Dedicated to Marius Tucsnak on the occasion of his 60^{th} *anniversary*

COMPUTATION OF MULTIPLE SCATTERING PROBLEMS BY CIRCULAR CYLINDERS

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The aim of this paper is to develop some well-adapted formulations for numerically solving the scattering problem of an incident time-harmonic wave by a cluster of circular cylinders. These formulations are illustrated by numerical simulations for different physical situations involving various kinds of boundary conditions.

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1. INTRODUCTION

Multiple scattering by clusters of obstacles is a complex physical and computational challenge. It indeed results from the nontrivial scattering interactions between the single scatterers that compose the structure, yielding then to complex patterns of the wave field that are difficult to predict. Related to this specific behavior emerge some new physical properties of waves that are used to create some technological devices or to explain some observations in the nature. From the perspective of numerical methods, quite a lot of recent computational approaches were developed over the years. We refer *e.g.* to [2, 3, 4, 8, 9, 14, 15, 17, 18, 22, 30, 32, 34] where methods are available to solve scattering by complex structures. In the present paper, we consider the two-dimensional case where we have M sound-soft, sound-hard or homogeneous dielectric/penetrable circular cylinders. Even if the shape of the obstacles is simple, numerically solving this problem for many obstacles and for large frequencies remains nontrivial. In particular, getting efficient and accurate computational solutions is of utmost importance for important applications (electromagnetics, optics, nanophotonics for instance) where a huge number of diffraction objects are considered to model structured or disordered media (see *e.g.* [7, 12, 13, 16, 20, 21, 22, 26, 29, 31, 33]).

To solve this class of problem, we first consider in Section 2 how to formulate the problem after writing the boundary-value problem. The formulation is developed following the Mie series expansion theory since the obstacles are circular cylinders. We then use the Dirichlet or Neumann boundary conditions to write the set of equations to solve. These equations are next suitably truncated and we explain quickly how to numerically solve them by spectral methods. We report some numerical examples (scattered/total fields and Radar Cross Sections) by using the Matlab toolbox μ-diff dedicated to solve such problems. Then we extend the methodology in Section 3 by using integral equation techniques in the case of dielectric circular cylinders combined with a reduced Schur complement, which is a main difference with the full approach that we developed in [30]. A numerical example is provided to illustrate the method. We end by a short conclusion in Section 4.

2. MULTIPLE SCATTERING BY A CLUSTER OF M CIRCULAR CYLINDERS: DIRICHLET AND NEUMANN BOUNDARY CONDITIONS

2.1. Formulation

Let us consider a homogeneous medium for the whole space $\mathbb{R}^2$ in which M disjoint scatterers $\Omega_1^-, \ldots, \Omega_M^-$ are included. We suppose that each of the scatterers Ω_p^-, $p = 1, \ldots, M$, is a bounded domain in $\mathbb{R}^2$ with boundary $\Gamma_p := \partial\Omega_p^-$. We define $\Omega^- = \cup_{p=1}^M \Omega_p^-$ as the global domain built from these separated obstacles. To start, we propose to analyze the case of the scattering problem of an incident plane wave $u^{\mathrm{inc}}(\mathbf{x}) = e^{ik\boldsymbol{\beta}\cdot\mathbf{x}}$ (with $\mathbf{x} = (x, y) \in \mathbb{R}^2$), with direction $\boldsymbol{\beta}$, by Ω^- (assuming that the time dependence has the form $e^{-i\omega t}$ and for a real-valued wavenumber k). Concretely, this means that we wish to solve the following boundary-value problem: compute the scattered field u solution to

$$(\mathcal{P}) \quad \begin{cases} \Delta u + k^2 u = 0, & \text{in } \Omega^+ := \mathbb{R}^2 \setminus \overline{\Omega^-}, \\ \gamma(u) = -\gamma(u^{\mathrm{inc}}), & \text{on } \Gamma := \partial\Omega^-, \\ \lim_{||\mathbf{x}|| \to +\infty} ||\mathbf{x}||^{1/2} (\nabla u \cdot \dfrac{\mathbf{x}}{||\mathbf{x}||} - iku) = 0. & \end{cases}$$

The operator $\Delta = \partial_x^2 + \partial_y^2$ is the Laplace operator and $(\Delta + k^2)$ is the Helmholtz operator. The gradient operator is ∇ and $||\mathbf{x}|| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$, where $\mathbf{x} \cdot \mathbf{y}$ is the scalar product of two vectors $\mathbf{x}$ and $\mathbf{y}$ of $\mathbb{R}^2$. In practice, for sound-soft obstacles, the boundary operator γ is given by the zeroth-order trace operator, i.e. the boundary condition writes: $u = -u^{\mathrm{inc}}$ on Γ. In the case of sound-hard obstacles, then the Neumann trace is involved leading to $\partial_{\mathbf{n}} u = -\partial_{\mathbf{n}} u^{\mathrm{inc}}$ on Γ, where $\mathbf{n}$ is the unit normal vector outwardly directed to the bounded scatterers Ω_p^-. The last equation is the well-known Sommerfeld's radiation condition at infinity which is needed to prove the existence of the solution [11, 27].

One possible and physically direct way to try solving the multiple scattering problem $(\mathcal{P})$ is to interpret it as a collection of M single scattering problems that are next coupled through multiple scattering phenomena. This means that we can rewrite equivalently the problem by introducing the scattered fields $u^1, \ldots, u^M$, considering that each field u^p is given as the wave reflected only by the obstacle p illuminated by the combination of the incident wave and the $M-1$ scattered waves u^q, for $q = 1, \ldots, M$, with $q \neq p$. In fact it can be proved [6] that the following result holds. We consider u as the solution to the multiple scattering problem $(\mathcal{P})$. Then, we can show that the M coupled scattering problems, for $p = 1, \ldots, M$, given by

$$(\mathcal{P}^p) \quad \begin{cases} \Delta u^p + k^2 u^p = 0, & \text{in } \mathbb{R}^2 \setminus \overline{\Omega_p^-}, \\ \gamma(u^p) = -\gamma(u^{\text{inc}} + \displaystyle\sum_{q=1, q\neq p}^{M} u^q) & \text{on } \Gamma_p, \\ \displaystyle\lim_{||\mathbf{x}|| \to +\infty} ||\mathbf{x}||^{1/2} (\nabla u \cdot \frac{\mathbf{x}}{||\mathbf{x}||} - iku) = 0, & \end{cases}$$

have a unique solution given by $(u^1, \ldots, u^M)$. In addition, we also have the following decomposition of the wave field

$$u = \sum_{p=1}^{M} u^p. \tag{1}$$

We now consider the specific case where the scatterers Ω_p^- are circular cylinders of radius a_p and centered at $\mathcal{O}_p = (x_p, y_p)$ of a given orthonormal system of coordinates $(\overrightarrow{\mathcal{O}x}, \overrightarrow{\mathcal{O}y})$. Let us introduce the following notations: for all $p = 1, \ldots, M$: $\mathbf{b}_p = \overrightarrow{\mathcal{O}\mathcal{O}_p}$, $b_p = |\mathbf{b}_p|$, $\alpha_p = Angle(\mathcal{O}x, \mathbf{b}_p)$ and for all $q = 1, \ldots, M$, with $q \neq p$: $\mathbf{b}_{pq} = \overrightarrow{\mathcal{O}_q\mathcal{O}_p}$, $b_{pq} = |\mathbf{b}_{pq}|$, $\alpha_{pq} = Angle(\mathcal{O}x, \mathbf{b}_{pq})$. Let us consider a point $\mathbf{P}$ of the plane described by its cartesian coordinates (x, y), or alternatively by its polar coordinates: $\mathbf{r} = \overrightarrow{\mathcal{O}\mathbf{P}}$, $r = |\mathbf{r}|$, $\theta = Angle(\mathcal{O}x, \mathbf{r})$. In the sequel, we will also need the local polar coordinates of $\mathbf{P}$ in the orthonormal system of coordinates associated to the p-th scatterer, *i.e.* $\mathbf{r}_p = \overrightarrow{\mathcal{O}_p\mathbf{P}}$, $r_p = |\mathbf{r}_p|$, $\theta_p = Angle(\mathcal{O}x, \mathbf{r}_p)$. We define, for all $m \in \mathbb{Z}$, the following cylindrical wave functions that are particular solutions to the Helmholtz equations for $r > 0$

$$\begin{cases} \psi_m(\mathbf{r}) = H_m^{(1)}(kr) e^{im\theta}, \\ \widehat{\psi}_m(\mathbf{r}) = J_m(kr) e^{im\theta}. \end{cases} \tag{2}$$

In the above relations, J_n is known as the n-th order Bessel function while $H_n^{(1)}$ is the n-th order first-kind Hankel function [1]. For all $m \in \mathbb{Z}$, it can be shown that ψ_m satisfies the outgoing Sommerfeld radiation condition. We also need, for all $m \in \mathbb{Z}$, the local cylindrical wave functions related to the p-th scatterer, for $p = 1, \dots, M$,

$$(3) \qquad \begin{cases} \psi_m^p(\mathbf{r}) = \psi_m(\mathbf{r}_p) = H_m^{(1)}(kr_p)e^{im\theta_p}, \\ \widehat{\psi}_m^p(\mathbf{r}) = \widehat{\psi}_m(\mathbf{r}_p) = J_m(kr_p)e^{im\theta_p}, \end{cases} \qquad \forall\, m \in \mathbb{Z}.$$

Because u^p is an outgoing solution to a single scattering problem outside a circular cylinder, we have the decomposition

$$(4) \qquad u^p(\mathbf{r}) = \sum_{m\in\mathbb{Z}} c_m^p \psi_m^p(\mathbf{r}), \qquad \forall\, p = 1, \dots, M,\ \forall r_p > a_p.$$

The complex coefficients $(c_m^p)_{m\in\mathbb{Z}}$ are obtained by simply using the boundary condition on Γ_p

$$(5) \qquad \gamma(u^p) = -\gamma(u^{\mathrm{inc}}) - \sum_{q=1, q\neq p}^{M} \gamma(u^q).$$

To get a more explicit expression of this relation, we need to write the fields u^{inc} and u^q, for $q \neq p$, in the local system of coordinates of the scatterer p. If $\boldsymbol{\beta} = (\cos\beta, \sin\beta)$, then we have [25]

$$(6) \qquad u^{inc}(\mathbf{r}) = \sum_{m\in\mathbb{Z}} d_m^p \widehat{\psi}_m^p(\mathbf{r}),$$

where $d_m^p = e^{ik\boldsymbol{\beta}\cdot\mathbf{b}_p}\, e^{im(\frac{\pi}{2}-\beta)}$. In addition, the separation theorem [25] implies that, for $1 \le p, q \le M$, with $p \neq q$, we have

$$(7) \qquad \psi_m^q(\mathbf{r}) = \begin{cases} \displaystyle\sum_{n\in\mathbb{Z}} S_{mn}(\mathbf{b}_{pq})\widehat{\psi}_n^p(\mathbf{r}) & \text{for } r_p < b_{pq}, \\ \displaystyle\sum_{n\in\mathbb{Z}} \widehat{S}_{mn}(\mathbf{b}_{pq})\psi_n^p(\mathbf{r}) & \text{for } r_p > b_{pq}, \end{cases} \qquad \forall\, m \in \mathbb{Z},$$

setting $S_{mn}(\mathbf{b}_{pq}) = \psi_{m-n}(\mathbf{b}_{pq})$ and $\widehat{S}_{mn}(\mathbf{b}_{pq}) = \widehat{\psi}_{m-n}(\mathbf{b}_{pq})$. The infinite dimensional matrices $\mathbb{S}^{p,q} = (S_{mn}(\mathbf{b}_{pq}))_{m,n\in\mathbb{Z}}$ and $\widehat{\mathbb{S}}^{p,q} = (\widehat{S}_{mn}(\mathbf{b}_{pq}))_{m,n\in\mathbb{Z}}$ are the so-called separation matrices. Based on (1), (4), (5) and (7), we obtain the infinite linear systems

$$(8) \qquad \mathbf{C}^p + \mathbb{D}^p \sum_{q=1, q\neq p}^{M} \left(\mathbb{S}^{p,q}\right)^{\mathbf{T}} \mathbf{C}^q = \mathbf{B}^p \qquad \forall\, p = 1, \dots, M,$$

where $\mathbf{C}^p = (c_n^p)_{n\in\mathbb{Z}}$ is the infinite vector containing the coefficients of the cylindrical decomposition (4) of u^p, $\left(\mathbb{S}^{p,q}\right)^{\mathbf{T}}$ denotes the transpose of the separation matrix $\mathbb{S}^{p,q}$ between the obstacles $\mathcal{B}_p$ and $\mathcal{B}_q$ defined by $\mathbb{S}^{p,q} = (\mathbb{S}^{p,q}_{mn})_{m\in\mathbb{Z},n\in\mathbb{Z}}$ and $\mathbb{S}^{p,q}_{mn} = \psi_{m-n}(\mathbf{b}_{pq})$, $\mathbb{D}^p = (\mathbb{D}^p_{mn})_{mn\in\mathbb{Z}}$ is the diagonal infinite matrix, with diagonal terms

$$\mathbb{D}^p_{m,m} = \begin{cases} \dfrac{J_n(ka_p)}{H_n^{(1)}(ka_p)} & \text{for sound-soft obstacles,} \\ \dfrac{J'_n(ka_p)}{H_n^{(1)'}(ka_p)} & \text{for sound-hard obstacles,} \end{cases} \tag{9}$$

$\mathbf{B}^p = -\mathbb{D}^p\mathbf{d}^p$, where $\mathbf{d}^p = (d^p_m)_{m\in\mathbb{Z}}$ is the infinite dimensional vector with coefficients (6) of the cylindrical incident field decomposition. Another way to write (8) is

$$\mathbb{A}\mathbf{C} = \mathbf{B} \tag{10}$$

where, if $\mathbb{I}$ is the identity operator on $\ell^2(\mathbb{C})$,

$$\mathbb{A} = \begin{bmatrix} \mathbb{I} & \mathbb{D}^1\left(\mathbb{S}^{1,2}\right)^{\mathbf{T}} & \dots & \mathbb{D}^1\left(\mathbb{S}^{1,M}\right)^{\mathbf{T}} \\ \mathbb{D}^2\left(\mathbb{S}^{2,1}\right)^{\mathbf{T}} & \mathbb{I} & \dots & \mathbb{D}^2\left(\mathbb{S}^{2,M}\right)^{\mathbf{T}} \\ \vdots & & \ddots & \\ \mathbb{D}^M\left(\mathbb{S}^{M,1}\right)^{\mathbf{T}} & \mathbb{D}^M\left(\mathbb{S}^{M,2}\right)^{\mathbf{T}} & \dots & \mathbb{I} \end{bmatrix},$$
$$\mathbf{C} = \begin{bmatrix} \mathbf{C}^1 \\ \mathbf{C}^2 \\ \vdots \\ \mathbf{C}^M \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \mathbf{B}^1 \\ \mathbf{B}^2 \\ \vdots \\ \mathbf{B}^M \end{bmatrix}. \tag{11}$$

Let us remark that the wave field u can be computed locally to Ω_p (for $r_p < \min\limits_{\substack{1\le q\le M \\ q\ne p}} b_{pq}$) by

$$u(\mathbf{r}) = \sum_{m\in\mathbb{Z}} c^p_m \psi^p_m(\mathbf{r}) + \sum_{m\in\mathbb{Z}} \Big(\sum_{q=1,q\ne p}^{M} \sum_{n\in\mathbb{Z}} S_{nm}(\mathbf{b}_{pq}) c^q_n \Big) \widehat{\psi}^p_m(\mathbf{r}), \tag{12}$$

while the scattering amplitude $a(\theta)$ such that

$$u(\mathbf{r}) = \frac{e^{ikr}}{\sqrt{r}} a(\theta) + \mathcal{O}\left(\frac{1}{r}\right),$$

for $r \to +\infty$, is

$$a(\theta) = e^{-i\pi/4}\sqrt{\frac{2}{\pi k}} \sum_{p=1}^{M} e^{-ib_p k\cos(\theta-\alpha_p)} (\sum_{n\in\mathbb{Z}} e^{in(\theta-\frac{\pi}{2})} c^p_n). \tag{13}$$

In addition, expressions for the traces and normal derivative traces can be obtained with respect to the boundary conditions.

2.2. **Computational aspects**

For a practical implementation, the linear system (8) must be truncated. Let us define the number of finite modes for the truncation of the solution u^p by $2N_p+1$, where $N_p \in \mathbb{N}$, $\mathcal{C}^p = (c_n^p)_{n=-N_p,\dots,N_p}$ as the finite vector with approximations of the first $2N_p+1$ coefficients c_n^p of the cylindrical decomposition (6) of u^p, $\mathcal{S}^{p,q}$ is the $(2N_p+1)\times(2N_q+1)$ finite dimensional separation matrix involving only the interactions between the first modes of Ω_p^- and Ω_q^- such that $\mathcal{S}^{p,q} = (\mathcal{S}_{mn}^{p,q})_{-N_p\le m\le N_p, -N_q\le n\le N_q}$, $\mathcal{S}_{mn}^{p,q} = \psi_{m-n}(\mathbf{b}_{pq})$, and $\mathcal{D}^p = (\mathcal{D}_{mn}^p)_{-N_p\le m\le N_p, -N_q\le n\le N_q}$ is the diagonal finite matrix, with diagonal terms $\mathcal{D}_{m,m}^p = \mathbb{D}_{m,m}^p$, $\mathcal{B}^p = -\mathcal{D}^p d^p$, where $d^p = (d_m^p)_{-N_p\le m\le N_p}$ contains the $2N_p+1$ first coefficients of the cylindrical incident field decomposition (4). The M coupled finite dimensional systems writes

$$\mathcal{A}\mathcal{C} = \mathcal{B}, \tag{14}$$

with $\mathcal{A} \in \mathbb{C}^{N,N}$ (of size $N = \sum_{p=1}^{M}(2N_p+1)$) ($\mathcal{I}^p$ is the identity matrix of $\mathbb{C}^{2N_p+1}$)

$$\begin{aligned}\mathcal{A} &= \begin{bmatrix} \mathcal{I}^1 & \mathcal{D}^1\left(\mathcal{S}^{1,2}\right)^{\mathbf{T}} & \dots & \mathcal{D}^1\left(\mathcal{S}^{1,M}\right)^{\mathbf{T}} \\ \mathcal{D}^2\left(\mathcal{S}^{2,1}\right)^{\mathbf{T}} & \mathcal{I}^2 & \dots & \mathcal{D}^2\left(\mathcal{S}^{2,M}\right)^{\mathbf{T}} \\ \vdots & & \ddots & \\ \mathcal{D}^M\left(\mathcal{S}^{M,1}\right)^{\mathbf{T}} & \mathcal{D}^M\left(\mathcal{S}^{M,2}\right)^{\mathbf{T}} & \dots & \mathcal{I}^M \end{bmatrix}, \\ \mathcal{C} &= \begin{bmatrix} \mathcal{C}^1 \\ \mathcal{C}^2 \\ \vdots \\ \mathcal{C}^M \end{bmatrix}, \mathcal{B} = \begin{bmatrix} \mathcal{B}^1 \\ \mathcal{B}^2 \\ \vdots \\ \mathcal{B}^M \end{bmatrix}.\end{aligned} \tag{15}$$

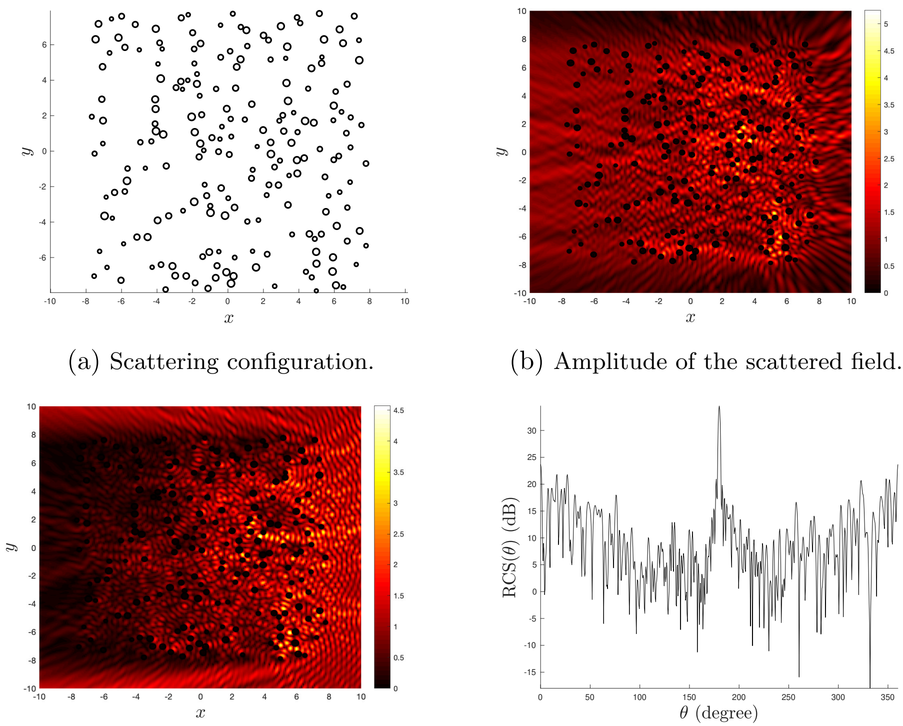
In practice, a suitable choice of the truncation parameter is given by (see e.g. [30])

$$N_p = \left[ka_p + \left(\frac{1}{2\sqrt{2}}\ln(2\sqrt{2}\pi k a_p \varepsilon^{-1})\right)^{\frac{2}{3}}(ka_p)^{1/3} + 1\right], \tag{16}$$

setting $[x]$ as the integer part of a real number x, and ε is the error bound on the Fourier coefficients. Finally, the numerical solution to the linear system (14) is obtained by a direct solver for dense linear systems or a preconditioned Krylov subspace iterative solver [28] combined with fast Toeplitz acceleration techniques [10]. In this case, a low memory storage of the system can be considered. The choice mainly depends on the physical configuration [30].

2.3. Numerical examples

To illustrate the method, we present two numerical examples based on the Matlab toolbox μ-diff [30]. Similar experiments can be reproduced based on available scripts that can be run on any computer. We consider the square box $]-8;8[^2$ in which we randomly distribute $M = 200$ circular scatterers with radii such that $0.1 \leq a_p \leq 0.2$, for $p = 1, ..., M$. The incident field is a plane wave with wavenumber $k = 10$ and angle of incidence $\beta = 180$ degrees. Let us remain that the circles are not sticky and that a minimal distance of 0.01 is fixed here between their boundary to avoid such a situation. We report on Figure 1 the results. On Figure 1a, we represent the scattering configuration for the $M = 200$ disks. On plots 1b and 1c, we draw the amplitude of the scattered and



(a) Scattering configuration. (b) Amplitude of the scattered field.

(c) Amplitude of the total field. (d) RCS (dB).

Fig. 1 – Scattering by $M = 200$ sound-soft disks randomly distributed in the box $]-8;8[^2$. The incident plane is fixed by $k = 10$ and $\boldsymbol{\beta} = (-1, 0)$.

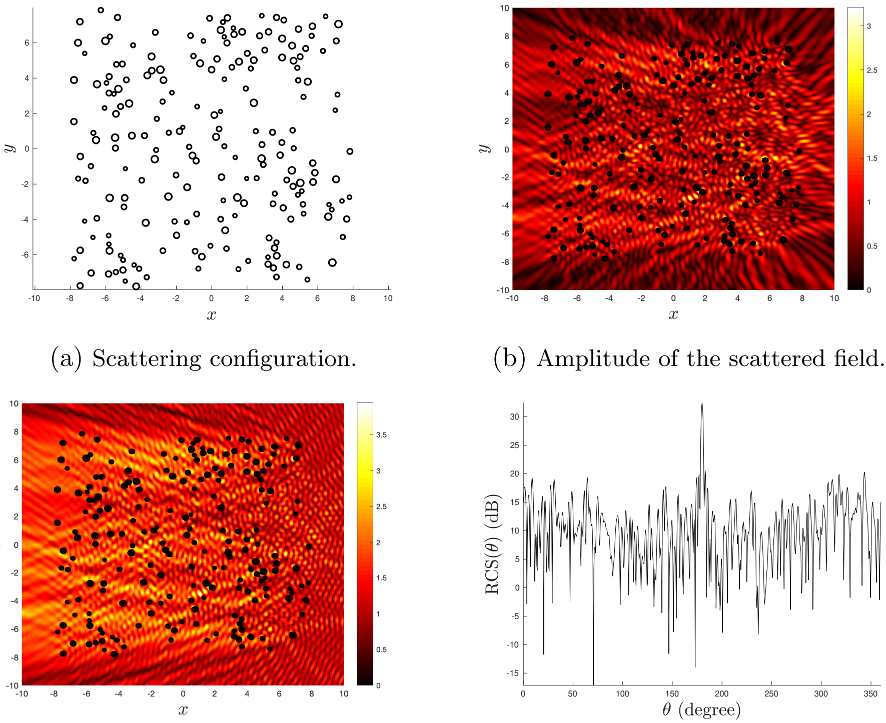
total wave fields, respectively. We can observe the strong interaction between the scatterers (with of course no penetration of the field inside the obstacles).

Finally, we give the bistatic Radar Cross Section (RCS) in the direction θ on Figure 1d which allows to analyze where the main lobs related to the diffraction problem arise. Its definition is given by

$$\mathrm{RCS}(\theta) := 10\log_{10}(|a(\theta)|^2) \quad (\mathrm{dB}),$$

where a is given by (13).

Similarly, with the same parameters (but for a different location of the scatterers), we report on Figures 2a−d the case of a Neumann problem which corresponds to the sound-hard case. We observe that the physical behavior of the wave field is strongly dependent on the kind of boundary condition that is involved. In particular, the Dirichlet boundary conditions reflects the field mainly in the forward direction while the field penetrates inside the cluster for the Neumann case, generating some important backscattering propagation.



(a) Scattering configuration.

(b) Amplitude of the scattered field.

(c) Amplitude of the total field.

(d) RCS (dB).

Fig. 2 – Scattering by $M = 100$ sound-hard disks randomly distributed in the box $]-8;8[^2$. The incident plane is fixed by $k = 2\pi$ and $\boldsymbol{\beta} = (-1, 0)$.

3. MULTIPLE SCATTERING BY A COLLECTION OF PENETRABLE CYLINDERS

3.1. Physical problem

We consider now M regular bounded and non intersecting circular dielectric *ElectroMagnetic* (EM) scatterers Ω_p^-, $p = 1, ..., M$. They are assumed to be distributed in $\mathbb{R}^2$, with boundary $\Gamma_p := \partial\Omega_p^-$. The global dielectric scatterer Ω^- is then built as the collection of the M separate single dielectric obstacles, i.e. $\Omega^- = \cup_{p=1}^M \Omega_p^-$, with boundary $\Gamma = \cup_{p=1}^M \Gamma_p$. The homogeneous isotropic exterior domain of propagation is $\Omega^+ = \mathbb{R}^2 \setminus \overline{\Omega^-}$. We assume that we have a time-harmonic incident plane wave $u^{\mathrm{inc}}(\mathbf{x}) = e^{ik^+\boldsymbol{\beta}\cdot\mathbf{x}}$ illuminating Ω^-, with incidence direction $\boldsymbol{\beta} = (\cos\beta, \sin\beta)$ and time dependence $e^{-i\omega t}$, where ω is the wave pulsation. The exterior wave number is $k^+ = \omega\sqrt{\varepsilon_0\mu_0}$ while the interior wavenumbers are such that $k_p^- = \omega\sqrt{\varepsilon_p\mu_p}$, defining (ε_0, μ_0) as (respectively (ε_p, μ_p)) the electric permittivity and magnetic permeability in the vacuum (respectively in the obstacle Ω_p^-). Let us introduce the wavenumber k defined as piecewise constant with value k^+ out of the single obstacles and k^- inside. Then, the problem writes: find the scattered field u^+ and the transmitted wave u^- solution to the transmission problem

$$
\left\{
\begin{array}{rcll}
\Delta u^- + (k^-)^2 u^- & = & 0, & \text{in } \Omega^-, \\
\Delta u^+ + (k^+)^2 u^+ & = & 0, & \text{in } \Omega^+, \\
u^+ - u^- & = & -u^{\mathrm{inc}}, & \text{on } \Gamma, \\
\partial_{\mathbf{n}} u^+ - \chi \partial_{\mathbf{n}} u^- & = & -\partial_{\mathbf{n}} u^{\mathrm{inc}}, & \text{on } \Gamma, \\
\multicolumn{4}{l}{\displaystyle\lim_{||\mathbf{x}||\to+\infty} ||\mathbf{x}||^{1/2} \left(\nabla u^+ \cdot \frac{\mathbf{x}}{||\mathbf{x}||} - ik^+ u^+ \right) = 0.}
\end{array}
\right. \tag{17}
$$

The total physical field u^{tot} writes: $u^{\mathrm{tot}} = u^+ + u^{\mathrm{inc}}$, outside and $u^{\mathrm{tot}} = u^-$ inside the obstacles. In the so-called Transverse-Magnetic mode (or Transverse-Electric mode), the unknown $u := u^{\pm}$ stands for the z-component H_z of the magnetic field $\mathbf{H}$ and the transmission condition has to be read with $\chi_p = \varepsilon_0\varepsilon_p^{-1}$, on the boundary Γ_p, while for TE modes (or TM mode), then u is linked to the electric field $\mathbf{E}$ through $u = E_z$ and the transmission condition is given by $\chi_p = \mu_0\mu_p^{-1}$. We define $\varepsilon|_{\Gamma_p} = \varepsilon_p$, $\mu|_{\Gamma_p} = \mu_p$, and $\chi|_{\Gamma_p} = \chi_p$, accordingly to the polarization.

3.2. An integral equation formulation of the scattering problem (17)

For system (17), we will use the theory of integral equations to write the set of equations to numerically resolve [5, 11, 27]. This can be done in the case of general obstacles thanks to the theory of potential. Let us remark that such an approach could also be used for Dirichlet, Neumann and even impedance boundary conditions.

Let k be a wavenumber. Let G be the two-dimensional free-space Green's function defined by

$$\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^2, \mathbf{x} \neq \mathbf{y}, \quad G(\mathbf{x}, \mathbf{y}) = \frac{i}{4} H_0^{(1)}(k\|\mathbf{x} - \mathbf{y}\|).$$

Integral equations are essentially based upon the Helmholtz integral representation formula [11]. Indeed, if v is a solution to the Helmholtz equation in the unbounded connected domain Ω^+ and satisfies the Sommerfeld radiation condition, then we have the following relation

$$(18) \qquad \int_\Gamma -G(\mathbf{x}, \mathbf{y})\partial_\mathbf{n} v(\mathbf{y}) + \partial_{\mathbf{n_y}} G(\mathbf{x}, \mathbf{y}) v(\mathbf{y}) \,\mathrm{d}\Gamma(\mathbf{y}) = \begin{cases} v(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega^+, \\ 0 & \text{otherwise.} \end{cases}$$

If v^- is now solution to the Helmholtz equation in the bounded domain Ω^-, then we have

$$(19) \quad \int_\Gamma -G(\mathbf{x}, \mathbf{y})\partial_\mathbf{n} v^-(\mathbf{y}) + \partial_{\mathbf{n_y}} G(\mathbf{x}, \mathbf{y}) v^-(\mathbf{y}) \,\mathrm{d}\Gamma(\mathbf{y}) = \begin{cases} 0 & \text{if } \mathbf{x} \in \Omega^+, \\ -v^-(\mathbf{x}) & \text{otherwise.} \end{cases}$$

The integrals on Γ has to be interpreted in the sense of the inner product in $L^2(\Gamma)$

$$\langle f, g \rangle_{H^{-1/2}, H^{1/2}} = \int_\Gamma fg \,\mathrm{d}\Gamma,$$

because both u^{inc} and Γ are regular. Let us define the volume single- and double-layer integral operators, respectively denoted by $\mathcal{L}$ and $\mathcal{M}$, as, $\forall \mathbf{x} \in \mathbb{R}^2 \backslash \Gamma$,

$$\begin{aligned} \mathcal{L} : \rho \quad &\rightarrow \quad \mathcal{L}\rho(\mathbf{x}) = \int_\Gamma G(\mathbf{x}, \mathbf{y})\rho(\mathbf{y}) \,\mathrm{d}\Gamma(\mathbf{y}), \\ \mathcal{M} : \lambda \quad &\rightarrow \quad \mathcal{M}\lambda(\mathbf{x}) = -\int_\Gamma \partial_{\mathbf{n_y}} G(\mathbf{x}, \mathbf{y})\lambda(\mathbf{y}) \,\mathrm{d}\Gamma(\mathbf{y}). \end{aligned}$$

Then we can express v and v^- based on (18)-(19) as

$$\begin{cases} v(\mathbf{x}) = -\mathcal{L}(\partial_\mathbf{n} v|_\Gamma)(\mathbf{x}) - \mathcal{M}(v|_\Gamma)(\mathbf{x}), & \forall \mathbf{x} \in \Omega^+, \\ v^-(\mathbf{x}) = \mathcal{L}(\partial_\mathbf{n} v^-|_\Gamma)(\mathbf{x}) + \mathcal{M}(v^-|_\Gamma)(\mathbf{x}), & \forall \mathbf{x} \in \Omega^-. \end{cases}$$

Furthermore, the single- and double-layer integral operators provide some outgoing solutions to the Helmholtz equation [11]. Indeed, for any densities $\rho \in H^{-1/2}(\Gamma)$ and $\lambda \in H^{1/2}(\Gamma)$, $\mathcal{L}\rho$ and $\mathcal{M}\lambda$, and any linear combination of both these functions yields to outgoing solutions to the Helmholtz equation in $\mathbb{R}^2 \backslash \Gamma$ for some boundary conditions. Let us now recall the jump relations

[11]. For any $\mathbf{x}$ in Γ, the trace and normal derivative traces of the operator $\mathcal{L}$ are given by (the signs means that $\mathbf{z}$ tends towards $\mathbf{x}$ from the exterior or the interior of Γ)

$$\text{(20)} \qquad \lim_{\mathbf{z}\in\Omega^{\pm}\to\mathbf{x}} \mathcal{L}\rho(\mathbf{z}) = L\rho(\mathbf{x}), \qquad \lim_{\mathbf{z}\in\Omega^{\pm}\to\mathbf{x}} \partial_{\mathbf{n_z}}\mathcal{L}\rho(\mathbf{z}) = \left(\mp\frac{1}{2}I + N\right)\rho(\mathbf{x}),$$

where I is the identity operator, for $\mathbf{x} \in \Gamma$, and where

$$L\rho(\mathbf{x}) = \int_{\Gamma} G(\mathbf{x},\mathbf{y})\rho(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}), \qquad N\rho(\mathbf{x}) = \int_{\Gamma} \partial_{\mathbf{n_x}} G(\mathbf{x},\mathbf{y})\rho(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}).$$

In the paper, we denote the boundary integral operators by a roman letter (e.g. L). To solve (17) for general smooth scatterers, we can consider the single-layer representation of u^+ and u^-

$$\text{(21)} \qquad u^+ = \mathcal{L}^+\rho^+ \quad \text{and} \quad u^- = \mathcal{L}^-\rho^-,$$

where $\mathcal{L}^+$ (respectively $\mathcal{L}^-$) is the single-layer operator associated with k^+ (respectively k^-). The unknown (ρ^+, ρ^-) is next solution to the integral equation

$$\text{(22)} \qquad \begin{pmatrix} L^+ & -L^- \\ \frac{I}{2} - N^+ & \chi(\frac{I}{2} + N^-) \end{pmatrix} \begin{pmatrix} \rho^+ \\ \rho^- \end{pmatrix} = \begin{pmatrix} -u^{\mathrm{inc}} \\ \partial_{\mathbf{n}} u^{\mathrm{inc}} \end{pmatrix},$$

that we rewrite

$$\text{(23)} \qquad A\boldsymbol{\rho} := \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix} \begin{pmatrix} \rho^+ \\ \rho^- \end{pmatrix} = \mathbf{b} := \begin{pmatrix} b^1 \\ b^2 \end{pmatrix}.$$

The plus and minus superscripts in L and N refers to as the wave numbers k^+ and k^-. Since $\Omega^- = \cup_{p=1}^{M}\Omega_p^-$ is multiply connected, all the integral operators can be written as blocks structured systems and (23) reads

$$\text{(24)} \qquad \begin{pmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,M} \\ A_{2,1} & A_{2,2} & \dots & A_{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ A_{M,1} & A_{M,2} & \dots & A_{M,M} \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_M \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_M \end{pmatrix},$$

where $A_{p,q}\rho_q = (A_q\rho_q)|_{\Gamma_p}$. After some manipulations and using a Schur complement then one gets ρ^+ (assuming that we are not on an interior resonance of one of the dielectric cylinders)

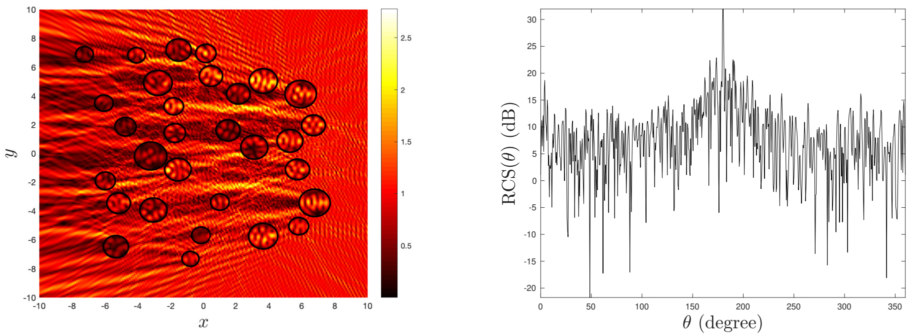
$$\text{(25)} \qquad \begin{aligned} A^{S,+}\rho^+ &:= (L^+ + \chi^{-1}L^-(\frac{I}{2} + N^-)^{-1}(\frac{I}{2} - N^+))\rho^+ \\ &= b^{S,+} := -u^{\mathrm{inc}} + \chi^{-1}L^-(\frac{I}{2} + N^-)^{-1}\partial_{\mathbf{n}}u^{\mathrm{inc}}, \end{aligned}$$

and ρ^- is obtained by: $\rho^- = A^{S,-}\rho^+ + b^{S,-} := (L^-)^{-1}(L^+\rho^+ + u^{\mathrm{inc}})$.

In the case of circular cylinders, then a numerical approach as the previous spectral one in Section 2 allows to simplify the expression of the integral operators and to solve the linear system efficiently by a direct or iterative solver [30].

3.3. Numerical example

We now provide on Figure 3 an example of computation based on the use of μ-diff. We consider $M = 30$ dielectric scatterers ($0.5 \leq a_p \leq 1$, $\varepsilon_p = 0.2\varepsilon_0$, $\mu_p = \mu_0$, $p = 1, ..., M$) illuminated by a plane wave of incidence $(-1, 0)$, with wave number $k = 20$. On plot 3a, we report the amplitude of the total field and see that the wave inside the scatterers propagates with a lower frequency, then producing a complex scattering pattern behind the cluster. In addition, we report the RCS on Figure 3b showing that there is an energy peak in the forward direction.



(a) Amplitude of the total field. (b) RCS (dB).

Fig. 3 – Scattering by $M = 30$ dielectric disks randomly distributed in the box $]-8; 8[^2$. The incident plane is fixed by $k = 20$ and $\boldsymbol{\beta} = (-1, 0)$.

4. CONCLUSION

We explained in this paper how to efficiently solve scattering by a cluster of separated circular cylinders. The methodology is relatively general and can handle many kinds of boundary conditions. In particular, using integral equations provides the access to a wide variety of formulations. We did not develop much the explanations according to the numerical schemes since they are available in the Matlab toolbox μ-diff. Because we are using the theory of integral equations, all the current developments can be directly applied to the scattering by spheres for three-dimensional problems, in particular for acoustic waves even if similar theories exist for the EM and elastic cases [5]. However, the numerical schemes must be more deeply investigated since the structure of the matrices defining the linear system to resolve strongly depends on the spherical harmonics expansions [19].

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