Dedicated to Marius Tucsnak on the occasion of his 60^{th} anniversary

NUMERICAL CONTROL OF THE WAVE EQUATION AND HUYGENS' PRINCIPLE

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The numerical control of the wave equation is investigated in dimension 2 and 3. Using Huygens' principle in dimension 3 (resp. the method of descent in dimension 2), we show that the exact controllability of the wave equation can be obtained numerically on a cuboid (resp. on a rectangle) in sharp time with a boundary control supported on adjacent sides, by directly solving the wave equation on a larger domain. As a consequence, the numerical control of the wave equation is not affected here by the appearance of high frequency spurious numerical waves, and no filtering is needed.

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1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded open set, and let $\Gamma_0 \subset \Gamma := \partial\Omega$ be an open set. The classical boundary control problem for the wave equation can be stated as follows: given $T > 0$ and $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, can we find a control input $g \in L^2(0,T,L^2(\Gamma_0))$ such that the solution of the boundary initial-value problem

$$\Box\, u := u_{tt} - \Delta u = 0 \quad \text{in } \Omega \times (0,T), \tag{1.1}$$

$$u = g\, 1_{\Gamma_0} \quad \text{on } \Gamma \times (0,T), \tag{1.2}$$

$$u(.,0) = u^0,\ u_t(.,0) = u^1, \tag{1.3}$$

satisfies

$$u(.,T) = u_t(.,T) = 0\,? \tag{1.4}$$

That issue has been extensively studied, both theoretically and numerically. Using duality arguments, the exact controllability of (1.1)–(1.4) was reduced to an observability estimate for the adjoint equation, that was in turn established with nonharmonic Fourier series in [25, 27, 35], with the multiplier method in [15, 18], and with microlocal analysis in [4, 5]. The study of the numerical control of the wave equation turned to be a delicate issue, due to the presence of high frequency spurious numerical waves. Various remedies (multi-grid methods, filtering, Tychonoff regularization, vanishing viscosity, mixed finite element methods, etc.) were introduced to overcome this problem, see *e.g.* [2, 3, 8, 12, 13, 22, 26, 29, 33, 37]. We refer the reader to [10, 38, 39] for recent and exhaustive surveys about the numerical control of the wave equation.

The numerical schemes in the references above use finite differences or finite elements methods combined with the *Hilbert Uniqueness Method* (HUM) introduced by J.-L. Lions. It should be mentioned that some other (more direct) approaches were also introduced since the seventies for the control of PDEs, using domain extensions [19, 35], fundamental solutions compactly supported in time [14, 34], resolution of ill-posed problems [20, 21], flatness outputs [16, 23, 24], or feedback controls achieving a finite-time stabilization [1, 30]. In [6], the numerical control of a semilinear wave equation on the interval $(0,1)$ is investigated by using a Picard iterative scheme and a transparent boundary condition at $x=1$.

In this paper, we are interested in developing a *direct* approach (*i.e.* not based on the observability of the adjoint equation) for the numerical control of (1.1)-(1.4). Our approach rests on the numerical integration (by using spectral methods) of a classical Cauchy problem for the wave equation in an extended domain for which Huygens' principle is valid. By Huygens' principle, we mean here the following property: for any $(x,t)\in\mathbb{R}^3\times\mathbb{R}$, we have

$$(1.5)\qquad (v^0(y), v^1(y), \nabla v^0(y)) = 0 \text{ for } |y-x| = |t| \quad\Rightarrow\quad v(x,t)=0$$

for any smooth solution v of

$$(1.6)\qquad \Box v = 0 \quad \text{in } \mathbb{R}^3\times\mathbb{R}, \qquad v(.,0)=v^0,\ v_t(.,0)=v^1.$$

A direct consequence of (1.5) is that if both v^0 and v^1 are supported in a compact set $K\subset\mathbb{R}^3$, and if $T>\text{diam }(K) := \sup_{x_1,x_2\in K}|x_1-x_2|$, then

$$(1.7)\qquad v(x,t)=0, \qquad \text{for } x\in K,\ t\geq T.$$

That property has important implications for the control problem (1.1)-(1.4). Assume first that $\Gamma_0=\Gamma$, and assume given $(u^0,u^1)\in L^2(\Omega)\times H^{-1}(\Omega)$. Let $(v^0,v^1)\in L^2(\mathbb{R}^3)\times H^{-1}(\mathbb{R}^3)$ denote the extension by 0 of (u^0,u^1), and let v denote the solution of (1.6). Then $u=v_{|\Omega\times(0,T)}$ and $g=v_{|\Gamma_0\times(0,T)}$ solve (1.1)-(1.4) provided that $T>\text{diam}(\Omega)$, by (1.7) [19].

When Γ_0 is a proper subset of Γ, the additional constraint

$$(1.8)\qquad u(x,t)=0 \qquad (x,t)\in(\Gamma\setminus\Gamma_0)\times(0,T)$$

imposes to modify the method as follows. We introduce, when it is possible, some unbounded smooth surface $\mathcal{S} \subset \mathbb{R}^3$ such that

(i) $\mathbb{R}^3 \setminus \mathcal{S} = \Omega_1 \cup \Omega_2$, Ω_1 and Ω_2 being two unbounded, connected open sets with $\partial\Omega_1 = \partial\Omega_2 = \mathcal{S}$;

(ii) $\Omega \subset \Omega_1$, $\Gamma \setminus \Gamma_0 \subset \mathcal{S}$;

(iii) (Huygens' principle in Ω_1) for any compact set $K \subset \overline{\Omega_1}$, there exists a time $T > 0$ such that for any solution v of

$$\Box v = 0 \quad \text{in } \Omega_1 \times \mathbb{R}, \tag{1.9}$$

$$v = 0 \quad \text{in } \mathcal{S} \times \mathbb{R}, \tag{1.10}$$

$$v(.,0) = v^0, \ v_t(.,0) = v^1 \tag{1.11}$$

with v^0, v^1 supported in K, it holds

$$v(x,t) = 0 \quad \text{for } (x,t) \in K \times [T, +\infty). \tag{1.12}$$

Next, given $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, we let $(v^0, v^1) \in L^2(\Omega_1) \times H^{-1}(\Omega_1)$ denote the extension by 0 of (u^0, u^1), and introduce the solution v of (1.9)-(1.11). Then $u = v_{|\Omega\times(0,T)}$ and $g = v_{|\Gamma_0\times(0,T)}$ solve (1.1)-(1.4)

Of course, this control method can be carried out at the only condition that Huygens' principle is valid and established. To the best knowledge of the authors, Huygens' principle in open sets as stated in (iii) was not investigated so far. It should be noticed that it is related to the Geometric Control Condition introduced in [4] for the control of the wave equation. Indeed, Huygens' principle in Ω_1 fails whenever there are some trapped rays in Ω_1 (see *e.g.* [32]). The validity of Huygens' principle cannot nevertheless be reduced to the absence of trapped rays in Ω_1. Indeed, Huygens' principle is likely false for the complement of a ball in $\mathbb{R}^3$, as it was noticed to the authors by G. Lebeau in [17].

Here, we shall assume that Ω_1 is an unbounded domain in which all the rays of geometric optics can escape to infinity.

In this paper, we shall prove that Huygens' principle as stated in (iii) is valid when Ω_1 is a half-space. The proof will be done by an extension of the classical solution of the wave equation in $\mathbb{R}^3$ by spherical means, the spherical wavefronts being replaced by wavefronts in Ω_1 taking into account the reflection of rays on $\mathcal{S}$. As an immediate application of the above method, we shall derive the exact controllability in optimal time of (1.1)-(1.3) when $\Omega = (0, L_1) \times (0, L_2) \times (0, L_3)$ and Γ_0 is the union of three adjacent sides. We refer to [25, 27] for similar results obtained by an Ingham-type approach, and to [3, 22, 37] for the numerical control of the wave equation on a square.

The main goal of this study is to develop efficient numerical schemes for the control problem (1.1)-(1.4). Even for simple geometries, fast numerical schemes are desirable to study the numerical control of semilinear wave equations (see [6]) or of some equations that may be obtained from the wave equation by the transmutation method (*e.g.* the heat equation and Schrödinger equation [9, 28, 31]).

In practice, when a pair $(\Omega_1, \mathcal{S})$ as above can be constructed, one solves the wave equation in an intermediate domain $\widehat{\Omega}$ such that $\Omega \subset \widehat{\Omega} \subset \Omega_1$, $\widehat{\Omega}$ is bounded, $\Gamma \setminus \Gamma_0 \subset \partial\widehat{\Omega}$, and $\mathrm{dist}(\partial\widehat{\Omega} \setminus \mathcal{S}, \Omega)$ is large enough so that bounces of waves on $\partial\widehat{\Omega} \setminus \mathcal{S}$ do not affect the solution v in $\Omega \times (0, T)$. (See Figure 1.)

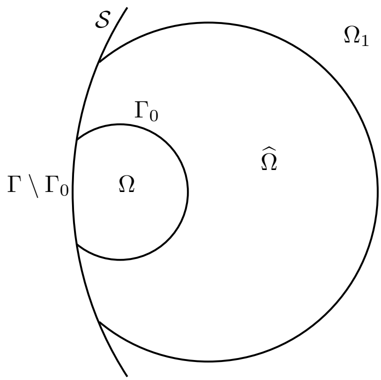

Fig. 1 – The open sets $\Omega \subset \widehat{\Omega} \subset \Omega_1$ and the surface $\mathcal{S}$.

If $\Gamma \setminus \Gamma_0$ is flat (examples include a cube, a half-ball, a pyramid, etc.), one can solve the wave equation in $\widehat{\Omega} = (-L, L)^3$ by using Fourier series.

All the results discussed so far concern domains $\Omega \subset \mathbb{R}^3$. It is well known that Huygens' principle fails in $\mathbb{R}^2$, so that the above method cannot be applied as it is for domains in $\mathbb{R}^2$. Some stabilization results for domains $\Omega \subset \mathbb{R}^2$ will be obtained by combining the above method with the classical method of descent.

The paper is outlined as follows. Section 2 is devoted to the derivation of Huygens' principle for the half-space. It is based on the extension of solutions of the wave equation to the whole space as odd functions in one coordinate and on the classical solution of the wave equation on $\mathbb{R}^3$ by spherical means. The same ideas are applied to solve (1.1)-(1.4) when Ω is a product of intervals. Section 3 contains some numerical experiments in dimensions 2 and 3. Section 4 provides some concluding remarks.

2. HUYGENS' PRINCIPLE FOR THE HALF-SPACE

The derivation of Huygens' principle in a half-space Ω_1 will follow the same pattern as for Huygens' principle in $\mathbb{R}^3$. It will be based on some extension of Kirchhoff formula, in which spherical means will be replaced by (algebraic) means over wavefronts.

2.1. Solution by spherical means and Huygens' principle in $\mathbb{R}^3$

For the reader's convenience, we briefly recall the main steps in the derivation of Kirchhoff formula for the solution of the wave equation in $\mathbb{R}^3$ (see *e.g.* [11] for the details). Let $(v^0, v^1) \in C^2(\mathbb{R}^3)^2$ be given, and let $v = v(x,t)$ denote the classical solution to (1.6). For given $x \in \mathbb{R}^3$, $r > 0$ and $t > 0$, let

$$V(x,r,t) = \frac{1}{4\pi r^2}\int_{\partial B(x,r)} v(y,t)\mathrm{d}\sigma(y), \tag{2.1}$$

$$\begin{aligned} V^0(x,r) &= \frac{1}{4\pi r^2}\int_{\partial B(x,r)} v^0(y)\mathrm{d}\sigma(y), \\ V^1(x,r) &= \frac{1}{4\pi r^2}\int_{\partial B(x,r)} v^1(y)\mathrm{d}\sigma(y). \end{aligned} \tag{2.2}$$

It can be proved by a direct computation that

$$V_r = \frac{1}{4\pi r^2}\int_{B(x,r)} \Delta v(y,t)\mathrm{d}y. \tag{2.3}$$

Using (1.6) this yields

$$r^2 V_r = \frac{1}{4\pi}\int_{B(x,r)} v_{tt}(y,t)\mathrm{d}y,$$

and hence

$$(r^2 V_r)_r = r^2 V_{tt}. \tag{2.4}$$

Let $\tilde V := rV$, $\tilde V^0 := rV^0$, and $\tilde V^1 := rV^1$. Then $\tilde V$ solves the 1D wave equation

$$\tilde V_{tt} - \tilde V_{rr} = 0 \quad \text{in } (0,+\infty)_r \times (0,+\infty)_t, \tag{2.5}$$

$$\tilde V = 0 \quad \text{in } \{0\}_r \times (0,+\infty)_t, \tag{2.6}$$

$$(\tilde V(.,0), \tilde V_t(.,0)) = (\tilde V^0, \tilde V^1) \tag{2.7}$$

so that, by d'Alembert formula applied to odd solutions of the 1D wave equation

$$\tilde V(x,r,t) = \frac{1}{2}(\tilde V^0(x,t+r) - \tilde V^0(x,t-r)) + \frac{1}{2}\int_{t-r}^{t+r} \tilde V^1(x,z)\mathrm{d}z \quad \text{for } 0 \le r \le t.$$

This yields with (2.1)-(2.2)

$$
\begin{aligned}
v(x,t) &= \lim_{r\to 0}\frac{\tilde V(x,r,t)}{r}\\
&= \tilde V^0_r(x,t)+\tilde V^1(x,t)\\
&= \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{\partial B(x,t)} v^0(y)\mathrm{d}\sigma(y)\right)+\frac{1}{4\pi t}\int_{\partial B(x,t)} v^1(y)\mathrm{d}\sigma(y). \qquad (2.8)
\end{aligned}
$$

Computing the derivative in (2.8) leads to Kirchhoff formula in $\mathbb{R}^3$:

$$
v(x,t)=\frac{1}{4\pi t^2}\int_{\partial B(x,t)}[\nabla v^0(y)\cdot(y-x)+v^0(y)+tv^1(y)]\mathrm{d}\sigma(y), \qquad (x,t)\in\mathbb{R}^3\times(0,+\infty). \qquad (2.9)
$$

Then (1.5) follows at once from (2.9).

Let us turn now to the solution of the wave equation in $\mathbb{R}^2$. In order to solve

$$
\Box w = 0 \quad \text{in } \mathbb{R}^2\times\mathbb{R}, \qquad (2.10)
$$

$$
(w(.,0),w_t(.,0)) = (w^0,w^1) \qquad (2.11)
$$

the method of descent consists in introducing the solution v of (1.6) for

$$
v^0(x_1,x_2,x_3)=w^0(x_1,x_2),\qquad v^1(x_1,x_2,x_3)=w^1(x_1,x_2).
$$

Since v does not depend on x_3, the solution w of (2.10)-(2.11) can be written as $w(x_1,x_2,t)=v(x_1,x_2,0,t)$. Combined with (2.9), this gives after some computations [11] Poisson's formula:

$$
w(x,t)=\frac{1}{2\pi t^2}\int_{B(x,t)}\frac{t\nabla w^0(y)\cdot(y-x)+tw^0(y)+t^2w^1(y)}{(t^2-|y-x|^2)^{\frac12}}\mathrm{d}y, \qquad (x,t)\in\mathbb{R}^2\times(0,+\infty). \qquad (2.12)
$$

From (2.12), it is clear (taking *e.g.* $w^0\equiv 0$ and $w^1=1_{B(0,1)}$) that Huygens' principle fails in $\mathbb{R}^2$.

2.2. **Solution of the wave equation on the half-space**

Assume that $\Omega_1=(0,+\infty)\times\mathbb{R}^2$, so that $\mathcal{S}=\{x_1=0\}$. To solve (1.9)-(1.11), we use an odd reflection w.r.t. x_1, *i.e.* we set

$$
\begin{aligned}
\tilde v^0(x_1,x_2,x_3) &:= \operatorname{sign}(x_1)v^0(|x_1|,x_2,x_3), \quad x\in\mathbb{R}^3,\\
\tilde v^1(x_1,x_2,x_3) &:= \operatorname{sign}(x_1)v^1(|x_1|,x_2,x_3), \quad x\in\mathbb{R}^3,\\
\tilde v(x_1,x_2,x_3,t) &:= \operatorname{sign}(x_1)v(|x_1|,x_2,x_3,t), \quad (x,t)\in\mathbb{R}^3\times(0,+\infty).
\end{aligned}
$$

Since $\tilde{v}$ solves (1.6) (with (v^0, v^1) replaced by $(\tilde{v}^0, \tilde{v}^1)$), then $\tilde{v}$ is given by (2.8) (or, equivalently (2.9)).

As Huygens' principle holds in $\mathbb{R}^3$, it holds as well in Ω_1. Indeed, if v^0 and v^1 are some functions which are smooth and supported in some compact set $K \subset \overline{\Omega_1}$, then if we pick $R > 0$ and $\bar{x} \in \mathcal{S}$ such that $K \subset B(\bar{x}, R)$, then for $T \geq 2R$, $x \in K$ and $y \in \partial B(x, T)$, we have $y \notin K$ (for $|y - \bar{x}| \geq R$), hence with (2.9)

$$v(x, T) = \tilde{v}(x, T) = 0.$$

Note that, by an easy density argument, the same conclusion holds if $(v^0, v^1) \in L^2(\Omega_1) \times H^{-1}(\Omega_1)$ is supported in the compact set K.

An extension of Kirchhoff formula can also be derived. For any $x = (x_1, x_2, x_3) \in \Omega_1$ and any $t > 0$, we set $s_1(x) = (-x_1, x_2, x_3)$ and

$$\Sigma_+(x, t) := \partial B(x, t) \cap \{x_1 > 0\}, \quad \Sigma_-(x, t) := \partial B(s_1(x), t) \cap \{x_1 > 0\},$$

$$\Sigma(x, t) := \begin{cases} \Sigma_+(x, t) & \text{if } 0 < t \leq x_1, \\ \Sigma_+(x, t) \cup \Sigma_-(x, t) & \text{if } t > x_1, \end{cases}$$

and for $y \in \Sigma(x, t)$

$$\varepsilon(x, y, t) := \begin{cases} 1 & \text{if } y \in \Sigma_+(x, t), \\ -1 & \text{if } y \in \Sigma_-(x, t). \end{cases}$$

(See Figure 2.)

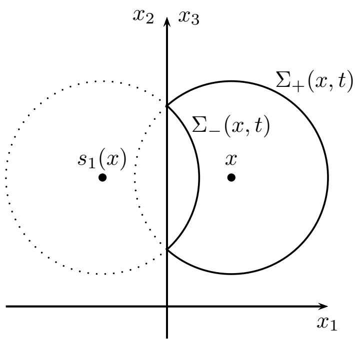

Fig. 2 – The wavefronts $\Sigma_+(x, t)$ and $\Sigma_-(x, t)$.

Since $\tilde{v}$ is odd w.r.t. x_1, we infer from (2.8) that

$$v(x, t) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{\Sigma(x,t)} \varepsilon(x, y, t) v^0(y) \mathrm{d}\sigma(y)\right) + \frac{1}{4\pi t}\int_{\Sigma(x,t)} \varepsilon(x, y, t) v^1(y) \mathrm{d}\sigma(y). \tag{2.13}$$

2.3. Control problem when $\Gamma \setminus \Gamma_0$ is flat

Assume that $\Omega \subset \mathbb{R}^3$ with $\Gamma \setminus \Gamma_0 \subset \{x_1 = 0\}$. Let $\mathcal{S} = \{x_1 = 0\}$ and $\Omega_1 = \{x_1 > 0\}$. Let $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$. Pick $\bar{x} = (0, \bar{x}_2, \bar{x}_3)$ and $R > 0$ such that $\Omega \subset B(\bar{x}, R)$, and let $T = 2R$ and $L > 3R$. Set

$$\widehat{\Omega} = (0, L) \times (\bar{x}_2 - L, \bar{x}_2 + L) \times (\bar{x}_3 - L, \bar{x}_3 + L). \tag{2.14}$$

Then the solution v of

$$\Box v = 0 \quad \text{in } \widehat{\Omega} \times \mathbb{R}, \tag{2.15}$$
$$v = 0 \quad \text{in } \partial\widehat{\Omega} \times \mathbb{R} \tag{2.16}$$
$$(v(.,0), v_t(.,0)) = (1_\Omega u^0, 1_\Omega u^1) \tag{2.17}$$

satisfies

$$v = v_t = 0 \quad \text{on} \quad \Omega \times \{t = T\}. \tag{2.18}$$

Indeed, since $L > 3R$, the solution of (2.15)-(2.17) coincide with those of (1.9)-(1.11) for $x \in \Omega$ and $0 \le t \le T$, and hence (2.18) holds according to Huygens' principle in $\{x_1 > 0\}$.

Therefore the pair $(u, g) = (v_{|\Omega \times (0,T)}, v_{|\Gamma_0 \times (0,T)})$ solves the control problem (1.1)-(1.4).

Assume now that $\Omega \subset \mathbb{R}^2$ is a bounded domain, and that Γ_0 is such that $\partial\Omega \setminus \Gamma_0 \subset \{x_1 = 0\}$. We shall use the classical *method of descent* (see *e.g.* [11]). Pick any $\delta > 0$ and set $\tilde{\Omega} := \Omega \times (-\delta, \delta)$. For given $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, let

$$v^0(x_1, x_2, x_3) := \begin{cases} u^0(x_1, x_2) & \text{if } (x_1, x_2, x_3) \in \tilde{\Omega}, \\ 0 & \text{otherwise}, \end{cases} \tag{2.19}$$

$$v^1(x_1, x_2, x_3) := \begin{cases} u^1(x_1, x_2) & \text{if } (x_1, x_2, x_3) \in \tilde{\Omega}, \\ 0 & \text{otherwise}. \end{cases} \tag{2.20}$$

Let $\bar{x} = (0, \bar{x}_2, 0) \in \mathbb{R}^3$ and $R > 0$ be such that $\tilde{\Omega} \subset B(\bar{x}, R)$. Let $T = 2R$, $L > 3R$, $\widehat{\Omega}$ be as in (2.14), and let v solve (2.15)-(2.17). Then it is easily seen that

$$u(x_1, x_2, t) := \frac{1}{2\delta} \int_{-L}^{L} v(x_1, x_2, x_3, t) \mathrm{d}x_3 \tag{2.21}$$

satisfies

$$u_{tt} - u_{x_1x_1} - u_{x_2x_2} = 0 \quad \text{in } \Omega \times (0, T), \tag{2.22}$$
$$u = 0 \quad \text{on } \{x_1 = 0\} \times (0, T), \tag{2.23}$$
$$(u(.,0), u_t(.,0)) = (u^0, u^1). \tag{2.24}$$

(For (2.22), we noticed that $v(x,t)=0$ for $t\in[0,T]$ and x in a neighborhood of $\Omega\times\{\pm L\}$.) Since the property

$$v(x_1,x_2,x_3,T)=0 \qquad \text{for } (x_1,x_2)\in\Omega \text{ and } x_3\in(-L,L)$$

is not necessarily true, we don't claim that $u=u_t=0$ for $(x_1,x_2)\in\Omega$ and $t=T$. What is expected from (2.9) is that $\|v(.,T)\|_{L^2(\Omega)}=O(T^{-1})$ (resp. $\|v(.,T)\|_{L^2(\Omega)}=O(T^{-2})$) if $v^1\neq 0$ (resp. $v^1=0$). Indeed, for $(x_1,x_2,x_3)\in\Omega\times(-L,L)$, the integration in (2.9) is performed over the set $\partial B(x,t)\cap\tilde\Omega$ whose Lebesgue measure is bounded.

2.4. Control problem when Ω is a box

2.4.1. $N=3$

We assume that

$$\begin{aligned}\Omega &= (0,L_1)\times(0,L_2)\times(0,L_3), \quad \Gamma=\partial\Omega,\\ \Gamma\setminus\Gamma_0 &= (\{x_1=0\}\cap\Gamma)\cup(\{x_2=0\}\cap\Gamma)\cup(\{x_3=0\}\cap\Gamma).\end{aligned}$$

For $k=(k_1,k_2,k_3)\in\mathbb{R}^3$, we denote $|k|=\sqrt{k_1^2+k_2^2+k_3^2}$ and $\|k\|_\infty=\sup(|k_1|,|k_2|,|k_3|)$.

Pick any $(u^0,u^1)\in L^2(\Omega)\times H^{-1}(\Omega)$ decomposed in sine Fourier series as

$$\begin{aligned}u^0(x_1,x_2,x_3) &= \sum_{k\in\mathbb{N}^{*3}} u_k^0\sin(\pi k_1x_1/L_1)\sin(\pi k_2x_2/L_2)\sin(\pi k_3x_3/L_3)\\ u^1(x_1,x_2,x_3) &= \sum_{k\in\mathbb{N}^{*3}} u_k^1\sin(\pi k_1x_1/L_1)\sin(\pi k_2x_2/L_2)\sin(\pi k_3x_3/L_3)\end{aligned}$$

with $\sum_{k\in\mathbb{N}^{*3}}(|u_k^0|^2+|k|^{-2}|u_k^1|^2)<\infty$. Let $A_\Omega=-\Delta$ with domain $D(A_\Omega)=H^2(\Omega)\cap H_0^1(\Omega)\subset L^2(\Omega)$. Then for any $s\geq 0$

$$(u^0,u^1)\in D(A_\Omega^{\frac{s}{2}})\times D(A_\Omega^{\frac{s-1}{2}}) \iff \sum_{k\in\mathbb{N}^{*3}}|k|^{2s}(|u_k^0|^2+|k|^{-2}|u_k^1|^2)<\infty.$$

Let $\varepsilon>0$ and let $\rho\in C^\infty(\mathbb{R})$ be an even function such that $\rho(z)=1$ for $|z|\leq 1$ and $\rho(z)=0$ for $|z|\geq 1+\varepsilon$. For $x\in\mathbb{R}^3$, let

$$\psi(x):=\rho(x_1/L_1)\rho(x_2/L_2)\rho(x_3/L_3) \tag{2.25}$$

and

(2.26)
$$v^0(x_1,x_2,x_3)=\psi(x)\sum_{k\in\mathbb{N}^{*3}} u_k^0\sin(\pi k_1x_1/L_1)\sin(\pi k_2x_2/L_2)\sin(\pi k_3x_3/L_3)$$

(2.27)
$$v^1(x_1,x_2,x_3)=\psi(x)\sum_{k\in\mathbb{N}^{*3}} u_k^1 \sin(\pi k_1x_1/L_1)\sin(\pi k_2x_2/L_2)\sin(\pi k_3x_3/L_3)$$

for $x=(x_1,x_2,x_3)\in\mathbb{R}^3$. Pick

$$L>2(1+\varepsilon)\sqrt{L_1^2+L_2^2+L_3^2}+(1+\varepsilon)\max(L_1,L_2,L_3)$$

and set $\widehat{\Omega}=(-L,L)^3$. Then for $(u^0,u^1)\in D(A_\Omega^{\frac{s}{2}})\times D(A_\Omega^{\frac{s-1}{2}})$, we have $(v^0,v^1)\in D(A_{\widehat{\Omega}}^{\frac{s}{2}})\times D(A_{\widehat{\Omega}}^{\frac{s-1}{2}})$, so that the solution v to

$$\begin{aligned}\Box v &= 0 \quad \text{in } \widehat{\Omega}\times\mathbb{R},\\ v &= 0 \quad \text{on } \partial\widehat{\Omega}\times\mathbb{R},\\ (v(.,0),v_t(.,0)) &= (v^0,v^1)\end{aligned}$$

satisfies $v\in C(\mathbb{R},D(A_{\widehat{\Omega}}^{\frac{s}{2}}))\cap C^1(\mathbb{R},D(A_{\widehat{\Omega}}^{\frac{s-1}{2}}))$. On the other hand, v is odd in x_1, x_2, and x_3. Decomposing v^0 and v^1 as

(2.28) $$v^0(x_1,x_2,x_3)=\sum_{k\in\mathbb{N}^{*3}} v_k^0\sin(\pi k_1x_1/L)\sin(\pi k_2x_2/L)\sin(\pi k_3x_3/L),$$

(2.29) $$v^1(x_1,x_2,x_3)=\sum_{k\in\mathbb{N}^{*3}} v_k^1\sin(\pi k_1x_1/L)\sin(\pi k_2x_2/L)\sin(\pi k_3x_3/L),$$

we have that

(2.30)
$$\begin{aligned}v(x,t)=\frac{1}{2}\sum_{k\in\mathbb{N}^{*3}}\Big[&(v_k^0-i\frac{Lv_k^1}{\pi|k|})e^{i\frac{\pi}{L}|k|t}\\ &+(v_k^0+i\frac{Lv_k^1}{\pi|k|})e^{-i\frac{\pi}{L}|k|t}\Big]\sin(\frac{\pi k_1x_1}{L})\sin(\frac{\pi k_2x_2}{L})\sin(\frac{\pi k_3x_3}{L}).\end{aligned}$$

Then $(u,g)=(v_{|\Omega\times(0,T)},v_{|\Gamma_0\times(0,T)})$ solves (1.1)-(1.4) for

$$T=2(1+\varepsilon)\sqrt{L_1^2+L_2^2+L_3^2}.$$

Indeed, if v is viewed as a solution of the wave equation on $\mathbb{R}^3$ with odd initial data supported in $\widehat{\Omega}$, then the above choice of T gives that $v(x,T)=v_t(x,T)=$ for all $x\in\Omega$ by (2.13), (2.25), (2.26) and (2.27), while the above choice of L ensures that the support of (v,v_t) does not meet the boundary of $\widehat{\Omega}$ at time T, so that this solution coincides with the solution of the wave equation with homogeneous Dirichlet boundary conditions on $\widehat{\Omega}$ given by (2.30).

To address the control problem from a numerical viewpoint, we let v_N (resp. g_N) denote the partial sum of the series in (2.30) (resp. its restriction to $\Gamma_0\times[0,T]$) restricted to $k_i\le N$ for $i=1,2,3$.

The following theorems give explicit error estimates in terms of the Fourier coefficients of the initial data.

THEOREM 2.1. *Let* $s \geq 0$, $(u^0, u^1) \in D(A_\Omega^{\frac{s}{2}}) \times D(A_\Omega^{\frac{s-1}{2}})$ *and* $N \geq 1$. *Then for any* $r \leq s$

$$(2.31)\quad ||v - v_N||_{C([0,T],H^r(\widehat{\Omega}))} \leq \frac{C}{N^{s-r}}\Big(\sum_{||k||_\infty > N} |k|^{2s}(|v_k^0|^2 + |k|^{-2}|v_k^1|^2)\Big)^{\frac{1}{2}}$$

$$(2.32)\quad \leq \frac{C}{N^{s-r}}\big(||u^0||_{H^s(\Omega)} + ||u^1||_{H^{s-1}(\Omega)}\big)$$

where $C > 0$ *is some universal constant.*

Proof. It follows from (2.30) that

$$\begin{aligned}
||v - v_N||^2_{C([0,T],D(A_{\widehat{\Omega}}^{\frac{r}{2}}))} &\leq C \sum_{||k||_\infty > N} |k|^{2r}\big(|v_k^0|^2 + |k|^{-2}|v_k^1|^2\big) \\
&\leq \frac{C}{N^{2(s-r)}} \sum_{||k||_\infty > N} |k|^{2s}\big(|v_k^0|^2 + |k|^{-2}|v_k^1|^2\big) \\
&\leq \frac{C}{N^{2(s-r)}}\big(||v^0||^2_{D(A_{\widehat{\Omega}}^{\frac{s}{2}})} + ||v^1||^2_{D(A_{\widehat{\Omega}}^{\frac{s-1}{2}})}\big) \\
&\leq \frac{C}{N^{2(s-r)}}\big(||u^0||^2_{D(A_\Omega^{\frac{s}{2}})} + ||u^1||^2_{D(A_\Omega^{\frac{s-1}{2}})}\big).
\end{aligned}$$

Since the norms $||\cdot||_{D(A_\Omega^{\frac{s}{2}})}$ and $||\cdot||_{H^s(\Omega)}$ are equivalent in $D(A_\Omega^{\frac{s}{2}})$, (2.32) follows. $\square$

Remark 2.2. 1. For $r = s$, we infer from (2.31) that

$$v_N \to v \ \text{ in } \ C([0,T], H^s(\widehat{\Omega})).$$

2. By Sobolev embedding, (2.32) yields for $1 \leq r \leq s$

$$||g - g_N||_{C([0,T],H^{r-\frac{1}{2}}(\Gamma_0))} \leq \frac{C}{N^{s-r}}\big(||u^0||_{H^s(\Omega)} + ||u^1||_{H^{s-1}(\Omega)}\big).$$

3. If $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, taking merely $\rho = 1_{(-1,1)}$ and defining v, u and g as above, we can solve (1.1)-(1.4) in the sharp time $T = 2\sqrt{L_1^2 + L_2^2 + L_3^2}$. Our approach allows us to recover the main results in [25] without using any Ingham inequality. Furthermore, the control input can be expressed explicitly in terms of the Fourier coefficients of the initial data. Note that the same approach gives the same sharp result in $\Omega = \prod_{i=1}^N (0, L_i)$ for any odd integer $N \geq 3$, for Huygens' principle is valid in $\mathbb{R}^N$ (see *e.g.* [11, Eq. (31) p. 77]).

2.4.2. $\boldsymbol{N = 2}$

We assume here that

$$\Omega = (0, L_1) \times (0, L_2),$$
$$\Gamma \setminus \Gamma_0 = (\{x_1 = 0\} \cap \Gamma) \cup (\{x_2 = 0\} \cap \Gamma).$$

Let $A_\Omega = -\Delta$ with domain $D(A_\Omega) = H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega)$. Pick any $(u^0, u^1) \in D(A_\Omega^{\frac{s}{2}}) \times D(A_\Omega^{\frac{s-1}{2}})$ $(s \geq 0)$ decomposed in sine Fourier series as

$$u^0(x_1, x_2) = \sum_{k \in \mathbb{N}^{*2}} u_k^0 \sin(\pi k_1 x_1 / L_1) \sin(\pi k_2 x_2 / L_2),$$
$$u^1(x_1, x_2) = \sum_{k \in \mathbb{N}^{*2}} u_k^1 \sin(\pi k_1 x_1 / L_1) \sin(\pi k_2 x_2 / L_2)$$

with $\sum_{k \in \mathbb{N}^{*2}} |k|^{2s} (|u_k^0|^2 + |k|^{-2} |u_k^1|^2) < \infty$. Pick $\varepsilon > 0$ and let $\rho \in C^\infty(\mathbb{R})$ be an even function such that $\rho(z) = 1$ for $|z| \leq 1$ and $\rho(z) = 0$ for $|z| \geq 1 + \varepsilon$. Pick $\delta > 0$ and denote $I_\delta = \int_{\mathbb{R}} \rho(z/\delta) dz$. For $x \in \mathbb{R}^2$, let

$$\psi(x) := \rho(x_1/L_1) \rho(x_2/L_2) \rho(x_3/\delta)$$

and

$$v^0(x_1, x_2, x_3) = \psi(x) \sum_{k \in \mathbb{N}^{*2}} u_k^0 \sin(\pi k_1 x_1 / L_1) \sin(\pi k_2 x_2 / L_2)$$
$$v^1(x_1, x_2, x_3) = \psi(x) \sum_{k \in \mathbb{N}^{*2}} u_k^1 \sin(\pi k_1 x_1 / L_1) \sin(\pi k_2 x_2 / L_2)$$

for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Note that v^0 and v^1 are odd in x_1, x_2, and even in x_3. Pick

$$L > 2(1+\varepsilon)\sqrt{L_1^2 + L_2^2 + \delta^2} + (1+\varepsilon)\max(L_1, L_2, \delta).$$

It is well-known that the family $(\sqrt{\frac{2}{\pi}} \cos((k - \frac{1}{2})x))_{k \in \mathbb{N}^*}$ is an orthonormal basis of $L^2(0, \pi)$. Decomposing v^0 and v^1 as

$$v^0(x_1, x_2, x_3) = \sum_{k \in \mathbb{N}^{*3}} v_k^0 \sin(\pi k_1 x_1 / L) \sin(\pi k_2 x_2 / L) \cos(\pi (k_3 - \frac{1}{2}) x_3 / L), \tag{2.33}$$

$$v^1(x_1, x_2, x_3) = \sum_{k \in \mathbb{N}^{*3}} v_k^1 \sin(\pi k_1 x_1 / L) \sin(\pi k_2 x_2 / L) \cos(\pi (k_3 - \frac{1}{2}) x_3 / L) \tag{2.34}$$

and denoting $\mu_k = \frac{\pi}{L}\sqrt{k_1^2 + k_2^2 + (k_3 - \frac{1}{2})^2}$, we have that the function

$w(x_1, x_2, t)$

$$= \frac{1}{2I_\delta}\int_{-L}^{L}\sum_{k\in\mathbb{N}^{*3}}\Big[(v_k^0 - i\frac{v_k^1}{\mu_k})e^{i\mu_k t}$$

$$+(v_k^0 + i\frac{v_k^1}{\mu_k})e^{-i\mu_k t}\Big]\sin(\frac{\pi k_1 x_1}{L})\sin(\frac{\pi k_2 x_2}{L})\cos(\pi(k_3-\frac{1}{2})x_3/L)\mathrm{d}x_3$$

$$= \sum_{k\in\mathbb{N}^{*3}}\Big[(v_k^0 - i\frac{v_k^1}{\mu_k})e^{i\mu_k t}$$

$$+(v_k^0 + i\frac{v_k^1}{\mu_k})e^{-i\mu_k t}\Big]\sin(\frac{\pi k_1 x_1}{L})\sin(\frac{\pi k_2 x_2}{L})\frac{(-1)^{k_3+1}L}{\pi(k_3-\frac{1}{2})I_\delta}$$

solves

$$\begin{aligned}\Box w &= 0 \quad \text{in } (-L,L)^2\times\mathbb{R},\\ w &= 0 \quad \text{on } \partial(-L,L)^2\times\mathbb{R},\\ (w(.,0),w_t(.,0)) &= (u^0,u^1).\end{aligned}$$

Then $(u,g) = (w_{|\Omega\times(0,T)}, w_{|\Gamma_0\times(0,T)})$ satisfies (1.1)-(1.3) for

$$T = 2(1+\varepsilon)\sqrt{L_1^2+L_2^2+\delta^2}.$$

Indeed, if v denotes the solution of the system

$$\begin{aligned}\Box v &= 0 \quad \text{in } (-L,L)^3\times\mathbb{R},\\ v &= 0 \quad \text{on } \partial(-L,L)^3\times\mathbb{R},\\ (v(.,0),v_t(.,0)) &= (v^0,v^1),\end{aligned}$$

then the support of (v,v_t) does not meet $\partial(-L,L)^3$ for $0\le t\le T$, and hence

$$w_{tt} - w_{x_1x_1} - w_{x_2x_2} = \frac{1}{I_\delta}\int_{-L}^{L}(v_{tt} - v_{x_1x_1} - v_{x_2x_2})dx_3 = \frac{1}{I_\delta}\int_{-L}^{L} v_{x_3x_3}dx_3 = 0.$$

We stress that u is expected to be "small" , but not necessarily zero, for $t=T$. Furthermore, we have that for some universal constant $C>0$ and for $r\le s$

$$\begin{aligned}||w-w_N||_{C([0,T],H^r(\Omega))} &\le \frac{C}{N^{s-r}}\Big(\sum_{||k||_\infty>N}|k|^{2s}(|v_k^0|^2+|k|^{-2}|v_k^1|^2)\Big)^{\frac{1}{2}}\\ &\le \frac{C}{N^{s-r}}\big(||u^0||_{H^s(\Omega)}+||u^1||_{H^{s-1}(\Omega)}\big)\end{aligned}$$

where

$$w_N(x,t) = \sum_{||k||_\infty\le N}\Big[(v_k^0 - i\frac{v_k^1}{\mu_k})e^{i\mu_k t}$$

$$+(v_k^0 + i\frac{v_k^1}{\mu_k})e^{-i\mu_k t}\Big]\sin(\frac{\pi k_1 x_1}{L})\sin(\frac{\pi k_2 x_2}{L})\frac{(-1)^{k_3+1}L}{\pi(k_3-\frac{1}{2})I_\delta}.$$

3. NUMERICAL EXPERIMENTS

3.1. Case $\Omega \subset \mathbb{R}^3$

For the sake of convenience, to avoid the (easy but tedious) scalings in space variables and in Fourier coefficients, we take

$$\Omega = (0, \frac{1}{2})^3 \subset \widehat{\Omega} = (-\pi, \pi)^3.$$

Thus $L_1 = L_2 = L_3 = \frac{1}{2}$ and $L = \pi$. Let

$$u^0(x_1, x_2, x_3) = \exp\left(-200[(x_1 - \frac{1}{4})^2 + (x_2 - \frac{1}{4})^2 + (x_3 - \frac{1}{4})^2]\right)$$

and $u^1(x_1, x_2, x_3) = 0$. We pick $\rho = 1_{(-1,1)}$ and $N = 50$. The $(50)^3$ Fourier coefficients v_k^0 in (2.28) are computed by using the trapezoid rule with $(20)^3$ points. The (free) evolution of the trajectory for $(x_1, x_2) \in (0, \pi)^2$ and $x_3 = 1/8$, $x_3 = 1/4$, and $x_3 = 3/8$ is given in Figures 3, 4 and 5, respectively.

Note that the sharp control time is $T = \sqrt{3} \approx 1.73$. The evolution of $U(t) := \|u(.,t)\|_{L^\infty(\Omega)}$ is displayed in Figure 6. We used a uniform mesh with 9^3 points to estimate $U(t)$ at each time t. We found that $U(1.73) \approx 5.36\ 10^{-5}$, which shows numerically that the solution vanishes at $t = T$ for $(x_1, x_2, x_3) \in (0, \frac{1}{2})^3$, as desired.

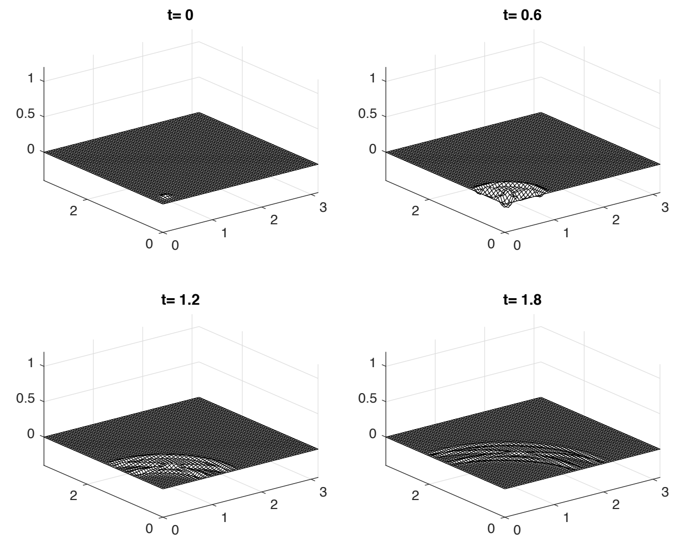

Fig. 3 – Control of the wave equation on $\Omega = (0, \frac{1}{2})^3$. $x_3 = 1/8$.

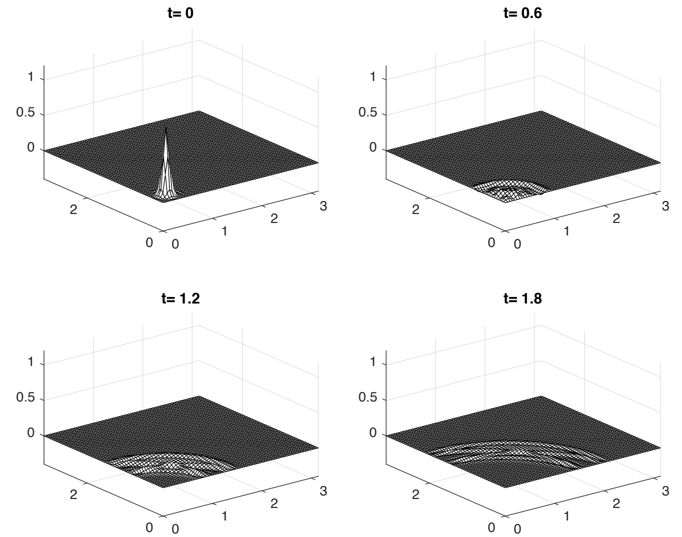

Fig. 4 – Control of the wave equation on $\Omega = (0, \frac{1}{2})^3$. $x_3 = 1/4$.

3.2. **Case** $\Omega \subset \mathbb{R}^2$

We take

$$\Omega = (0, \frac{1}{2})^2 \quad \text{and} \quad \widehat{\Omega} = (-\pi, \pi)^3,$$

so that $L_1 = L_2 = \frac{1}{2}$ and $L = \pi$.

Let $u^0(x_1, x_2) = \exp\left(-200[(x_1 - \frac{1}{4})^2 + (x_2 - \frac{1}{4})^2]\right)$ and $u^1(x_1, x_2) = 0$. We pick $\rho = 1_{(-1,1)}$, $\delta = 0.15$ and $N = 50$. The $(50)^3$ Fourier coefficients v_k^0 in (2.33) are computed by using the trapezoid rule with $(20)^3$ points. Letting $x_3 = 0$ and $(x_1, x_2) \in \Omega$ yields the desired controlled trajectory. The evolution of the trajectory in $(0, \pi)^2$ is given in Figure 7. As expected, we observe the propagation of waves from Ω into $(0, L)^2$. At time $t = 1.5$, waves are localized in $(0, L)^2 \setminus \Omega$. The evolution of $U(t) := \|u(., t)\|_{L^\infty(\Omega)}$ is displayed in Figure 8. We recall that the sharp control time is $T = \sqrt{2} \approx 1.44$. We used a uniform mesh with 9^2 points to numerically compute $U(t)$ at each time t. We obtained $U(1.44) \approx 1.5\ 10^{-3}$, a result which is less good than in the case $\Omega = (0, \frac{1}{2})^3$ with the same number of Fourier coefficients. This is consistent with the fact that only a stabilization result is obtained in 2D by applying the method of descent.

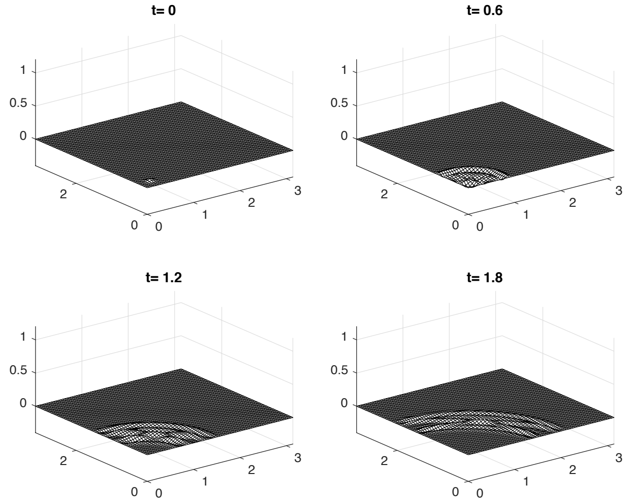

Fig. 5 – Control of the wave equation on $\Omega = (0, \frac{1}{2})^3$. $x_3 = 3/8$.

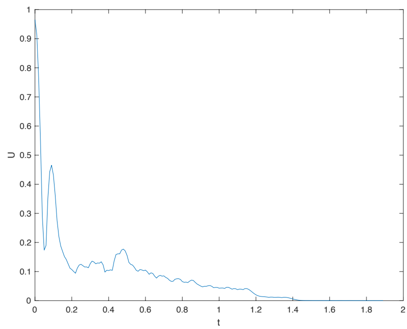

Fig. 6 – Evolution of $U(t) = \|u(.,t)\|_{L^\infty((0,\frac{1}{2})^3)}$.

4. CONCLUDING REMARKS

Using Huygens' principle in dimension 3, the control of the wave equation on a cuboid can be reduced to a classical Cauchy problem in a larger domain. A simple numerical scheme, based on Fourier series, can then be used to solve numerically the control problem without incorporating dissipation or filtering.

Our control method amounts to associate with a pair (Ω, Γ_0) an open set $\Omega_1 \subset \mathbb{R}^3$ with $\Omega \subset \Omega_1$, $\Gamma \setminus \Gamma_0 \subset \partial\Omega_1$, so that Huygens' principle is satisfied in Ω_1, and next to extend the initial data by 0 on $\Omega_1 \setminus \Omega$ and to take the restriction to Ω of the free solution of the wave equation on Ω_1 with Dirichlet boundary conditions.

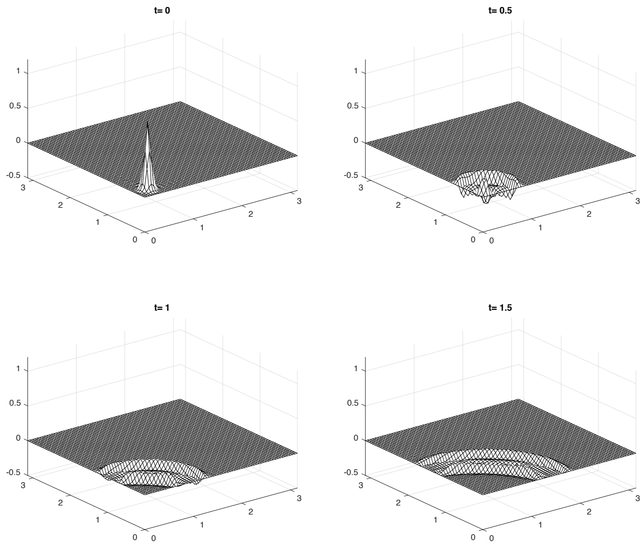

Fig. 7 – Control of the wave equation on $\Omega = (0, \frac{1}{2})^2$.

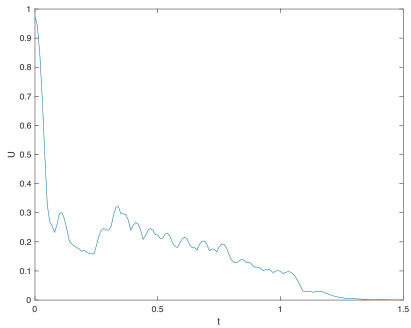

Fig. 8 – Evolution of $U(t) = \|u(., t)\|_{L^\infty((0,\frac{1}{2})^2)}$.

So far, the above method is limited to geometries for which the complement of the control region on the boundary is both flat and connected. It would be interesting to remove those assumptions. This requires first to extend Huygens' principle on unbounded domains limited by (convenient) surfaces. One way would be to extend Kirchhoff formula (2.8) as in (2.13) by replacing spheres by wavefronts.

Combining Huygens' principle in dimension 3 with the descent method, we obtain numerically a stabilization result, rather than an exact controllability result. It would be interesting to modify the above approach in order to obtain an exact controllability result in sharp time.

We expect that our approach can be applied in some situations when the complement of the control region is connected, but not necessarily flat. This will be investigated elsewhere.

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