mu \int_D \nabla \left(\mathbf{y}_{(g+\lambda r)}^\epsilon - \mathbf{y}_g^\epsilon \right) : \nabla \mathbf{v} \,\mathrm{d}\mathbf{x}$$

$$= -\frac{1}{\epsilon}\int_D \left[H^\epsilon(g+\lambda r)\mathbf{y}^\epsilon_{(g+\lambda r)} - H^\epsilon(g)\mathbf{y}^\epsilon_g\right]\cdot \mathbf{v}\,\mathrm{d}\mathbf{x}$$

Clearly, the above parenthesis strongly converges to 0 in $L^2(D)^2$, as $\lambda \to 0$, since ϵ is fixed. We may fix above $\mathbf{v} = \mathbf{y}^\epsilon_{(g+\lambda r)} - \mathbf{y}^\epsilon_g$ and the ellipticity property gives the desired conclusion. □

The next proposition summarizes differentiability properties proved in [14, Prop. 3], [13], that remain valid for divergence free functions. See [7] as well.

Proposition 3.5. *For $g, r \in \mathcal{F}$, there exists $\boldsymbol{\zeta} \in V$ such that $\lim_{\lambda\to 0} \frac{\mathbf{y}^\epsilon_{(g+\lambda r)}-\mathbf{y}^\epsilon_g}{\lambda} = \boldsymbol{\zeta}$ weakly in $H^1(D)^2$ and $\boldsymbol{\zeta}$ is the unique solution of*

$$\mu\int_D \nabla\boldsymbol{\zeta} : \nabla\mathbf{v}\,\mathrm{d}\mathbf{x} + \frac{1}{\epsilon}\int_D H^\epsilon(g)\,\boldsymbol{\zeta}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} = -\frac{1}{\epsilon}\int_D (H^\epsilon)'(g) r\,\mathbf{y}^\epsilon_g\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}, \tag{3.7}$$

for any $\mathbf{v} \in V$. Moreover, since $\frac{H^\epsilon(g+\lambda r)-H^\epsilon(g)}{\lambda} \to (H^\epsilon)'(g)r$ uniformly in $\mathcal{C}(\overline{D})$, then $\lim_{\lambda\to 0} \frac{\mathbf{y}^\epsilon_{(g+\lambda r)}-\mathbf{y}^\epsilon_g}{\lambda} = \boldsymbol{\zeta}$ strongly in $H^1(D)^2$.

As a consequence, we obtain:

Proposition 3.6. *Assume that $j(\mathbf{x},\cdot,\cdot)$ is in $\mathcal{C}^1(\mathbb{R}^2\times\mathbb{R}^4)$, such that*

$$|j(\mathbf{x},\mathbf{y},\mathbf{z})| \le a(\mathbf{x}) + c\left(|\mathbf{y}|^2_{\mathbb{R}^2} + |\mathbf{z}|^2_{\mathbb{R}^4}\right),\ \text{a.e. in } D,\ \forall \mathbf{y},\mathbf{z} \tag{3.8}$$

$$|\partial_2 j(\mathbf{x},\mathbf{y},\mathbf{z})| \le a_2(\mathbf{x}) + c_2\left(|\mathbf{y}|_{\mathbb{R}^2} + |\mathbf{z}|_{\mathbb{R}^4}\right)\ \text{a.e. in } D,\ \forall \mathbf{y},\mathbf{z} \tag{3.9}$$

$$|\partial_3 j(\mathbf{x},\mathbf{y},\mathbf{z})| \le a_3(\mathbf{x}) + c_3\left(|\mathbf{y}|_{\mathbb{R}^2} + |\mathbf{z}|_{\mathbb{R}^4}\right)\ \text{a.e. in } D,\ \forall \mathbf{y},\mathbf{z} \tag{3.10}$$

where $a \in L^1(D)$ $a_2, a_3 \in L^2(D)$, $c, c_2, c_3 \in \mathbb{R}_+$. Let $\mathbf{y}^\epsilon_g \in H^1(D)^2$ be the solution of (3.3) and let $\boldsymbol{\zeta} \in V$ be the solution of (3.7). Then, the directional derivative of the objective function (3.2) in the direction $r \in \mathcal{F}$ is:

$$\begin{aligned} J'(g)r =& -\int_D (H^\epsilon)'(g) r\, j\left(\mathbf{x},\mathbf{y}^\epsilon_g(\mathbf{x}),\nabla\mathbf{y}^\epsilon_g(\mathbf{x})\right)\mathrm{d}\mathbf{x} \\ &+\int_D (1-H^\epsilon(g))\,\partial_2 j\left(\mathbf{x},\mathbf{y}^\epsilon_g(\mathbf{x}),\nabla\mathbf{y}^\epsilon_g(\mathbf{x})\right)\cdot\boldsymbol{\zeta}(\mathbf{x})\mathrm{d}\mathbf{x} \\ &+\int_D (1-H^\epsilon(g))\,\partial_3 j\left(\mathbf{x},\mathbf{y}^\epsilon_g(\mathbf{x}),\nabla\mathbf{y}^\epsilon_g(\mathbf{x})\right):\nabla\boldsymbol{\zeta}(\mathbf{x})\mathrm{d}\mathbf{x}. \end{aligned} \tag{3.11}$$

Here, $\partial_2 j, \partial_3 j$ denote the derivatives with respect to the second group of variables, respectively to the third group of variables.

Proof. We employ the Lebesgue dominated convergence theorem, see [4], p. 54. For $\lambda \to 0$, we have $\mathbf{y}^\epsilon_{(g+\lambda r)} \to \mathbf{y}^\epsilon_g$, $\frac{\mathbf{y}^\epsilon_{(g+\lambda r)}-\mathbf{y}^\epsilon_g}{\lambda} \to \boldsymbol{\zeta}$ strongly in

$L^2(D)^2$ and $\nabla \mathbf{y}^\epsilon_{(g+\lambda r)} \to \nabla \mathbf{y}^\epsilon_g$, $\frac{\nabla(\mathbf{y}^\epsilon_{(g+\lambda r)} - \mathbf{y}^\epsilon_g)}{\lambda} \to \nabla \boldsymbol{\zeta}$ strongly in $L^2(D)^4$. On a subsequece, see [4], Theorem IV.9, p. 58, we have

$$
\begin{aligned}
&\mathbf{y}^\epsilon_{(g+\lambda r)}(\mathbf{x}) \to \mathbf{y}^\epsilon_g(\mathbf{x}),\ |\mathbf{y}^\epsilon_{(g+\lambda r)}(\mathbf{x})|_{\mathbb{R}^2} \le \mathbf{h}^1(\mathbf{x}),\ a.e.\ in\ D,\\
&\nabla \mathbf{y}^\epsilon_{(g+\lambda r)}(\mathbf{x}) \to \nabla \mathbf{y}^\epsilon_g(\mathbf{x}),\ |\nabla \mathbf{y}^\epsilon_{(g+\lambda r)}(\mathbf{x})|_{\mathbb{R}^4} \le \mathbf{h}^2(\mathbf{x}),\ a.e.\ in\ D,\\
&\frac{\mathbf{y}^\epsilon_{(g+\lambda r)}(\mathbf{x}) - \mathbf{y}^\epsilon_g(\mathbf{x})}{\lambda} \to \boldsymbol{\zeta}(\mathbf{x}),\ \left|\frac{\mathbf{y}^\epsilon_{(g+\lambda r)}(\mathbf{x}) - \mathbf{y}^\epsilon_g(\mathbf{x})}{\lambda}\right|_{\mathbb{R}^2} \le \mathbf{h}^3(\mathbf{x}),\ a.e.\ in\ D,\\
&\frac{\nabla(\mathbf{y}^\epsilon_{(g+\lambda r)} - \mathbf{y}^\epsilon_g)(\mathbf{x})}{\lambda} \to \nabla\boldsymbol{\zeta}(\mathbf{x}),\ \left|\frac{\nabla(\mathbf{y}^\epsilon_{(g+\lambda r)} - \mathbf{y}^\epsilon_g)(\mathbf{x})}{\lambda}\right|_{\mathbb{R}^4} \le \mathbf{h}^4(\mathbf{x}),\ a.e.\ in\ D,
\end{aligned}
$$

where $\mathbf{h}^1, \mathbf{h}^3 \in L^2(D)^2$ and $\mathbf{h}^2, \mathbf{h}^4 \in L^2(D)^4$.

Adding and subtracting one term, we obtain

$$
\begin{aligned}
&\int_D \frac{(1-H^\epsilon(g+\lambda r))j\left(\mathbf{x}, \mathbf{y}^\epsilon_{(g+\lambda r)}, \nabla \mathbf{y}^\epsilon_{(g+\lambda r)}\right) - (1-H^\epsilon(g))j\left(\mathbf{x}, \mathbf{y}^\epsilon_g, \nabla \mathbf{y}^\epsilon_g\right)}{\lambda} \mathrm{d}\mathbf{x}\\
&= \int_D \frac{-H^\epsilon(g+\lambda r) + H^\epsilon(g)}{\lambda} j\left(\mathbf{x}, \mathbf{y}^\epsilon_{(g+\lambda r)}, \nabla \mathbf{y}^\epsilon_{(g+\lambda r)}\right) \mathrm{d}\mathbf{x}\\
&+ \int_D (1-H^\epsilon(g)) \frac{j\left(\mathbf{x}, \mathbf{y}^\epsilon_{(g+\lambda r)}, \nabla \mathbf{y}^\epsilon_{(g+\lambda r)}\right) - j\left(\mathbf{x}, \mathbf{y}^\epsilon_g, \nabla \mathbf{y}^\epsilon_g\right)}{\lambda} \mathrm{d}\mathbf{x}.
\end{aligned}
$$

We have $\frac{-H^\epsilon(g+\lambda r)+H^\epsilon(g)}{\lambda} \to -(H^\epsilon)'(g)r$ uniformly in $\mathcal{C}(\overline{D})$, $\left|\frac{-H^\epsilon(g+\lambda r)+H^\epsilon(g)}{\lambda}\right| \le M$ (due to the Lipschitz property of H^ϵ). Since $j(\mathbf{x}, \cdot, \cdot)$ is in $\mathcal{C}^1(\mathbb{R}^2 \times \mathbb{R}^4)$, on a subsequence

$$
j\left(\mathbf{x}, \mathbf{y}^\epsilon_{(g+\lambda r)}(\mathbf{x}), \nabla \mathbf{y}^\epsilon_{(g+\lambda r)}(\mathbf{x})\right) \to j\left(\mathbf{x}, \mathbf{y}^\epsilon_g(\mathbf{x}), \nabla \mathbf{y}^\epsilon_g(\mathbf{x})\right)
$$

and from (3.8), we get

$$
|j\left(\mathbf{x}, \mathbf{y}^\epsilon_{(g+\lambda r)}(\mathbf{x}), \nabla \mathbf{y}^\epsilon_{(g+\lambda r)}(\mathbf{x})\right)| \le a(\mathbf{x}) + c\left(|\mathbf{h}^1(\mathbf{x})|^2_{\mathbb{R}^2} + |\mathbf{h}^2(\mathbf{x})|^2_{\mathbb{R}^4}\right),\ a.e.\ in\ D.
$$

By the Lebesgue dominated convergence theorem, we obtain

$$
\begin{aligned}
&\int_D \frac{-H^\epsilon(g+\lambda r) + H^\epsilon(g)}{\lambda} j\left(\mathbf{x}, \mathbf{y}^\epsilon_{(g+\lambda r)}, \nabla \mathbf{y}^\epsilon_{(g+\lambda r)}\right) \mathrm{d}\mathbf{x}\\
&\to\ -\int_D (H^\epsilon)'(g) r\, j\left(\mathbf{x}, \mathbf{y}^\epsilon_g, \nabla \mathbf{y}^\epsilon_g\right) \mathrm{d}\mathbf{x}
\end{aligned}
$$

which is the first line of (3.11).

Since $j(\mathbf{x}, \cdot, \cdot)$ is in $\mathcal{C}^1(\mathbb{R}^2 \times \mathbb{R}^4)$, by the Mean Value Theorem we get

$$
\frac{j\left(\mathbf{x}, \mathbf{y}^\epsilon_{(g+\lambda r)}(\mathbf{x}), \nabla \mathbf{y}^\epsilon_{(g+\lambda r)}(\mathbf{x})\right) - j\left(\mathbf{x}, \mathbf{y}^\epsilon_g(\mathbf{x}), \nabla \mathbf{y}^\epsilon_g(\mathbf{x})\right)}{\lambda}
$$

$$\begin{aligned}
&= \partial_2 j\left(\mathbf{x},\mathbf{y_x},\mathbf{z_x}\right)\cdot\frac{\mathbf{y}^\epsilon_{(g+\lambda r)}(\mathbf{x})-\mathbf{y}^\epsilon_g(\mathbf{x})}{\lambda}\\
&\quad+\partial_3 j\left(\mathbf{x},\mathbf{y_x},\mathbf{z_x}\right):\frac{\nabla(\mathbf{y}^\epsilon_{(g+\lambda r)}-\mathbf{y}^\epsilon_g)(\mathbf{x})}{\lambda}\\
&\to \partial_2 j\left(\mathbf{x},\mathbf{y}^\epsilon_g(\mathbf{x}),\nabla\mathbf{y}^\epsilon_g(\mathbf{x})\right)\cdot\boldsymbol{\zeta}(\mathbf{x})\\
&\quad+\partial_3 j\left(\mathbf{x},\mathbf{y}^\epsilon_g(\mathbf{x}),\nabla\mathbf{y}^\epsilon_g(\mathbf{x})\right):\nabla\boldsymbol{\zeta}(\mathbf{x}),\ \textit{a.e. in } D
\end{aligned}$$

where $\mathbf{y_x}=(1-\theta)\mathbf{y}^\epsilon_g(\mathbf{x})+\theta\mathbf{y}^\epsilon_{(g+\lambda r)}(\mathbf{x})$ and $\mathbf{z_x}=(1-\theta)\nabla\mathbf{y}^\epsilon_g(\mathbf{x})+\theta\nabla\mathbf{y}^\epsilon_{(g+\lambda r)}(\mathbf{x})$, $0<\theta<1$ depending on $\mathbf{x}$. Using the hypotheses (3.9), (3.10) and the above estimates involving $\mathbf{h}^1,\mathbf{h}^2,\mathbf{h}^3,\mathbf{h}^4$, we get

$$\begin{aligned}
&\left|\partial_2 j\left(\mathbf{x},\mathbf{y_x},\mathbf{z_x}\right)\cdot\frac{\mathbf{y}^\epsilon_{(g+\lambda r)}(\mathbf{x})-\mathbf{y}^\epsilon_g(\mathbf{x})}{\lambda}\right|\\
+&\left|\partial_3 j\left(\mathbf{x},\mathbf{y_x},\mathbf{z_x}\right):\frac{\nabla(\mathbf{y}^\epsilon_{(g+\lambda r)}-\mathbf{y}^\epsilon_g)(\mathbf{x})}{\lambda}\right|\\
\le&\left[a_2(\mathbf{x})+c_2\left(|\mathbf{h}^1(\mathbf{x})|_{\mathbb{R}^2}+|\mathbf{h}^2(\mathbf{x})|_{\mathbb{R}^4}\right)\right]\mathbf{h}^3(\mathbf{x})\\
+&\left[a_3(\mathbf{x})+c_3\left(|\mathbf{h}^1(\mathbf{x})|_{\mathbb{R}^2}+|\mathbf{h}^2(\mathbf{x})|_{\mathbb{R}^4}\right)\right]\mathbf{h}^4(\mathbf{x}).
\end{aligned}$$

Moreover, $|1-H^\epsilon\left(g(\mathbf{x})\right)|\le 1$ for $\mathbf{x}\in D$, using again the Lebesgue dominated convergence theorem, we obtain the 2nd and the 3rd terms of (3.11). Since the limit is unique, then the convergence is valid without taking subsequences. □

Let us introduce the adjoint state system: find $\mathbf{z}^\epsilon_g\in V$ such that

$$\begin{aligned}
(3.12)\qquad &\mu\int_D\nabla\mathbf{z}^\epsilon_g:\nabla\mathbf{v}\,d\mathbf{x}+\frac{1}{\epsilon}\int_D H^\epsilon(g)\,\mathbf{z}^\epsilon_g\cdot\mathbf{v}\,d\mathbf{x}\\
=&\int_D(1-H^\epsilon(g))\,\partial_2 j\left(\mathbf{x},\mathbf{y}^\epsilon_g(\mathbf{x}),\nabla\mathbf{y}^\epsilon_g(\mathbf{x})\right)\cdot\mathbf{v}(\mathbf{x})d\mathbf{x}\\
&+\int_D(1-H^\epsilon(g))\,\partial_3 j\left(\mathbf{x},\mathbf{y}^\epsilon_g(\mathbf{x}),\nabla\mathbf{y}^\epsilon_g(\mathbf{x})\right):\nabla\mathbf{v}(\mathbf{x})d\mathbf{x},\quad\forall\mathbf{v}\in V.
\end{aligned}$$

As in Proposition 3.1, the adjoint state system has a unique solution in V.

Proposition 3.7. *Assume that $j(\mathbf{x},\cdot,\cdot)$ is in $\mathcal{C}^1(\mathbb{R}^2\times\mathbb{R}^4)$ and (3.8)-(3.10) hold. Let $\mathbf{y}^\epsilon_g$ be the solution of (3.3) and let $\mathbf{z}^\epsilon_g$ be the solution of (3.12). Then, the directional derivative of the objective function (3.2) in the direction $r\in\mathcal{F}$ is:*

$$\begin{aligned}
(3.13)\qquad J'(g)r &= -\int_D(H^\epsilon)'(g)r\,j\left(\mathbf{x},\mathbf{y}^\epsilon_g(\mathbf{x}),\nabla\mathbf{y}^\epsilon_g(\mathbf{x})\right)d\mathbf{x}\\
&\quad-\frac{1}{\epsilon}\int_D(H^\epsilon)'(g)r\,\mathbf{y}^\epsilon_g(\mathbf{x})\cdot\mathbf{z}^\epsilon_g(\mathbf{x})d\mathbf{x}.
\end{aligned}$$

Proof. Putting $\mathbf{v} = \mathbf{z}_g^\epsilon$ in (3.7) and $\mathbf{v} = \boldsymbol{\zeta}$ in (3.12), we get

$$\begin{aligned} &\int_D (1 - H^\epsilon(g))\, \partial_2 j\left(\mathbf{x}, \mathbf{y}_g^\epsilon(\mathbf{x}), \nabla \mathbf{y}_g^\epsilon(\mathbf{x})\right) \cdot \boldsymbol{\zeta}(\mathbf{x}) \mathrm{d}\mathbf{x} \\ &+ \int_D (1 - H^\epsilon(g))\, \partial_3 j\left(\mathbf{x}, \mathbf{y}_g^\epsilon(\mathbf{x}), \nabla \mathbf{y}_g^\epsilon(\mathbf{x})\right) : \nabla \boldsymbol{\zeta}(\mathbf{x}) \mathrm{d}\mathbf{x} \\ = \quad & -\frac{1}{\epsilon} \int_D (H^\epsilon)'(g) r\, \mathbf{y}_g^\epsilon(\mathbf{x}) \cdot \mathbf{z}_g^\epsilon(\mathbf{x}) \mathrm{d}\mathbf{x}. \end{aligned}$$

Using Proposition 3.6, we get (3.13). $\square$

We close this section with some examples and comments on special cases. Since $(H^\epsilon)' \geq 0$ in D, we can use as descent direction

$$\widetilde{r} = \epsilon\, j\left(\mathbf{x}, \mathbf{y}_g^\epsilon(\mathbf{x}), \nabla \mathbf{y}_g^\epsilon(\mathbf{x})\right) + \mathbf{y}_g^\epsilon(\mathbf{x}) \cdot \mathbf{z}_g^\epsilon$$

if $\widetilde{r} \in \mathcal{F}$. In the case of minimization of the dissipated energy $j\left(\mathbf{x}, \mathbf{y}_g^\epsilon(\mathbf{x}), \nabla \mathbf{y}_g^\epsilon(\mathbf{x})\right) = \|\mathbf{e}(\mathbf{y}_g^\epsilon(\mathbf{x}))\|_F^2$, we get

$$\widetilde{r} = \epsilon \|\mathbf{e}(\mathbf{y}_g^\epsilon)\|_F^2 + \mathbf{y}_g^\epsilon \cdot \mathbf{z}_g^\epsilon. \tag{3.14}$$

Next, we present a different descent direction, without using the adjoint state, when $\boldsymbol{\phi} = 0$. If

$$J(g) = \int_D \mathbf{f} \cdot \mathbf{y}_g^\epsilon \mathrm{d}\mathbf{x} + \int_{\Gamma_N} \boldsymbol{\psi} \cdot \mathbf{y}_g^\epsilon \mathrm{d}s \tag{3.15}$$

we can deduce that $J'(g)r = \int_D \mathbf{f} \cdot \boldsymbol{\zeta} \mathrm{d}\mathbf{x} + \int_{\Gamma_N} \boldsymbol{\psi} \cdot \boldsymbol{\zeta} \mathrm{d}s$. Putting $\mathbf{v} = \mathbf{y}_g^\epsilon$ in (3.7), (we are working with $\boldsymbol{\phi} = 0$) and $\mathbf{v} = \boldsymbol{\zeta}$ in (3.3), we get

$$J'(g)r = -\frac{1}{\epsilon} \int_D (H^\epsilon)'(g) r\, \mathbf{y}_g^\epsilon \cdot \mathbf{y}_g^\epsilon \mathrm{d}\mathbf{x}$$

and then

$$\widetilde{r} = \mathbf{y}_g^\epsilon \cdot \mathbf{y}_g^\epsilon \tag{3.16}$$

is a descent direction for (3.15).

If $\mathbf{y}_\Omega \in H^2(\omega)^2$ and using that $\nabla \cdot \mathbf{y}_\Omega = 0$ in ω, we can obtain that $2\nabla \cdot \mathbf{e}(\mathbf{y}_\Omega) = \Delta \mathbf{y}_\Omega$ in ω. If we replace (1.4), by

$$2\mu \mathbf{e}(\mathbf{y}_\Omega)\mathbf{n} - p_\Omega \mathbf{n} = \boldsymbol{\psi}, \text{ on } \Gamma_N, \tag{3.17}$$

the weak variational formulation (1.6) has to be replaced by: find $\mathbf{y}_\Omega \in y_{\boldsymbol{\phi}} + V_\Omega$

$$2\mu \int_\omega \mathbf{e}(\mathbf{y}_\Omega) : \mathbf{e}(\mathbf{v}_\Omega)\, \mathrm{d}\mathbf{x} = \int_\omega \mathbf{f} \cdot \mathbf{v}_\Omega\, \mathrm{d}\mathbf{x} + \int_{\Gamma_N} \boldsymbol{\psi} \cdot \mathbf{v}_\Omega\, \mathrm{d}s, \ \forall \mathbf{v}_\Omega \in V_\Omega. \tag{3.18}$$

If $\boldsymbol{\phi} = 0$, we can put $\mathbf{v}_\Omega = \mathbf{y}_\Omega$ before and get

$$2\mu \int_\omega \|\mathbf{e}(\mathbf{y}_\Omega)\|_F^2\, \mathrm{d}\mathbf{x} = \int_\omega \mathbf{f} \cdot \mathbf{y}_\Omega\, \mathrm{d}\mathbf{x} + \int_{\Gamma_N} \boldsymbol{\psi} \cdot \mathbf{y}_\Omega\, \mathrm{d}s.$$

Since $\mathbf{y}_g^\epsilon$ is "almost" zero in Ω_g, the objective function (3.15) may be also used for minimizing the dissipated energy.

4. NUMERICAL RESULTS

We have employed the software FreeFem++, [10]. The computational domain is $D = \{(x_1, x_2) \in \mathbb{R}^2; -\frac{L}{2} < x_1 < \frac{L}{2},\ -\frac{l}{2} < x_2 < \frac{l}{2}\}$ with $L = 1.5$, $l = 1$, the viscosity is $\mu = 1$ and the body forces are $\mathbf{f} = (0,0)$. On the left and right sides, we impose Neumann boundary condition (1.4) with $\boldsymbol{\psi} = (30,0)$ and on the top and bottom sides, we impose Dirichlet boundary condition (1.3) with $\boldsymbol{\phi} = (0,0)$.

We use a mesh of 34020 triangles and 17261 vertices. The Stokes equations are solved using mixed finite element formulation, see [6]. For the velocity, we employ the finite elements $\mathbb{P}_1 + bubble$ and for the pressure, we employ $\mathbb{P}_1$. The penalization parameter is $\epsilon = 0.0001$. As descent direction, for minimizing the dissipated energy $j = \|\mathbf{e}(\mathbf{y}_g^\epsilon(\mathbf{x}))\|_F^2$, we use (3.14). The algorithm is $g_{n+1} = g_n + \lambda_n r_n$, where r_n is the descent direction and $\lambda_n \in \arg\min_{\lambda>0} J(g_n + \lambda r_n)$. Practically, we use $\lambda = \rho^i$, $\rho \in (0,1)$, $i = 0, 1, \dots, 9$ and $\rho = 0.6$.

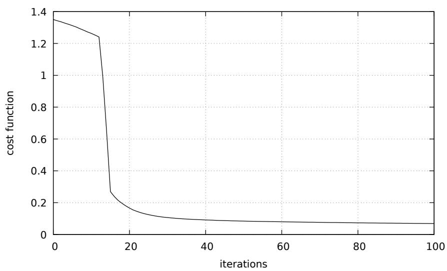

Fig. 1 – Convergence history of the objective functions.

The initial domain is given by $g_0 = -0.3 - \sin(4\pi x_1)\sin(3\pi(x_2 - 0.5))$. The initial value of the objective function is 1.35046 and the value after 100 iterations is 0.0688113. The history is plotted in Figure 1. We notice a significant decrease before iteration 20. After that, the decrease is slow. The initial, intermediate and the final obstacle domains are presented in Figure 2. We observe that the final obstacle is connected and there exist two zones where the fluid is confined.

We have also tested for the the objective function (3.15) and the descent direction (3.16) with the same numerical parameters as before. The evolution of the objective function and the obstacle domain after 100 iterations are presented in Figure 3. We point out that (3.15) is an approximation of the previous objective function multiplied by the factor 2μ (see the equality after (3.18)).

The volume of the initial obstacle is 0.422077 and after 100 iterations it grows to 0.543312 in the case of objective function (3.15) (more than 0.47344 obtained for the domain obstacle presented in Figure 2, at the bottom, right).

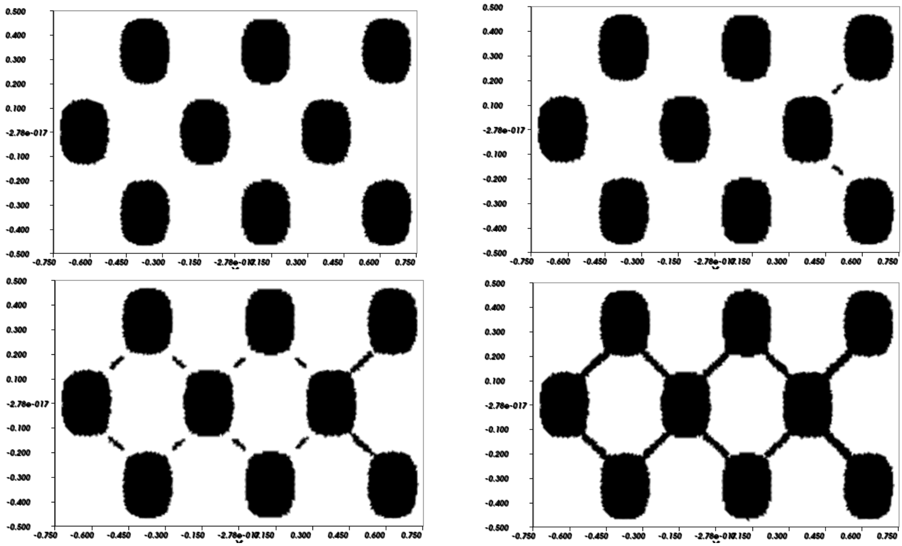

Fig. 2 – Initial (top, left), intermediate and final (bottom, right, after 100 iterations) obstacle domains.

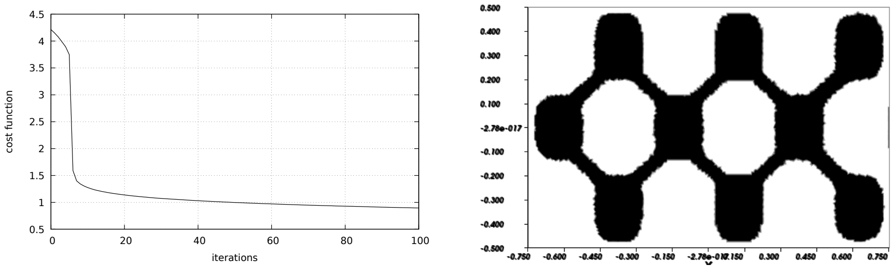

Fig. 3 – Convergence history of the objective function (3.15) and the obstacle domain after 100 iterations using the descent direction (3.16).

REFERENCES

[1] R. Adams, *Sobolev spaces.* Academic Press, New York-London, 1975.

[2] S. Amstutz, *The topological asymptotic for the Navier-Stokes equations.* ESAIM Control Optim. Calc. Var. **11** (2005), *3*, 401–425.

[3] F. Boyer and P. Fabrie, *Mathematical tools for the study of the incompressible Navier-Stokes equations and related models.* Applied Mathematical Sciences Vol. 183. New York, Springer, 2013.

[4] H. Brezis, *Analyse fonctionnelle. Théorie et applications.* Dunod, 2005.

[5] C. Dapogny, P. Frey, F. Omnès, and Y. Privat, *Geometrical shape optimization in fluid mechanics using FreeFem++.* Structural and Multidisciplinary Optimization **58** (2018), 2761–2788.

[6] V. Girault and P.A. Raviart, *Finite element methods for Navier-Stokes equations. Theory and algorithms.* Springer Series in Computational Mathematics Vol. 5. Springer-Verlag, Berlin, 1986.

[7] A. Halanay and D. Tiba, *Shape optimization for stationary Navier-Stokes equations.* Control and Cybernetics **38** (2009) No. 4B.

[8] A. Halanay, C.M. Murea, and D. Tiba, *Existence of a steady flow of Stokes fluid past a linear elastic structure using fictitious domain*, J. Math. Fluid Mech. **18** (2016), *2*, 397–413.

[9] M. Hassine, S. Jan, and M. Masmoudi, *From differential calculus to 0-1 topological optimization.* SIAM J. Control Optim. **45** (2007), *6*, 1965–1987.

[10] F. Hecht, *New development in FreeFem++.* J. Numer. Math. **20** (2012), 251–265.

[11] M.W. Hirsch, S. Smale, and R.L. Devaney, *Differential Equations, Dynamical Systems, and an Introduction to Chaos.* Elsevier, Academic Press, San Diego, 2014.

[12] V. Maz'ya and J. Rossmann, *Lp estimates of solutions to mixed boundary value problems for the Stokes system in polyhedral domains*, Math. Nachr. **280** (2007), *7*, 751–793.

[13] C.M. Murea and D. Tiba, *Topological optimization via cost penalization.* Topological Methods in Nonlinear Analysis **54** (2019), *2B*, 1023–1050.

[14] C.M. Murea and D. Tiba, *Topological optimization and minimal compliance in linear elasticity.* Evolution Equations and Control Theory **9** (2020), *4*, 1115-1131.

[15] P. Neittaanmäki, J. Sprekels, and D. Tiba, *Optimization of elliptic systems. Theory and applications.* Springer Monographs in Mathematics. Springer, New York, 2006.

[16] P. Neittaanmäki, A. Pennanen, and D. Tiba, *Fixed domain approaches in shape optimization problems with Dirichlet boundary conditions.* Inverse Problems, **25** (2009), 1–18.

[17] P. Neittaanmäki and D. Tiba, *Fixed domain approaches in shape optimization problems.* Inverse Problems **28** (2012) 1–35.

[18] A. Novotny and J. Sokolowski, *Topological derivatives in shape optimization.* Springer, Heidelberg, 2013.

[19] S. Osher and J.A. Sethian, *Fronts propagating with curvature-dependent speed: algorithms based on Hamilton-Jacobi formulations.* J. Comput. Phys. **79** (1988), *1*, 12–49.

[20] P. Plotnikov and J. Sokolowski, *Compressible Navier-Stokes equations. Theory and shape optimization.* Birkhauser, Springer, Basel, 2012.

[21] L.S. Pontryagin, *Equations Differentielles Ordinaires.* MIR, Moscow, 1968.

[22] L. Stupelis, *Navier-Stokes Equations in Irregular Domains.* Springer, 1995.

[23] R. Temam, *Navier-Stokes equations. Theory and numerical analysis.* Third edition. Studies in Mathematics and its Applications Vol. 2. North-Holland Publishing Co., Amsterdam, 1984.

[24] R. Temam, *Navier-Stokes equations and nonlinear functional analysis.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1995.

[25] D. Tiba, *Domains of class C: properties and application.* Ann. of the Univ. of Bucharest (Ser. Math.) **4** (LXII) (2013), 89–102.

[26] D. Tiba, *Iterated Hamiltonian type systems and applications.* J. Differential Equations **264** (2018), *8*, 5465–5479.

[27] D. Tiba, *A penalization approach in shape optimization*, Atti della Accademia Peloritana dei Pericolanti - Classe di Scienze Fisiche, Matematiche e Naturali **96** (2018), *1*, A8.

[28] D. Tiba, *Implicit parametrizations and applications in optimization and control*, Mathematical Control and Related Fields **10**, *3* (2020), 455–470.

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