Dedicated to Marius Tucsnak on the occasion of his 60^{th} *anniversary*

SOLVING FORWARD AND CONTROL PROBLEMS FOR TELEGRAPHER'S EQUATIONS ON METRIC GRAPHS

G. M. ALAM, S. A. AVDONIN, and N. B. AVDONINA

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In this paper, we solve forward and control problems for telegrapher's equation on metric graphs. The forward problem is considered on general graphs, and an efficient algorithm for solving the equations for a constant inductance and capacitance and for a variable resistance and conductance is developed. The control problem is considered on tree graphs, *i.e.* graphs without cycles, with some restrictions on the coefficients. In particular, we consider equations with constant coefficients which do not depend on the edge. We obtained the necessary and sufficient conditions of the exact controllability and indicate the minimal control time.

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1. **INTRODUCTION**

The telegrapher's equations, also known as transmission line equations, are coupled, linear first-order partial differential equations that describe the change of voltage and current on an electrical transmission line with distance and time. It first appeared in a paper by Kirchhoff [17] in 1857, and subsequently, by Heaviside [16] in 1876. The telegraph equation attracted close attention when it was treated by Poincaré [21] in 1893. It is widely used in the study of the propagation of electric signals in a cable transmission line as well as in wave phenomena. The transmission line is thought to be composed of millions of tiny little circuit elements, such as distributed resistance R per unit length, distributed inductance L per unit length, and the distributed capacitance between the conductors of shunt capacitance C per unit length. Meanwhile the leakage conductance of the dielectric material separating the two conductors is denoted by a conductance G per unit length. If the line voltage is denoted by $V(x,t)$ and the current — by $I(x,t)$, where x is the displacement and t is the time, then the relationship between the voltage $V(x,t)$

and the current $I(x,t)$ along the transmission line can be described by the following coupled equations

$$L\partial_t I + \partial_x V + RI = 0, \tag{1}$$

$$C\partial_t V + \partial_x I + GV = 0. \tag{2}$$

In this paper, we consider forward and control problems for the telegraph equation networks or, in other words, telegraph equations on metric graphs. The forward problem is solved on a non-homogeneous network, that is, for a constant inductance and capacitance and for a variable resistance and conductance. On the other hand, the control problem is solved in the case of constant parameters.

Networks of the telegrapher equations have been recently used for modeling electrical circuits, see, *e.g.* [1, 19, 23], and arterial blood flows [13, 14, 24]. Very little is known about the controllability of such networks. Some results of this kind were obtained in [24] for star graphs of three edges. Exact controllability results for tree graphs, *i.e.* graphs without cycles, of the telegraph equations were obtained in [20] without an estimate of the controllability time. On the other hand, controllability of the wave equation on trees is studied pretty well, see, *e.g.* monographs [18, 8, 15], surveys [4, 25] and references therein.

In the present paper, we demonstrate that, for homogeneous networks, the control problem for the telegraph equation can be reduced to a control problem for the wave equation for current. For the current equation, we obtain the Neumann control problem with non-standard, so-called delta-prime matching conditions. We study this problem developing the method recently proposed in [12]. We give constructive algorithms to solve control problems on a tree graphs. We prove that the systems of current and telegraph equations are exactly controllable if and only if the control is supported at all or at all but one of the boundary vertices of the tree. We also obtain a sharp estimate of the controllability time. Some of these results (without the detailed proofs) were presented in [3].

If the coefficients of the equations are constants, then eliminating $V(x,t)$ in the system (1), (2) we obtain the second order equation for I :

$$(CL)I_{tt} - I_{xx} + (CR + LG)I_t + (RG)I = 0. \tag{3}$$

The same equation is valid for V. If $CL \neq 0$, then using a simple change of variables this equation can be transformed to the wave equation with potential:

$$u_{tt} - u_{xx} + qu = 0 \tag{4}$$

with some constant q. Controllability of this equation on graphs (for q dependent on x) was studied in many papers, see e. g. [18, 8, 15, 4, 10, 25] and references therein.

If there is no inductance in the transmission line, *i.e.* $L = 0$, then equation (3) takes the form

$$u_t - \frac{1}{RC}u_{xx} + \frac{G}{C}u = 0. \tag{5}$$

This occurs in the case of transmission line of axons and dendrites of nerve cells. In that case (5) is called the cable equation; it describes the dynamics of trans-membrane potential $u(x,t)$. Control and inverse problems for this equation on tree graphs were studied in [5] and [6]. In the case of no resistance and leakage, equation (3) takes the form

$$u_{tt} - c^2 u_{xx} = 0, \tag{6}$$

which is the equation of wave motion with the phase speed of $\frac{1}{\sqrt{CL}}$. On the other hand, when the inductance is negligible compared with the resistance and there is no leakage, equation (3) takes the form

$$u_t - k u_{xx} = 0, \tag{7}$$

which is the equation of diffusion with diffusivity $k = \frac{1}{RC}$. The control problems for equations (6) and (7) on graphs were considered, *e.g.* in [10], for the case of variable coefficients.

2. PRELIMINARIES

Let $\Omega = \{V, E\}$ be a finite compact and connected metric graph, where $V = \{v_1, v_2, \ldots, v_M\}$ is a set of vertices and $E = \{e_1, e_2, \ldots, e_N\}$ is a set of edges. We recall that a graph is called a *metric graph* if every edge $e_k \in E$, $k = 1, \ldots, N$, is identified with an interval $(0, l_k)$ of the real line with a positive length l_k. We denote the boundary vertices (*i.e.* vertices of degree one) by $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$. We write $k \in J(v)$ if $e_k \in E(v)$, where $E(v)$ is the set of edges incident to v. The graph Ω determines naturally the Hilbert space of square integrable functions $\mathcal{H} := L^2(\Omega) = \oplus_k L^2(e_k)$. When convenient, we will denote the restriction of a function w on Ω to e_k by w_k. Let Γ be a union of two disjoint sets: $\Gamma_1 = \{\gamma_1, ..., \gamma_{m_1}\}$, $\Gamma_0 = \{\gamma_{m_1+1}, ..., \gamma_m\}$, and Γ_0 may be empty. We consider a system described by the following initial boundary value problem (IBVP) on Ω with Kirchhoff's conditions at each internal vertex v_j:

$$L_k \partial_t I_k + \partial_x V_k + R_k I_k = 0,\ (x,t) \in (0, l_k) \times (0, T),\ k = 1, \ldots, N, \tag{8}$$

$$C_k \partial_t V_k + \partial_x I_k + G_k V_k = 0,\ (x,t) \in (0, l_k) \times (0, T),\ k = 1, \ldots, N, \tag{9}$$

$$V_k|_{t=0} = I_k|_{t=0} = 0,\ x \in (0, l_k),\ k = 1, \ldots, N, \tag{10}$$

$$V_i(v_j, t) = V_k(v_j, t),\ i, k \in J(v_j),\ v_j \in V \setminus \Gamma,\ t \in (0, T), \tag{11}$$

$$\sum_{k\in J(v_j)} \varkappa_{kj} I_k(v_j,t) = 0,\ v_j \in V\setminus\Gamma,\ t\in(0,T), \tag{12}$$

$$V_k(\gamma_j,t) = f_j(t),\ k\in J(\gamma_j),\ j=1,\dots,m_1,\ t\in(0,T), \tag{13}$$

$$V_k(\gamma_j,t) = 0,\ k\in J(\gamma_j),\ j=m_1+1,\dots,m,\ t\in(0,T). \tag{14}$$

Here T is arbitrary positive number, $\varkappa_{kj}=1$ if v_j coincides with 0 and $\varkappa_{kj}=-1$ if v_j coincides with l_k in the representation of the edge $e_k=(0,l_k)$, and $f_j\in L^2(0,T)$ for all k and j. The coefficients $R_k(x), G_k(x), L_k(x), C_k(x)$ represent distributed resistance, inductance, conductance and capacitance respectively. The function $I_k(x,t)$ represents current and $V_k(x,t)$ represents voltage on each edge e_k; the vector function $f=\{f_j\}\in L^2(0,T;\mathbb{R}^{m_1}) =: \mathcal{F}^T$ is referred to as boundary control. We assume that $R_k(x), G_k(x)\in C[0,l_k]$, and $L_k(x), C_k(x)\in C^1[0,l_k]$ such that $L_k(x), C_k(x)\geq 0$.

The well-posedness of this system was studied *e.g.* in [20] (see also [9]; it was proved that for any $f\in\mathcal{F}^T$, there exists a unique (generalized) solution of the IBVP (8)–(14) such that $V, I\in C([0,T];\mathcal{H})$. It means that $V(\cdot,t)$ and $I(\cdot,t)$ belong to $\mathcal{H}$ for any $t\in[0,T]$ and continuously depend on t in $\mathcal{H}$ norm. In the next section we propose a new proof of this result together with a constructive way solving the problem. We consider also the solution of the IBVP (8)–(14) for more smooth controls, $f_k\in H^1(0,T)$, $f_k(0)=0$, $\forall k$.

3. FORWARD PROBLEM ON GRAPHS

We start with the discussion of the solution of the system (8)–(14) on a finite interval and a star-shaped network.

3.1. Telegrapher's Equation on a Finite Interval $[0,l]$

Consider the system (8)–(14) where Ω is a single interval $[0,l]$ and the control $f(t)$ is applied at $x=0$. Introducing the transformations $\xi(x)=\int_0^x\sqrt{C(s)L(s)}\mathrm{d}s$, $I(x,t)=\sqrt{C(x)}u(\xi(x),t)$, $V(x,t)=\sqrt{L(x)}y(\xi(x),t)$ and putting $U(\xi,t)=\begin{pmatrix}u(\xi,t)\\ y(\xi,t)\end{pmatrix}$ we obtain the IBVP:

$$\partial_t U(\xi,t)+A\partial_\xi U(\xi,t)+QU(\xi,t) = 0,\ (\xi,t)\in(0,\ell)\times(0,T), \tag{15}$$

$$u|_{t=0}=y|_{t=0} = 0,\ \xi\in(0,\ell), \tag{16}$$

$$y(0,t) = \frac{1}{\sqrt{L(0)}}f(t),\ t\in(0,T), \tag{17}$$

$$y(\ell,t) = 0,\ t\in(0,T). \tag{18}$$

Here, $\ell = \xi(l)$, $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Q(\xi) = \begin{pmatrix} q_{11}(\xi) & q_{12}(\xi) \\ q_{21}(\xi) & q_{22}(\xi) \end{pmatrix}$ with $q_{11}(\xi) = \frac{R(x(\xi))}{L(x(\xi))}$, $q_{12}(\xi) = \frac{C'(x(\xi))}{2C(x(\xi))\sqrt{L(x(\xi))C(x(\xi))}}$, $q_{21}(\xi) = \frac{L'(x(\xi))}{2L(x(\xi))\sqrt{L(x(\xi))C(x(\xi))}}$, and $q_{22}(\xi) = \frac{G(x(\xi))}{C(x(\xi))}$. For simplicity of presentation, we denote the right hand side of (17) by $f(t)$ again. We use the notation $U^{f^-}(\xi,t) = \begin{pmatrix} u^{f^-}(\xi,t) \\ y^{f^-}(\xi,t) \end{pmatrix}$ to represent the solution if the control $f(t)$ is applied at the left end $\xi = 0$.

By direct substitution one can prove the following proposition.

PROPOSITION 3.1. *If $0 \le t < T \le \ell$, $Q \in C^1(0,\ell)$, and $f \in L^2(0,T)$ then the system (15)–(18) has a unique generalized solution $U \in C([0,T]; L^2(0,\ell;\mathbb{R}^2))$ given by* $\begin{pmatrix} u^{f^-}(\xi,t) \\ y^{f^-}(\xi,t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ *for $t \le \xi$ and*

$$\begin{pmatrix} u^{f^-}(\xi,t) \\ y^{f^-}(\xi,t) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} f(t-\xi) + \int_\xi^t \begin{pmatrix} w_1(\xi,s) \\ w_2(\xi,s) \end{pmatrix} f(t-s)\mathrm{d}s, \ \xi < t, \tag{19}$$

where the vector kernel, $W(\xi,s) = \begin{pmatrix} w_1(\xi,s) \\ w_2(\xi,s) \end{pmatrix}$, is the solution of the Goursat problem

$$\partial_s W(\xi,s) + A\partial_\xi W(\xi,s) + QW(\xi,s) = 0, \ 0 < \xi < s \le T, \tag{20}$$

$$w_1|_{s<\xi} = w_2|_{s<\xi} = 0, \ 0 < \xi < T, \tag{21}$$

$$w_2(0,s) = 0, \ 0 < s < T, \tag{22}$$

with

$$w_2(\xi,\xi) - w_1(\xi,\xi) = \frac{1}{2}\left(q_{11}(\xi) + q_{12}(\xi) - q_{21}(\xi) - q_{22}(\xi)\right). \tag{23}$$

The system (20) – (23) can be solved by the standard iteration method after diagonalizing the matrix A and then using the transformation $\xi = s - x$ and $\eta = s + x$ for smooth Q (see, [2] for details). For the solution of scalar Goursat problem, see, *e.g.*, [22] for smooth q and [11] for $q \in L_1(0,\ell)$.

To find the solution of (15) – (18) for $T > \ell$ we use the idea proposed in [12] for the wave equation. This method does not work on Telegrapher's equation with variable L and C. As a result, L and C are considered constants for solutions on finite intervals if $T > \ell$ and on metric graphs. We extend $Q(\xi)$ to the semi-axis $\xi > 0$ by the rule $Q(2n\ell \pm \xi) = Q(\xi)$ for all $n \in \mathbb{N}$ and solve the problem (20) – (23) with extended $Q(\xi)$. Then the solution of the system (15)–(18) is given by the following proposition which can be proved by substitution.

Proposition 3.2. *If $Q \in C^1(0,\ell)$, $f \in L^2(0,T)$, and $t \geq 0$, then the system (15)–(18) has a unique generalized solution $U \in C([0,T];L^2(0,\ell;\mathbb{R}^2))$ given by the formula*

$$\begin{aligned}\begin{pmatrix} u^{f^-}(\xi,t) \\ y^{f^-}(\xi,t)\end{pmatrix} &= \sum_{n=0}^{\lfloor \frac{t-\xi}{2\ell}\rfloor} \Bigg[\begin{pmatrix}1\\1\end{pmatrix} f(t-2n\ell-\xi) \\ &\quad + \int_{2n\ell+\xi}^{t} \begin{pmatrix} w_1(2n\ell+\xi,s) \\ w_2(2n\ell+\xi,s)\end{pmatrix} f(t-s)\mathrm{d}s\Bigg] \\ &\quad + \sum_{n=1}^{\lfloor \frac{t+\xi}{2\ell}\rfloor} \Bigg[\begin{pmatrix}1\\-1\end{pmatrix} f(t-2n\ell+\xi) \\ &\quad + \int_{2n\ell-\xi}^{t} \begin{pmatrix} w_1(2n\ell-\xi,s) \\ -w_2(2n\ell-\xi,s)\end{pmatrix} f(t-s)\mathrm{d}s\Bigg],\end{aligned} \tag{24}$$

where $\lfloor\cdot\rfloor$ is the floor function.

Here and everywhere below we assume that control functions are extended by zero to the negative semi-axis, *i.e.* $f(t) = 0$ for $t < 0$.

Now we consider the case when the control function $f(t)$ is applied at the right endpoint $\xi = \ell$:

$$\partial_t U(\xi,t) + A\partial_\xi U(\xi,t) + QU(\xi,t) = 0,\ (\xi,t) \in (0,\ell)\times(0,T), \tag{25}$$

$$u|_{t=0} = y|_{t=0} = 0,\ \xi \in (0,\ell), \tag{26}$$

$$y(0,t) = 0,\ t \in (0,T), \tag{27}$$

$$y(\ell,t) = f(t),\ t \in (0,T). \tag{28}$$

In order to solve the system (25)–(28) we construct $\tilde{Q}(\xi) = Q(\ell-\xi)$ and denote by $K(\xi,s) = \begin{pmatrix} k_1(\xi,s) \\ k_2(\xi,s)\end{pmatrix}$ the solution to the Goursat problem

$$\partial_s K(\xi,s) + A\partial_\xi K(\xi,s) + \tilde{Q}K(\xi,s) = 0,\ 0 < \xi < s \leq T, \tag{29}$$

$$k_1|_{s<\xi} = k_2|_{s<\xi} = 0,\ 0 < \xi < T, \tag{30}$$

$$k_2(0,s) = 0,\ 0 < s < T, \tag{31}$$

with

$$k_2(\xi,\xi) - k_1(\xi,\xi) = \frac{1}{2}\left(\tilde{q}_{11}(\xi) + \tilde{q}_{12}(\xi) - \tilde{q}_{21}(\xi) - \tilde{q}_{22}(\xi)\right). \tag{32}$$

We extend $\tilde{Q}$ to the semi-axis by letting $\tilde{Q}(2n\ell \pm \xi) = \tilde{Q}(\xi)$ for all $n \in \mathbb{N}$. Let the vector kernel K satisfies the Goursat system (29)–(31) with extended $\tilde{Q}(\xi)$. We use the notation $U^{f^+}(\xi,t) = \begin{pmatrix} u^{f^+}(\xi,t) \\ y^{f^+}(\xi,t)\end{pmatrix}$ to represent the solution if the control is applied at the right end $\xi = \ell$. Then the solution of the system (25)–(28) is given by the following proposition.

PROPOSITION 3.3. *If $Q \in C^1(0,\ell)$, $f \in L^2(0,T)$, and $t \geq 0$ then the system (25)–(28) has a unique generalized solution $U \in C([0,T]; L^2(0,\ell;\mathbb{R}^2))$ given by* $\begin{pmatrix} u^{f^+}(\xi,t) \\ y^{f^+}(\xi,t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ *for $t \leq \ell - \xi$ and*

$$\begin{aligned}
\begin{pmatrix} u^{f^+}(\xi,t) \\ y^{f^+}(\xi,t) \end{pmatrix} &= \sum_{n=0}^{\lfloor \frac{t-\ell+\xi}{2\ell} \rfloor} \left[\begin{pmatrix} -1 \\ 1 \end{pmatrix} f(t-(2n+1)\ell+\xi) \right. \\
&+ \left. \int_{(2n+1)\ell-\xi}^{t} \begin{pmatrix} -k_1((2n+1)\ell-\xi,s) \\ k_2((2n+1)\ell-\xi,s) \end{pmatrix} f(t-s)\mathrm{d}s \right] \\
&- \sum_{n=0}^{\lfloor \frac{t-\ell-\xi}{2\ell} \rfloor} \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} f(t-(2n+1)\ell-\xi) \right. \\
&- \left. \int_{(2n+1)\ell+\xi}^{t} \begin{pmatrix} k_1((2n+1)\ell+\xi,s) \\ k_2((2n+1)\ell+\xi,s) \end{pmatrix} f(t-s)\mathrm{d}s \right], \ell-\xi < t \leq T.
\end{aligned} \tag{33}$$

In the next section we will discuss the forward problem on a star-shaped network.

3.2. **Telegrapher's Equation on a Star-shaped Network**

Consider a star-shaped network $\Omega = \{V, E\}$ with $V = \{v, \gamma_1, \gamma_2, \ldots, \gamma_m\}$, $\Gamma = \{\gamma_1, ..., \gamma_m\}$, $\Gamma_1 = \{\gamma_1\}$ and $\Gamma_0 = \{\gamma_2, ..., \gamma_m\}$. Each edge e_k, $k = 1, \ldots, m$, is identified with the interval $(0, l_k)$, the vertex v is identified with $x = 0$ and γ_k is identified with $x = l_k$. The transformed system on Ω takes the following form:

$$\partial_t U_k(\xi,t) + A\partial_\xi U_k(\xi,t) + QU_k(\xi,t) = 0, (\xi,t) \in (0,\ell_k) \times (0,T), \quad k = 1,\ldots,m, \tag{34}$$

$$u_k|_{t=0} = y_k|_{t=0} = 0,\ \xi \in (0,\ell_k),\ k = 1,\ldots,m, \tag{35}$$

$$\alpha_i y_i(0,t) = \alpha_k y_k(0,t),\ i,k \in J(v), t \in (0,T), \tag{36}$$

$$\sum_{k \in J(v)} \beta_k u_k(0,t) = 0,\ t \in (0,T), \tag{37}$$

$$y_1(\ell_1,t) = f(t), t \in (0,T), \tag{38}$$

$$y_k(\ell_k,t) = 0,\ k = 2,3,\ldots,m,\ t \in (0,T). \tag{39}$$

Here, $\ell_k = \xi(l_k), \alpha_i = \sqrt{L_i}$, and $\beta_k = \sqrt{C_k}$. We put $h_k(t) := y_k(0,t)$, $k = 1,2,3,\dots,m$. Using the notations of the previous section we present the solution of the system (34)–(39) in the form:

$$U_1(\xi,t) = U_1^{h_1^-}(\xi,t) + U_1^{f^+}(\xi,t), \tag{40}$$

$$U_k(\xi,t) = U_k^{h_k^-}(\xi,t), \text{ for } k = 2,3,\dots,m. \tag{41}$$

Using (24) and (33) for computing $U_1(\xi,t)$, we obtain:

$$\begin{aligned} u_1(0,t) &= h_1(t) + \int_0^t w_{11}(0,s)h_1(t-s)\mathrm{d}s \\ &+ 2\sum_{n=1}^{\left\lfloor \frac{t}{2\ell_1}\right\rfloor}\left[h_1(t-2n\ell_1) + \int_{2n\ell_1}^t w_{11}(2n\ell_1,s)h_1(t-s)\mathrm{d}s\right] \\ &- 2\sum_{n=0}^{\left\lfloor \frac{t-\ell_1}{2\ell_1}\right\rfloor}\left[f(t-(2n+1)\ell_1) + \int_{(2n+1)\ell_1}^t k_{11}((2n+1)\ell_1,s)f(t-s)\mathrm{d}s\right]. \end{aligned} \tag{42}$$

Here w_{11} is the first component of the solution of the system (20)–(22) with extended Q_1 and k_{11} is the first component of the solution of the system (29)–(31) with extended $\tilde{Q}_1$. On $e_k, k = 2,3,\dots,m$, at $v(\xi = 0)$ we have

$$\begin{aligned} u_k(0,t) &= h_k(t) + \int_0^t w_{1k}(0,s)h_k(t-s)\mathrm{d}s \\ &+ 2\sum_{n=1}^{\left\lfloor \frac{t}{2\ell_k}\right\rfloor}\left[h_k(t-2n\ell_k) + \int_{2n\ell_k}^t w_{1k}(2n\ell_k,s)h_k(t-s)\mathrm{d}s\right]. \end{aligned}$$

Here w_{1k} is the first component of the solution of the system (20)–(22) with extended Q_k. The vertex matching conditions (36) and (37) give us

$$h_k(t) = \frac{\alpha_1}{\alpha_k}h_1(t)(k = 2,3,\dots,m) \tag{43}$$

and

$$ah_1(t) + \int_0^t G_1(0,s)h_1(t-s)\mathrm{d}s = F(t) \tag{44}$$

where, $a = \sum_{k=1}^m \frac{\beta_k}{\alpha_k}$, $G_1(0,s) = \sum_{k=1}^m \frac{\beta_k}{\alpha_k}w_{1k}(0,s)$ and

$$F(t) = -2a\sum_{n=1}^{\left\lfloor \frac{t}{2\ell_k}\right\rfloor}\left[h_1(t-2n\ell_k) + \int_{2n\ell_k}^t w_{1k}(2n\ell_k,s)h_1(t-s)\mathrm{d}s\right]$$

$$+2\frac{\beta_1}{\alpha_1}\sum_{n=0}^{\left\lfloor\frac{t-\ell_1}{2\ell_1}\right\rfloor}\left[f(t-(2n+1)\ell_1)+\int_{(2n+1)\ell_1}^{t}k_{11}((2n+1)\ell_1,s)f(t-s)\mathrm{d}s\right].$$

If in lieu of one boundary control $f(t)$ at γ_1, m_1 boundary controls $f_k(t)$ are applied at the vertices γ_k, $k=1,2,3,\dots,m_1$, then the RHS of the equation (44) takes the form

$$\begin{aligned}F(t)=&-2a\sum_{n=1}^{\left\lfloor\frac{t}{2\ell_k}\right\rfloor}\left[h_1(t-2n\ell_k)+\int_{2n\ell_k}^{t}w_{1k}(2n\ell_k,s)h_1(t-s)\mathrm{d}s\right]\\&+\frac{2}{\alpha_1}\sum_{k=1}^{m_1}\beta_k\sum_{n=0}^{\left\lfloor\frac{t-\ell_k}{2\ell_k}\right\rfloor}\Bigg[f_k(t-(2n+1)\ell_k)\\&\qquad+\int_{(2n+1)\ell_k}^{t}k_{1k}((2n+1)\ell_k,s)f_k(t-s)\mathrm{d}s\Bigg].\end{aligned}\tag{45}$$

The coefficients L_k and C_k are strictly positive for all k, so $\alpha_k=\sqrt{L_k}>0$, $\beta_k=\sqrt{C_k}>0\,\forall k$. As a result the coefficient a in equation (44) is positive. Equation (44) is a delay integral equation for h_1, its RHS depends on h_1 of the delayed argument. We will demonstrate now that this equation can be solved by the method of steps.

Proposition 3.4. *Suppose that in the IBVP* (34)–(39), $f_k(t)\in L^2_{loc}(0,\infty)$, $k=1,2,\dots,m_1$. *Then h_1 can be computed from* (44), *and* $h_1\in L^2_{loc}(0,\infty)$.

Proof. Let $\Delta:=\underset{k=1,2,\dots,m}{min}\ \ell_k$. In equation (44), the kernel $G_1(0,s)$ is known and F depends h_1 with arguments delayed by at least 2Δ and on f_k, $k=1,2,\dots,m_1$. Since $f_k(s)$ are known for all s, if $h_1(s)$ is known for $0\le s\le t-2\Delta$, then for h_1 on the interval $[t-2\Delta,t]$ we obtain the Volterra integral equation of the second kind. Taking into account that $h_1(t)=0$ for $t\le\Delta$, one can solve equation (44) in steps with a step size of 2Δ. $\square$

The other controls h_k, $k=2,3,\dots,m_1$, can be obtained from the equation (43). Finally the solution of the system (34)–(39) on the star graph is given by (40) and (41).

We are now ready to solve the forward problem on a general metric graph.

3.3. Telegrapher's Equation on a Metric Graph

The transformed telegrapher's equations with Kirchhoff's conditions at each internal vertex v_j take the form:

$$\partial_t U_k(\xi,t) + A\partial_\xi U_k(\xi,t) + Q_k U_k(\xi,t) = 0,\ (\xi,t)\in(0,\ell_k)\times(0,T),\quad k=1,\dots,N, \tag{46}$$

$$u_k|_{t=0} = y_k|_{t=0} = 0,\ \xi\in(0,\ell_k),\ k=1,\dots,N, \tag{47}$$

$$\alpha_{ij} y_i(v_j,t) = \alpha_{kj} y_k(v_j,t),\ i,k\in J(v_j), v_j\in V\setminus\Gamma, t\in(0,T), \tag{48}$$

$$\sum_{k\in J(v_j)} \varkappa_{kj}\beta_{kj} u_k(v_j,t) = 0,\ v_j\in V\setminus\Gamma,\ t\in(0,T), \tag{49}$$

$$y_1(\gamma_1,t) = f(t),\quad t\in(0,T), \tag{50}$$

$$y_k(\gamma_k,t) = 0,\ k=2,3,\dots,m,\ t\in(0,T) \tag{51}$$

where $\ell_k = \xi(l_k)$, $\alpha_{kj} = \sqrt{L_k(v_j)}$, and $\beta_{kj} = \sqrt{C_k(v_j)}$. The new graph is denoted by $\tilde\Omega$. Let U_k be the solution of the system (46) – (51) on the edge e_k of $\tilde\Omega$. If the values of $y_k(v_j)$ are known for all k and j, one can use (24) and (33) and the principle of superposition to find the solution U on each edge e_k of $\tilde\Omega$. Since these values are known for boundary vertices, it remains to find them on $V\setminus\Gamma$. Consider an edge e_k with vertices v_i and v_j oriented in a way such that the vertex v_i is identified as $\xi=0$ and the vertex v_j is identified as $\xi=\ell_k$. Define operators $\mathcal{S}_k^\pm : L^2(0,T)\mapsto C([0,T];L^2(0,\ell_k;\mathbb{R}^2))$ by

$$(\mathcal{S}_k^- f)(\xi,t) = U_k^{f^-}(\xi,t) \text{ and } (\mathcal{S}_k^+ f)(\xi,t) = U_k^{f^+}(\xi,t).$$

Here $(\mathcal{S}_k^- f)(\xi,t)$ represents the solution on the edge e_k applying the control $f(t)$ at v_i and similarly $(\mathcal{S}_k^+ f)(\xi,t)$ represents the solution on the edge e_k applying the control $f(t)$ at v_j. The operators $\mathcal{O}_k^\mp : C([0,T];L^2(0,\ell_k;\mathbb{R}^2))\mapsto L^2(0,T)$ are defined by

$$\mathcal{O}_k^-(U_k) = u_k(v_i,\cdot) \text{ and } \mathcal{O}_k^+(U_k) = u_k(v_j,\cdot).$$

Thus we have four combinations of $\mathcal{O}_k^\mp$ and $\mathcal{S}_k^\mp$ on an edge e_k:

$$\begin{aligned}(\mathcal{O}_k^-\mathcal{S}_k^-)f &= u_k^{f^-}(v_i,t)\\ &= f(t) + \int_0^t w_{1k}(0,s)f(t-s)\mathrm{d}s\\ &+ 2\sum_{n=1}^{\left\lfloor \frac{t}{2\ell_k}\right\rfloor}\left[f(t-2n\ell_k) + \int_{2n\ell_k}^t w_{1k}(2n\ell_k,s)f(t-s)\mathrm{d}s\right],\end{aligned}$$

$$
\begin{aligned}
(\mathcal{O}_k^+\mathcal{S}_k^-)f &= u_k^{f^-}(v_j,t) \\
&= -2\sum_{n=0}^{\left\lfloor \frac{t-\ell_k}{2\ell_k}\right\rfloor}\Bigg[f(t-(2n+1)\ell_k) \\
&\qquad + \int_{(2n+1)\ell_k}^{t} w_{1k}((2n+1)\ell_k,s)f(t-s)\mathrm{d}s\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{O}_k^-\mathcal{S}_k^+)f &= u_k^{f^+}(v_i,t) \\
&= 2\sum_{n=0}^{\left\lfloor \frac{t-\ell_k}{2\ell_k}\right\rfloor}\Bigg[f(t-(2n+1)\ell_k) \\
&\qquad + \int_{(2n+1)\ell_k}^{t} k_{1k}((2n+1)\ell_k,s)f(t-s)\mathrm{d}s\Bigg],
\end{aligned}
$$

and,

$$
\begin{aligned}
(\mathcal{O}_k^+\mathcal{S}_k^+)f &= u_k^{f^+}(v_j,t) \\
&= -f(t)-\int_0^t k_{1k}(0,s)f(t-s)ds \\
&\quad -2\sum_{n=1}^{\left\lfloor \frac{t}{2\ell_k}\right\rfloor}\left[f(t-2n\ell_k)+\int_{2n\ell_k}^{t} k_{1k}(2n\ell_k,s)f(t-s)\mathrm{d}s\right].
\end{aligned}
$$

We introduce now vector $h(t)$ with components $h_i(t)$, $i=1,\ldots,M$:

$$
h_i(t) := y(v_i,t),\ v_i\in\Gamma;\ \ h_i(t) := y_{k_i}(v_i,t),\ k_i=\min\{k\in J(v_i)\},\ v_i\in V\setminus\Gamma,
$$

and put $\gamma_{ki} := \alpha_{k_i}/\alpha_{ki}$, $k\in J(v_i)$. Now we define an $N\times M$ matrix operator $\mathcal{U}$ such that it has one row for each edge and one column for each vertex. The entries of $\mathcal{U}$ are defined by analogy to the entries of the incidence matrix of Ω : if there is an edge e_k from v_i to v_j, then $\mathcal{U}_{ki}=\gamma_{ki}\mathcal{S}_k^-$ and $\mathcal{U}_{kj}=\gamma_{kj}\mathcal{S}_k^+$. All other entries of $\mathcal{U}$ are zero. According to the matching conditions (48), the k-th entry of the vector $\mathcal{U}h$ is U_k.

Next we define an $M\times N$ matrix operator $\mathcal{P}$ such that it has one row for each vertex and one column for each edge. Its entries are defined by analogy to the entries of the transpose to the incidence matrix of Ω : if there is an edge e_k from vertex v_i to v_j, then $\mathcal{P}_{ik}=\varkappa_{ki}\beta_{ki}\mathcal{O}_k^-$ and $\mathcal{P}_{jk}=\varkappa_{kj}\beta_{kj}\mathcal{O}_k^+$. All other entries of $\mathcal{P}$ are zero. Now $\mathcal{P}\mathcal{U}h$ is a column vector with M entries. The i-th entry represents the LHS of equation (49). The $M\times M$ diagonal matrix D is defined to choose the interior vertices. That is, $D_{ij} = 1$ if $i = j$ and $v_i \in V \setminus \Gamma$ and $D_{ij} = 0$ otherwise. Then the matching condition (49) can be represented by

$$DP\mathcal{U}h = 0. \tag{52}$$

Equation (52) is a system of $|V \setminus \Gamma|$ Volterra integral equations of 2nd kind

$$Ah(t) + \int_o^t G(0,s)h(t-s)\mathrm{d}s = F(t) \tag{53}$$

where $A = \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_{|V_c|} \end{bmatrix}$, $h = \begin{bmatrix} h_1 \\ \vdots \\ h_{|V_c|} \end{bmatrix}$, $G = \begin{bmatrix} G_1 & & \\ & \ddots & \\ & & G_{|V_c|} \end{bmatrix}$, $F = \begin{bmatrix} F_1 \\ \vdots \\ F_{|V \setminus \Gamma|} \end{bmatrix}$

with

$$a_j = \sum_{k \in J(v_j)} \gamma_{kj}\beta_{kj}, \quad G_j(0,s) = \sum_{k \in J(v_j)} \gamma_{kj}\beta_{kj}\eta_k(0,s),$$

and

$$F_j(t) = -2 \sum_{k \in J(v_j)} \gamma_{kj}\beta_{kj} \sum_{n=1}^{\left\lfloor \frac{t}{2\ell_k} \right\rfloor} \left[h_j(t - 2n\ell_k) + \int_{2n\ell_k}^t \eta_k(2n\ell_k, s)h_j(t-s)\mathrm{d}s \right]$$

$$+2 \sum_{k \in J(v_j) \text{ and } k \in J(v_r)} \beta_{kj} \sum_{n=0}^{\left\lfloor \frac{t-\ell_k}{2\ell_k} \right\rfloor} \left[h_r(t - (2n+1)\ell_k) \right.$$

$$\left. + \int_{(2n+1)\ell_k}^t \eta_k((2n+1)\ell_k, s)h_r(t-s)\mathrm{d}s \right].$$

Here we have put $h_r(t) := y_k(v_r, t)$ at the vertex v_r on the edge e_k such that $k \in J(v_j)$ (v_r is a neighbouring vertex to v_j on edge e_k) and η_k are given by

$$\eta_k = \begin{cases} w_{1k}, \text{if } \varkappa_{kj} = 1 \\ k_{1k}, \text{if } \varkappa_{kj} = -1 \end{cases}$$

for any k. Equation (53) is a system of Volterra integral equation where each equation is of the form

$$a_j h_j(t) + \int_0^t G_j(0,s)h_j(t-s)\mathrm{d}s = F_j(t). \tag{54}$$

In the system (53), the kernel $G(0,s)$ is known and the RHS, F depends on h with arguments delayed by at least Δ, where Δ is previously defined. If $h(s)$ is known for $0 \le s \le t - \Delta$ then $F(s)$ is known for $0 \le s \le t$. That is, if h on $[0, t-\Delta]$ is known then one can find h on $[t - \Delta, t]$ in steps with a step size of Δ. Solving the system (52) we obtain h_j at the internal vertex $v_j (j = 1, 2, \ldots, |V \setminus \Gamma|)$ along an edge e_{k_j} incident to v_j, where $k_j = \min\{k \in J(v_j)\}$, $v_j \in V \setminus \Gamma$. The value of h_k on the edge $e_k, k \in J(v_j), k \neq k_j$, that is, $u_k(v_j, t), k \in J(v_j), k \neq k_j$ are obtained from the matching condition (48). Since the boundary controls are known one can find the solution on each edge e_k with vertices v_i and v_j. The solution of the system (46)–(51) on each edge e_k of the metric graph $\tilde{\Omega}$ is given by:

$$U_k(\xi, t) = U_k^{h_i^-}(\xi, t) + U_k^{h_j^+}(\xi, t).$$

It should be noted that if v_i or v_j is a boundary vertex then h_i or h_j coincides with the given boundary controls. Finally the solution of the system (8)–(14) is given by

$$I_k(x, t) = \sqrt{C_k}\, u_k(\xi(x), t) \quad \text{and} \quad V_k(x, t) = \sqrt{L_k}\, y_k(\xi(x), t).$$

We have proved that for L^2 boundary control, $f \in \mathcal{F}^T$, the IBVP (8)–(14) has a unique solution $V, I \in C([0, T]; \mathcal{H})$. In the next section we study control problems for this system, and it will be convenient to use more smooth controls. More specifically, we consider the control space $\mathcal{F}_1^T := \{f = \{f_k\}, f_k \in H^1(0, T), f_k(0) = 0, k = 1, \ldots, m_1\}$. In this case, we can prove existence and uniqueness of a more regular solution.

To describe these solutions we introduce the following Sobolev-type spaces of functions on Ω. Let $\mathcal{H}^1$ is the space of functions ϕ on Ω such that $\phi_k \in H^1(e_k)\, \forall k$. Let $\mathcal{H}_v^1$ be a space of functions from $\mathcal{H}^1$ continuous on Ω and $\mathcal{H}_c^1$ — a space of functions from $\mathcal{H}^1$ such that

$$\sum_{k \in J(v_j)} \varkappa_{kj} \phi_k(v_j) = 0, \ \forall\, v_j \in V \setminus \Gamma.$$

If $f \in \mathcal{F}_1^T$, then following the scheme described in this section one can construct the solution of the IBVP (8)–(14) such that $V \in C([0, T]; \mathcal{H}_v^1)$ and $I \in C([0, T]; \mathcal{H}_c^1)$. The details of the proof are left to the reader.

4. CONTROL PROBLEMS FOR TELEGRAPHER'S EQUATIONS

In this section we prove the exact controllability of telegrapher's equation on metric tree graphs. For that purpose we establish its relations with controllability of the wave equations for the current. We begin with several definitions.

Definition 4.1. The system (8)–(14) is called (V, I)-controllable in time T if, given arbitrary functions $\varphi \in \mathcal{H}_v^1$, $\psi \in \mathcal{H}_c^1$ one can find $f \in \mathcal{F}_1^T$ such that $V(\cdot, T) = \varphi$ and $I(\cdot, T) = \psi$.

Definition 4.2. The system (8)–(14) is called (I, I_t)-controllable in time T if, given arbitrary functions $\phi \in \mathcal{H}_c^1$, $\psi \in \mathcal{H}$ one can find $f \in \mathcal{F}_1^T$, such that $I(\cdot, T) = \phi$ and $I_t(\cdot, T) = \psi$.

In this section we assume that all parameters, C_k, L_k, R_k, G_k, are independent of x. The following observation is important for constructions of this section.

PROPOSITION 1. *If the system (8)–(14) is (I, I_t)-controllable in time T, it is (V, I)-controllable in the same time interval.*

Proof. Let us choose arbitrary functions $\phi \in \mathcal{H}_v^1, \psi \in \mathcal{H}_c^1$ and prove that there exists a function $f \in \mathcal{F}_1^T$ such that

$$V(x,T) = \phi(x), \quad I(x,T) = \psi(x). \tag{55}$$

Keeping in mind equation (8), we define function $\zeta \in \mathcal{H}$ by

$$\zeta(x) = -\frac{1}{L}\left[\phi'(x) + R\psi(x)\right]. \tag{56}$$

Since the system (8)–(14) is exactly (I, I_t)-controllable in time T, there exists $f \in \mathcal{F}_1^T$ such that

$$I(x,T) = \psi(x), \quad I_t(x,T) = \zeta(x). \tag{57}$$

Then, according to (56) and (8), $\partial_x V(x,T) = \phi'(x)$. Taking into account boundary conditions (13) and (14), we obtain (55). $\square$

Proposition 1 allows us to reduce the question about controllability of the telegraph equations on graphs to the corresponding question for the wave equation of current. If a graph Ω has cycles, the wave or telegraph equation is not exactly boundary controllable in any time. For the wave equation it was proved in [8, Ch. 7]; the same argument works also for the telegraph equation. We will consider the exact controllability question for trees, *i.e.* graphs without cycles. To reduce a control problem for the system (8)–(14) to the problem for the wave equation of current we need to impose some constraints on the coefficients of our equations. We have already assumed that all C_k, L_k, R_k, and G_k are constant, *i.e.* independent of x. It allows to eliminate V_k from the telegraph equations and rewrite them in the form (3) or (4). To eliminate V_k form the matching conditions (11) we assume additionally that the coefficients are independent of k. We denote them by C, L, R, and G. Then the system (8)–(14) can be transformed to the IBVP for $u_k(x,t) := I_k(x/\sqrt{CL}, t)\exp\{t(R/2L + G/2C)\}$:

$$\begin{aligned}
&(58) \quad \partial_t^2 u_k - \partial_x^2 u_k + q u_k &= \;& 0,\ (x,t) \in (0,\ell_k) \times (0,T),\ k = 1, \dots, N,\\
&(59) \quad u_k|_{t=0} = \partial_t u_k|_{t=0} &= \;& 0,\ x \in (0,\ell_k),\ k = 1, \dots, N,
\end{aligned}$$

$$\partial u_i(v_j,t) = \partial u_k(v_j,t),\ i,k \in J(v_j),\ v_j \in V \setminus \Gamma,\ t \in (0,T), \tag{60}$$

$$\sum_{k \in J(v_j)} \varkappa_{kj} u_k(v_j,t) = 0,\ v_j \in V \setminus \Gamma,\ t \in (0,T), \tag{61}$$

$$\partial u_k(\gamma_j,t) = g_j(t),\ k \in J(\gamma_j),\ j = 1,\dots,m_1,\ t \in (0,T), \tag{62}$$

$$\partial u_k(\gamma_j,t) = 0,\ k \in J(\gamma_j),\ j = m_1+1,\dots,m,\ t \in (0,T). \tag{63}$$

Here $q = \frac{RG}{CL} - \frac{1}{4}\left(\frac{R}{L} + \frac{G}{C}\right)^2$, $\ell_k = \sqrt{CL}l_k$ and $\partial u_i(v_j,\cdot)$ is derivative of u at the vertex v_j along the edge e_i in the direction outward the vertex. For simplicity we keep the same notations for the vertices of a new graph and its vertices and edges: $\Omega = (V,E)$. We notice that the lengths of the edges of the new graph are equal to $\sqrt{CL}\,l_k$, $k = 1,\dots,N$. The functions g_j are connected with f_j by the equalities $g_j(t) = -Cf_j'(t) - Gf_j(t)$, and therefore, a new set of control functions $g := \{g_j\}$ belongs to the space $\mathcal{F}^T$ if $f \in \mathcal{F}_1^T$.

Definition 4.3. The system (58)–(63) is called exactly controllable in time T if, given arbitrary functions $\phi \in \mathcal{H}_c^1$, $\psi \in \mathcal{H}$, one can find $g \in \mathcal{F}^T$, such that $u(\cdot,T) = \phi$ and $u_t(\cdot,T) = \psi$.

Clearly, the system (8)–(14) is (I, I_t)-controllable in time T if the system (58)–(63) is exactly controllable in the same time interval. In the next section we will find the conditions for exact controllability of the system (58)–(63). The matching conditions (60), (61) are nonstandard matching conditions, and the authors are not aware about results on controllability of such systems in the literature. However, the methods developed in [12] for the wave equation on graphs with standard matching conditions can be applied to handle the current situation.

5. CONTROLLABILITY OF THE WAVE EQUATION OF CURRENT ON METRIC TREES

Our approach to control problem for the system (58)–(63) is based on the relationship between exact controllability, on one hand, and shape and velocity controllability on the other hand. First we prove the shape and velocity controllability using the dynamical method — we reduce these problems to the Volterra integral equations of the second kind. Then we prove exact controllability using the spectral approach — the method of moments and properties of exponential families. This approach was used in [12] for tree graphs, with Dirichlet boundary controls and standard (Kirchhoff–Neumann) matching conditions. In the present paper we consider Neumann type controls and nonstandard matching conditions (60), (61).

Let U be a union of disjoint paths (except for the end points) on Ω. Each path $P(\gamma)$ starts from a controlled boundary vertex $\gamma \in \Gamma_1$ and ends anywhere on Ω, and $\cup_{\gamma\in\Gamma_1} P(\gamma) = \Omega$. In [12] it was proved that such a representation is possible if and only if the controls act at all or all but one of the boundary vertices.

The following result concerning controllability of the system (58)–(63) is valid.

THEOREM 5.1. *Let Ω be a tree graph where $|\Gamma_1| \geq |\Gamma| - 1$. Let U be the described above path union representation of $\Omega : \Omega = \cup_{\gamma\in\Gamma_1} P(\gamma)$, and let $T_* = \max_{P\in U} \operatorname{length} P(\gamma)$. Then*

1. *For any $T > T_*$ and any $\phi \in \mathcal{H}_c^1$, there exists a boundary control $g \in \mathcal{F}^T$ such that $u^g(\cdot, T) = \phi(\cdot)$.*

2. *For any $T \geq T_*$ and any $\psi \in \mathcal{H}$, there exists a boundary control $g \in \mathcal{F}^T$ such that $u_t^g(\cdot, T) = \psi(\cdot)$.*

The property (1) is called shape controllability and property (2) — velocity controllability. Similar result was proved in [12] for the wave equation with Dirichlet boundary controls and standard matching conditions, and all main steps the proof work for the system (58)–(63). The only difference is that the system with Neumann boundary control is not necessarily shape controllable in the critical time interval. This is well known in the case of one interval of length ℓ, since $u(T, T) = 0$ for $T \leq \ell$.

We note that our result is valid for x-dependent potential $q|_{e_k} \in C[0, \ell_k]$.

Now we will prove that the shape and velocity controllability imply the exact controllability.

THEOREM 5.2. *Suppose that the system (58)–(63) is both shape and velocity controllable in time T. Then the system is exactly controllable in time $2T$.*

Proof. We consider the following eigenvalue problem on the graph Ω:

$$
\begin{aligned}
& -\varphi''(x) + q(x)\varphi(x) = \omega^2 \varphi(x); \\
& \sum_{k\in J(v_j)} \varkappa_{kj} \varphi|_{e_k}(v_j) = 0 \quad \forall\, v_j \in V \setminus \Gamma; \\
& \partial\varphi|_{e_i}(v_j) = \partial\varphi|_{e_k}(v_j), \ i, k \in J(v_j), \quad \forall\, v_j \in V \setminus \Gamma; \\
& \partial\varphi|_{\Gamma} = 0.
\end{aligned}
$$

It is known that the spectrum $\{\omega_n\}_{n\in\mathbb{N}}$ of this problem is purely discrete and the eigenfunctions $\{\varphi_n\}_{n\in\mathbb{N}}$ form an orthonormal basis in $\mathcal{H}$. The solution $u(\cdot,t)$ of (58)–(63) can be represented in a form of a series with respect to $\{\varphi_n\}$ (see, for example, [8, 9]).

Control problems are reduced to moment problems using the Fourier method. The shape controllability is equivalent to the solvability of the moment problem

$$a_n = \langle f, \mathfrak{s}_n\rangle_{\mathcal{F}^T}, \quad n\in\mathbb{N}, \quad \mathfrak{s}_n(t) := \varphi_n|_{\Gamma_1} \sin\omega_n(T-t). \tag{64}$$

(see, *e.g.* [8, Ch. 3]). Solvability means that for any $\{a_n\}\in l^2$, there exists $f\in\mathcal{F}^T$ satisfying (64). For simplicity we assume here that $\omega_n \neq 0 \ \forall n$. If $\omega_n = 0$ for some n, we use t to replace $\omega_n^{-1}\sin\omega_n(T-t)$ in the corresponding moment equality.

The velocity controllability in time T is equivalent to the solvability of the moment problem

$$b_n = \langle g, \mathfrak{c}_n\rangle_{\mathcal{F}^T}, \quad n\in\mathbb{N}, \quad \mathfrak{c}_n(t) := \varphi_n|_{\Gamma_1} \cos\omega_n(T-t). \tag{65}$$

Denote by $f_-(t)$ the odd extension with respect to T of $f(t)$ from $[0,T]$ to $[0,2T]$ and by $g_+(t)$ the even extension of $g(t)$. We observe that the function

$$h(t) = \frac{f_-(t)+g_+(t)}{2}$$

solves both moment problems

$$a_n = \langle h, \mathfrak{s}_n\rangle_{\mathcal{F}^{2T}}, \quad b_n = \langle h, \mathfrak{c}_n\rangle_{\mathcal{F}^{2T}}, \tag{66}$$

where $\mathcal{F}^{2T} := L^2(0,2T;\mathbb{R}^{m_1})$. It means that the moment problem (66) is solvable for any sequences $\{a_n\},\{b_n\}\in l^2$. Therefore, the family $\{\mathfrak{s}_n,\mathfrak{c}_n\}$ forms a Riesz sequence in $\mathcal{F}^{2T}$ [8, Ch. 1]. It implies that both families

$$\left\{\varphi_n|_{\Gamma_1} e^{\pm i\omega_n(T-t)}\right\} \quad \text{and} \quad \left\{\varphi_n|_{\Gamma_1} e^{\pm i\omega_n t}\right\}$$

also form Riesz sequences in $\mathcal{F}^{2T}$, and by Theorem III.3.10 of [8] the system (58)–(63) is exactly controllable in time $2T$. □

So we proved that if Ω is a tree and control functions g_j act at all or at all but one of the boundary vertices, the system (58)–(63) is exactly controllable in any time T greater than $2T_*$. This controllability time estimate is sharp, *i.e.* it guarantees controllability of any tree and, generally, it cannot be improved.

Taking into account the relations between exact controllability of the system (58)–(63) and (V,I) controllability of the original telegraph equation, we can now formulate the main result of the paper.

Theorem 1. *If Ω is a tree, control functions act at all or at all but one boundary vertices, and the coefficients C_k, L_k, R_k, G_k are independent of k, the system (8)–(14) is (V, I)-controllable in time T greater than $2T_*\sqrt{CL}$.*

We note that T_* in this theorem is defined by the path representation of the original graph, where the system (8)–(14) is defined, not by the graph corresponding the transformed system (58)–(63).

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G. M. Alam and N. B. Avdonina
University of Alaska Fairbanks
Department of Mathematics and Statistics
Fairbanks, AK 99775-6660, USA
gmalam@alaska.edu, navdonina@alaska.edu

S. A. Avdonin
University of Alaska Fairbanks
Department of Mathematics and Statistics
Fairbanks, AK 99775-6660, USA
and
Moscow Center for Fundamental
and Applied Mathematics,
Moscow 119991, Russia
saavdonin@alaska.edu