Dedicated to Marius Tucsnak on the occasion of his 60^{th} anniversary

APPROXIMATION OF FEEDBACK GAINS STABILIZING VISCOUS INCOMPRESSIBLE FLUID FLOWS USING THE PSEUDO-COMPRESSIBILITY METHOD

MEHDI BADRA, JEAN-PIERRE RAYMOND

Communicated by Günter Leugering

AMS 2010 Subject Classification: 93B52, 49N10, 76D55, 93C20, 93C95.

Key words: Feedback control, pseudo-compressibility method, approximation of feedback laws, Navier-Stokes equations, uniform stabilizability.

1. INTRODUCTION

We consider the Oseen system and the linearized Boussinesq system with a distributed control, and their approximation obtained by the pseudo-compressibility method. We want to show that these systems fit with the abstract setting recently introduced in [2]. Let us first introduce the Oseen system.

Let Ω be a bounded domain in $\mathbb{R}^d$, $d = 2, 3$, with boundary Γ of class C^2, and (v_s, q_s) a stationary solution to the Navier-Stokes equations

$$(1)\quad \begin{aligned} &(v_s \cdot \nabla)v_s - \nu\Delta v_s + \nabla q_s = f_s, \quad \operatorname{div} v_s = 0 \quad \text{in} \quad \Omega,\\ &v_s = g_s \quad \text{on} \quad \Gamma. \end{aligned}$$

We consider the control Navier-Stokes system

$$(2)\quad \begin{aligned} &\frac{\partial v}{\partial t} + (v \cdot \nabla)v - \nu\Delta v + \nabla q = f_s + \chi_{\mathcal{O}}\, u, \quad \operatorname{div} v = 0 \quad \text{in} \quad Q \stackrel{\text{def}}{=} \Omega \times (0, \infty),\\ &v = g_s \quad \text{on} \quad \Sigma \stackrel{\text{def}}{=} \Gamma \times (0, \infty),\\ &v(0) = v_0 = v_s + y_0 \quad \text{on} \quad \Omega, \end{aligned}$$

M. Badra and J.-P. Raymond are partially supported by the ANR-Project IFSMACS (ANR 15-CE40.0010).

where $\mathcal{O}$ is a nonempty open domain in Ω. In this setting, u is the control variable, y_0 is a perturbation in the initial condition. The characteristic function of $\mathcal{O}$, denoted by $\chi_{\mathcal{O}}$, is used to localize the control in $\mathcal{O}$. The nonlinear system satisfied by $(y,p)=(v,q)-(v_s,q_s)$ is

$$(3)\qquad \begin{aligned} &\frac{\partial y}{\partial t}+(y\cdot\nabla)v_s+(v_s\cdot\nabla)y+\kappa\,(y\cdot\nabla)y-\nu\Delta y+\nabla p=\chi_{\mathcal{O}}\,u \ \text{ in } Q,\\ &\operatorname{div} y=0 \text{ in } Q,\quad y=0 \ \text{ on } \Sigma,\\ &y(0)=y_0 \ \text{ in } \Omega, \end{aligned}$$

with $\kappa=1$. The Navier-Stokes system linearized around v_s corresponds to system (3) with $\kappa=0$. We approximate system (3) with $\kappa=0$ by the pseudo-compressibility method, which consists of replacing the divergence condition $\operatorname{div} y=0$ by $\operatorname{div} y_\varepsilon+\varepsilon p_\varepsilon=0$, with $\varepsilon\in(0,1)$:

$$(4)\qquad \begin{aligned} &\frac{\partial y_\varepsilon}{\partial t}-\nu\Delta y_\varepsilon+(y_\varepsilon\cdot\nabla)v_s^\varepsilon+(v_s^\varepsilon\cdot\nabla)y_\varepsilon+\nabla p_\varepsilon=\chi_{\mathcal{O}}\,u \ \text{ in } Q,\\ &\operatorname{div} y_\varepsilon+\varepsilon p_\varepsilon=0 \text{ in } Q,\quad y_\varepsilon=0 \ \text{ on } \Sigma,\quad y_\varepsilon(0)=y_0 \ \text{ in } \Omega. \end{aligned}$$

We already know that system (3) with $\kappa=0$ can be stabilized, with any prescribed exponential decay rate, by a control u in feedback form. Such feedback laws $K\in\mathcal{L}(L^2(\Omega;\mathbb{R}^d))$ can be determined by solving optimal control problems (see, *e.g.*, [4]). In this paper, we would like to show that feedback laws stabilizing the pseudo-compressible Oseen system may be used to stabilize the Oseen system. For that, we are going to use results proved in [2] for abstract parabolic systems. If we denote by P the orthogonal projection in $L^2(\Omega;\mathbb{R}^d)$ onto $Z=\{z\in L^2(\Omega;\mathbb{R}^d)\mid \operatorname{div} z=0,\ z\cdot n=0 \text{ on } \Gamma\}$, where n is the unit normal to Γ exterior to Ω, and if $Py_0=y_0\in Z$, we know that the solution y to system (3) with $\kappa=0$ obeys $y=Py$, and that y is the solution of a control system of the form (see (35))

$$(5)\qquad y'=Ay+Bu \quad \text{in } (0,\infty),\quad y(0)=y_0\in Z.$$

In this setting, the Oseen operator $(A,\mathcal{D}(A))$ is the infinitesimal generator of an analytic semigroup $(e^{tA})_{t\geq 0}$ on Z, the control operator B is a bounded operator from the control space space $U=L^2(\Omega;\mathbb{R}^d)$ into Z.

Similarly y_ε, the solution to (4), is the solution of an equation of the form

$$(6)\qquad y_\varepsilon'=A_\varepsilon y_\varepsilon+B_\varepsilon u \quad \text{in } (0,\infty),\quad y_\varepsilon(0)=y_0\in Z_\varepsilon.$$

Here, $(A_\varepsilon,\mathcal{D}(A_\varepsilon))$ is the infinitesimal generator of an analytic semigroup $(e^{tA_\varepsilon})_{t\geq 0}$ on $Z_\varepsilon=L^2(\Omega;\mathbb{R}^d)$, the control operator B_ε is a bounded operator from U into Z_ε.

In order to study the approximation of feedback laws stabilizing (A,B) in Z by feedback laws stabilizing $(A_\varepsilon,B_\varepsilon)$ in Z_ε, it is convenient to use the abstract functional framework introduced in [2], dealing with the case of nonconforming approximation, that is the case when $Z_\varepsilon \not\subset Z$. It consists of introducing a larger Hilbert space H containing both Z and Z_ε and a projector P from H onto Z and a projector P_ε from H onto Z_ε satisfying some assumptions.

In the particular case we are dealing with, the natural choice consists of choosing $H = L^2(\Omega;\mathbb{R}^d) = Z_\varepsilon$, P the Leray projector already introduced above, and P_ε the identity in H.

In [2], we have shown that when the triplets (A, B, P) and $(A_\varepsilon, B_\varepsilon, P_\varepsilon)$ satisfy Assumptions $(H_1) - (H_4)$ stated in Section 2, then the stabilizability of the pair (A, B) in Z implies the stabilizability of the pair $(A_\varepsilon, B_\varepsilon)$ in Z_ε, uniformly with respect to $\varepsilon \in (0, \varepsilon_0)$, for $\varepsilon_0 > 0$ small enough. We recall this result in Theorem 2.1-(i). We have also shown that Riccati based feedback stabilizing $(A_\varepsilon, B_\varepsilon)$ in Z_ε may be used to stabilize (A, B) in Z. This is a consequence of Theorems 2.1-(ii) and 2.2.

In order to apply these results to the Oseen system and the pseudo-compressible Oseen system, we show in Section 3 that the different assumptions $(H_1)-(H_4)$ of Section 2 are satisfied by the triplets (A, B, P) and $(A_\varepsilon, B_\varepsilon, P_\varepsilon)$.

We extend these results to the Boussinesq system and to the pseudo-compressible Boussinesq system in Section 4. The approximation of the Boussinesq system by the pseudo-compressibility method has been recently studied in [8]. It is a control system very similar to the Oseen system. The uniform stabilizability is established in [8] as a consequence of uniform Carleman estimates. Here, we establish both the uniform stabilizability of the approximate Oseen system and convergence rates for the approximation of stabilizing feedbacks of the Oseen system by stabilizing feedbacks of the approximate Oseen system. Thus, our results complete those obtained in [8].

2. ABSTRACT SETTING

Throughout the paper, C denotes a generic constant which may vary from one line to another one, but is independent of the parameter ε, of $\lambda \in \mathbb{C}$, and of $t \in (0, \infty)$.

In this section, H and U are Hilbert spaces, and Z and Z_ε are Hilbert subspaces of H. The operator $P \in \mathcal{L}(H)$ is the orthogonal projector in H onto Z, and $P_\varepsilon \in \mathcal{L}(H)$ is the orthogonal projector in H onto Z_ε. (This case is a simplification of the functional setting studied in [2] where we consider the case where P and P_ε are not necessarily orthogonal projectors.)

We will use the following identifications

$$H' \equiv H, \quad Z' = Z, \quad Z'_\varepsilon = Z'_\varepsilon, \quad U' = U.$$

We assume that the triplet (A, B, P) and its approximation $(A_\varepsilon, B_\varepsilon, P_\varepsilon)$ satisfy the approximations stated below.

(H_1) **Analyticity of $(A, \mathcal{D}(A))$ and uniform analyticity of $(A_\varepsilon, \mathcal{D}(A_\varepsilon))$.** There exists $(\omega_0, \delta) \in \mathbb{R}\times]0, \pi/2[$ such that:

$$\begin{aligned} &\{\omega_0\} + \mathbb{S}_{\pi/2+\delta} \subset \rho(A), \\ &\|(\lambda I - A)^{-1}\|_{\mathcal{L}(Z)} \le \frac{C}{|\lambda - \omega_0|}, \quad \forall \lambda \in \{\omega_0\} + \mathbb{S}_{\pi/2+\delta}, \end{aligned} \tag{7}$$

and

$$\begin{aligned} &\{\omega_0\} + \mathbb{S}_{\pi/2+\delta} \subset \rho(A_\varepsilon), \\ &\|(\lambda I - A_\varepsilon)^{-1}\|_{\mathcal{L}(Z_\varepsilon)} \le \frac{C}{|\lambda - \omega_0|}, \quad \forall \lambda \in \{\omega_0\} + \mathbb{S}_{\pi/2+\delta}, \quad \forall \varepsilon \in (0, 1), \end{aligned} \tag{8}$$

where, for $\delta \in]0, \pi/2[$, the subset $\mathbb{S}_{\pi/2+\delta} \subset \mathbb{C}$ denotes the sector $\{\lambda \in \mathbb{C} \mid |\arg(\lambda)| < \pi/2 + \delta\}$, and $\rho(A)$ and $\rho(A_\varepsilon)$ are the resolvent sets of A and A_ε respectively.

We set

$$\widehat{A} \overset{\text{def}}{=} A - \lambda_0 I \quad \text{and} \quad \widehat{A}_\varepsilon \overset{\text{def}}{=} A_\varepsilon - \lambda_0 I, \quad \text{with } \lambda_0 > \omega_0.$$

(H_2) **Convergence rate of A_ε towards A.** The pair $(\widehat{A}, \widehat{A}_\varepsilon)$ satisfies the following approximation assumption

$$\|\widehat{A}^{-1} P - \widehat{A}_\varepsilon^{-1} P_\varepsilon\|_{\mathcal{L}(H)} \le C\varepsilon^s, \quad \forall \varepsilon \in (0, 1), \quad \text{with } s > 0. \tag{9}$$

(H_3) **Uniform bound for B_ε.** The control operator B belongs to $\mathcal{L}(U, Z)$. The family of operators $B_\varepsilon \in \mathcal{L}(U, Z_\varepsilon)$ satisfies the uniform bound

$$\sup_{\varepsilon \in (0,1)} \|B_\varepsilon\|_{\mathcal{L}(U,H)} < +\infty. \tag{10}$$

(H_4) **Convergence rate of B_ε towards B.** The pair (B, B_ε) satisfies the following approximation assumption

$$\|\widehat{A}^{-1} B - \widehat{A}_\varepsilon^{-1} B_\varepsilon\|_{\mathcal{L}(U,H)} \le C\varepsilon^r \quad \text{for all } \varepsilon \in (0, 1), \quad \text{with } 0 < r \le s. \tag{11}$$

In [2], we have proved the following result.

Theorem 2.1. *Let us assume that $(H_1) - (H_4)$ are satisfied.*

(i) *Let us assume that there exist $F \in \mathcal{L}(Z, U)$ and $\omega_F > 0$ such that $A + \omega_F I + BF$ is the infinitesimal generator of an exponentially stable strongly* *continuous semigroup on Z, and that $F_\varepsilon \in \mathcal{L}(Z_\varepsilon, U)$, with $\varepsilon \in (0,1)$, is a family satisfying*

$$\|FP - F_\varepsilon\|_{\mathcal{L}(Z_\varepsilon,U)} \leq \sigma(\varepsilon), \tag{12}$$

where σ is a continuous function from $\mathbb{R}^+$ into $\mathbb{R}^+$ satisfying $\sigma(0) = 0$. We set $A_F \overset{def}{=} A + BF$ and $A_{\varepsilon,F_\varepsilon} \overset{def}{=} A_\varepsilon + B_\varepsilon F_\varepsilon$.

Then, there exist $\varrho > 0$ and $\varepsilon_0 \in (0,1)$ such that, for all $t > 0$ and all $\varepsilon \in (0,\varepsilon_0)$, we have:

$$\|e^{A_{\varepsilon,F_\varepsilon} t}\|_{\mathcal{L}(Z_\varepsilon)} \leq Ce^{-\omega_{F,\varepsilon} t}, \quad \forall \varepsilon \in (0,\varepsilon_0), \tag{13}$$

$$\|e^{A_F t}P - e^{A_{\varepsilon,F_\varepsilon} t}P_\varepsilon\|_{\mathcal{L}(H)} \leq C\frac{e^{-\omega_{F,\varepsilon} t}}{t^{r/s}}(\varepsilon^r + \sigma(\varepsilon)), \quad \forall \varepsilon \in (0,\varepsilon_0), \tag{14}$$

with $\omega_{F,\varepsilon} \overset{def}{=} \omega_F - \varrho(\varepsilon^r + \sigma(\varepsilon))$. In particular, (13) *and* (14) *hold to be true for $F_\varepsilon = FP$, with $\sigma \equiv 0$.*

(ii) *Let us assume that $(F_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ of $\mathcal{L}(Z_\varepsilon, U)$, with $0 < \varepsilon_0 \leq 1$, is a family satisfying*

$$\|F_\varepsilon P_\varepsilon\|_{\mathcal{L}(Z)} \leq C, \quad \forall \varepsilon \in (0,\varepsilon_0), \tag{15}$$

and that $(F^{(\varepsilon)})_{0<\varepsilon<\varepsilon_0}$ is family of $\mathcal{L}(Z,U)$ satisfying

$$\|F_\varepsilon P_\varepsilon - F^{(\varepsilon)}\|_{\mathcal{L}(Z,U)} \leq \sigma(\varepsilon), \quad \forall \varepsilon \in (0,\varepsilon_0), \tag{16}$$

where σ is a continuous function from $\mathbb{R}^+$ into $\mathbb{R}^+$ satisfying $\sigma(0) = 0$.

In addition, we assume that there exists $\omega_F > 0$ such that the family $((e^{t(A_\varepsilon + \omega_F I + B_\varepsilon F_\varepsilon)})_{t\geq 0})_{0<\varepsilon<\varepsilon_0}$, of strongly continuous semigroups on Z_ε, is exponentially stable on Z_ε, uniformly in $\varepsilon \in (0,\varepsilon_0)$.

As in part (i), *we set $A_{F^{(\varepsilon)}} \overset{def}{=} A + BF^{(\varepsilon)}$, $A_{\varepsilon,F_\varepsilon} \overset{def}{=} A_\varepsilon + B_\varepsilon F_\varepsilon$, and $\omega_{F,\varepsilon} \overset{def}{=} \omega_F - \varrho(\varepsilon^r + \sigma(\varepsilon))$.*

Then, there exist $\varrho > 0$ and $\widetilde{\varepsilon}_0 \in (0,1)$ such that

$$\|e^{A_{F^{(\varepsilon)}} t}\|_{\mathcal{L}(Z_\varepsilon)} \leq Ce^{-\omega_{F,\varepsilon} t}, \quad \forall \varepsilon \in (0,\widetilde{\varepsilon}_0), \tag{17}$$

$$\|e^{A_{\varepsilon,F_\varepsilon} t}P - e^{A_{F^{(\varepsilon)}} t}P_\varepsilon\|_{\mathcal{L}(H)} \leq C\frac{e^{-\omega_{F,\varepsilon} t}}{t^{r/s}}(\varepsilon^r + \sigma(\varepsilon)), \quad \forall \varepsilon \in (0,\widetilde{\varepsilon}_0). \tag{18}$$

In particular, (17) *and* (18) *hold to be true for $F^{(\varepsilon)} = F_\varepsilon P_\varepsilon$, with $\sigma \equiv 0$.*

In order to apply the results stated in Theorem 2.1 to systems (5) and (6), we have to find a feedback law K for the pair (A, B) and a feedback law K_ε for the pair $(A_\varepsilon, B_\varepsilon)$ for which (12) will be satisfied.

For that, the idea in [2] is to determine K and K_ε by the so-called LQR theory. We consider an output operator $\mathcal{C} \in \mathcal{L}(H)$. We assume that

$$\begin{array}{l}\text{The pair } (A,B) \text{ is stabilizable in } Z,\\ \text{The pair } (A, \mathcal{C}\,|_Z) \text{ is detectable in } Z.\end{array} \tag{19}$$

We consider the algebraic Riccati equation

$$(20)\qquad \begin{aligned} &\Pi \in \mathcal{L}(Z), \quad \Pi = \Pi^* \geq 0, \quad B^*\Pi \in \mathcal{L}(Z,U),\\ &\Pi A + A^*\Pi - \Pi B B^*\Pi + P^*\mathcal{C}^*\mathcal{C}P = 0. \end{aligned}$$

Notice that here, we are in the case where $P = P^*$. According to [2, Section 5], this equation admits a unique solution $\Pi \in \mathcal{L}(Z)$. Moreover, the semigroup generated by $A_\Pi \stackrel{\text{def}}{=} A - BB^*\Pi$ is analytic and exponentially stable on Z, and we choose $\omega_\Pi > 0$ such that

$$(21)\qquad \text{The semigroup } e^{(A_\Pi + \omega_\Pi I)t} \text{ is exponentially stable on } Z.$$

The algebraic Riccati equation of the approximate system is

$$(22)\qquad \begin{aligned} &\Pi_\varepsilon \in \mathcal{L}(Z_\varepsilon), \quad \Pi_\varepsilon = \Pi_\varepsilon^* \geq 0, \quad B_\varepsilon^*\Pi_\varepsilon \in \mathcal{L}(Z_\varepsilon,U),\\ &\Pi_\varepsilon A_\varepsilon + A_\varepsilon^*\Pi_\varepsilon - \Pi_\varepsilon B_\varepsilon B_\varepsilon^*\Pi_\varepsilon + P_\varepsilon^*\mathcal{C}^*\mathcal{C}P_\varepsilon = 0. \end{aligned}$$

Notice that here, $P_\varepsilon^* = P_\varepsilon$. Since $(A, \mathcal{C}\,|_Z)$ is detectable in Z, the pair $(A^*, P^*\mathcal{C}^*)$ is stabilizable in $Z' \equiv Z$. Thus, using Theorem 2.1(i), we can show that $(A_\varepsilon^*, P_\varepsilon^*\mathcal{C}^*)$ is stabilizable in $Z_\varepsilon' \equiv Z_\varepsilon$, uniformly with respect to $\varepsilon \in (0, \varepsilon_0)$, for some $\varepsilon_0 \in (0,1)$. Therefore, the pair $(A_\varepsilon, \mathcal{C}\,|_{Z_\varepsilon})$ is detectable in Z_ε, uniformly with respect to $\varepsilon \in (0, \varepsilon_0)$.

In [2] (in the case a bounded control operator $B \in \mathcal{L}(U,Z)$ and a uniformly bounded family of approximate control operators $\sup_{\varepsilon\in(0,1)} \|B_\varepsilon\|_{\mathcal{L}(U,H)} < \infty$), we have proved the following results.

THEOREM 2.2. *Let us assume that* $(H_1) - (H_4)$ *are satisfied. There exist* $\omega_\Pi^* > 0$ *and* $\varepsilon_0 \in (0,1)$ *such that*

$$(23)\qquad \sup_{\varepsilon\in(0,\varepsilon_0)} \|e^{A_{\varepsilon,\Pi_\varepsilon} t}\|_{\mathcal{L}(Z_\varepsilon)} \leq C e^{-\omega_\Pi^* t}, \quad \forall t \geq 0.$$

If $0 < r < s$, *we have*

$$(24)\qquad \|\Pi P - \Pi_\varepsilon P_\varepsilon\|_{\mathcal{L}(H)} \leq C\varepsilon^r, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

If $0 < r = s$, *we have*

$$(25)\qquad \|\Pi P - \Pi_\varepsilon P_\varepsilon\|_{\mathcal{L}(H)} \leq C\varepsilon^s\,|\ln(\varepsilon)|, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

Moreover, if $0 < r \leq s$, *we have*

$$(26)\qquad \|B^*\Pi P - B_\varepsilon^*\Pi_\varepsilon P_\varepsilon\|_{\mathcal{L}(H,U)} \leq C\varepsilon^r |\ln\varepsilon|, \quad \text{for all } \varepsilon \in (0, \varepsilon_0).$$

The uniform estimate (23) is proved in [2] when the assumption $(19)_2$ is replaced by the stronger one

The operator $\mathcal{C}\mathcal{C}^*$ is invertible in H.

However, in the case when $B \in \mathcal{L}(U, Z)$, and when (10) is satisfied, the proof of (23) is well known, see, *e.g.*, [7, Theorem 2.1].

In the special case that we consider here, the estimate (26), deduced from [2], can be improved because we have $B^* = \chi_{\mathcal{O}} P$ for the Oseen system, and $B^*_\varepsilon = \chi_{\mathcal{O}}$ for the pseudo-compressible Oseen system, and similar expressions for the Boussinesq system and the pseudo-compressible Boussinesq system. Because of that we have

$$
\begin{aligned}
&\|B^*\Pi P - B^*_\varepsilon \Pi_\varepsilon P_\varepsilon\|_{\mathcal{L}(H,U)} = \|\chi_{\mathcal{O}}\Pi P - \chi_{\mathcal{O}}\Pi_\varepsilon P_\varepsilon\|_{\mathcal{L}(H,U)} \\
&\le C\|\Pi P - \Pi_\varepsilon P_\varepsilon\|_{\mathcal{L}(H)}.
\end{aligned}
$$

Therefore, we have

$$
\|B^*\Pi P - B^*_\varepsilon \Pi_\varepsilon P_\varepsilon\|_{\mathcal{L}(H,U)} \le C\varepsilon^r \quad \text{if } 0 < r < s, \quad \text{for all } \varepsilon \in (0, \varepsilon_0). \tag{27}
$$

If $0 < r = s$, we do not improve (26).

3. THE OSEEN SYSTEM AND THE PSEUDO-COMPRESSIBLE OSEEN SYSTEM

We recall that Ω is a bounded domain in $\mathbb{R}^d$, $d = 2, 3$, of class C^2.

We assume that the solution (v_s, q_s) to equation (1) satisfies

$$
v_s \in H^1(\Omega; \mathbb{R}^d) \cap L^\infty(\Omega; \mathbb{R}^d) \quad \text{and} \quad q_s \in L^2(\Omega). \tag{28}
$$

3.1. The Oseen system

We denote with boldface spaces of vector fields: $\mathbf{H}^\ell(\Omega) = H^\ell(\Omega; \mathbb{R}^d)$, $\mathbf{H}^\ell(\Gamma) = H^\ell(\Gamma; \mathbb{R}^d)$ for $\ell \ge 0$, $\mathbf{L}^k(Q) = L^k(Q; \mathbb{R}^d)$ for $1 \le k \le \infty$. We introduce the spaces

$$
\begin{aligned}
&V^0_n(\Omega) = \Big\{ z \in \mathbf{L}^2(\Omega) \mid \ \operatorname{div} z = 0 \text{ in } \Omega,\ z \cdot n = 0 \text{ on } \Gamma \Big\}, \\
&V^1_0(\Omega) = \mathbf{H}^1_0(\Omega) \cap V^0_n(\Omega), \\
&V^\ell(\Gamma) = \Big\{ z \in \mathbf{H}^\ell(\Gamma) \mid \int_\Gamma z \cdot n \, \mathrm{d}x = 0 \Big\}, \quad \text{for } \ell \ge 0, \\
&L^2_0(\Omega) = \Big\{ q \in L^2(\Omega) \mid \int_\Omega q \, \mathrm{d}x = 0 \Big\}, \\
&\mathcal{H}^\ell(\Omega) = H^\ell(\Omega) \cap L^2_0(\Omega), \quad \text{for } \ell \ge 0.
\end{aligned}
$$

We have the following orthogonal decomposition

$$
\mathbf{L}^2(\Omega) = V^0_n(\Omega) \oplus \operatorname{grad} H^1(\Omega).
$$

The orthogonal projector in $\mathbf{L}^2(\Omega)$ onto $V_n^0(\Omega)$ will be denoted by P, and called the Leray projector for the above decomposition.

We are going to use the functional framework of Section 2 with

$$H = \mathbf{L}^2(\Omega), \quad Z = V_n^0(\Omega), \quad Z_\varepsilon = \mathbf{L}^2(\Omega), \quad P_\varepsilon = I_H \text{ (the identity in } H).$$

For all w_s satisfying

$$\|w_s\|_{\mathbf{H}^1(\Omega)} \le \|v_s\|_{\mathbf{H}^1(\Omega)} + 1, \tag{29}$$

we set

$$\begin{aligned} &A_{w_s} z = P(\nu\Delta z - (z\cdot\nabla)w_s - (w_s\cdot\nabla)z) \\ &\text{and} \quad \mathcal{D}(A_{w_s}) = V^2(\Omega)\cap V_0^1(\Omega). \end{aligned} \tag{30}$$

For any $w_s \in \mathbf{H}^1(\Omega)$, we introduce the continuous bilinear form a_{w_s} on $\mathbf{H}^1(\Omega)\times\mathbf{H}^1(\Omega)$ defined by

$$a_{w_s}(z,\zeta) = \int_\Omega (\nu\nabla z : \nabla\zeta + (w_s\cdot\nabla)z\cdot\zeta + (z\cdot\nabla)w_s\cdot\zeta)\,\mathrm{d}x. \tag{31}$$

According to [3, Section 3.1], we can choose $\omega_0 > 0$ such that

$$\omega_0\|z\|^2_{\mathbf{L}^2(\Omega)} + a_{w_s}(z,z) \ge \frac{\nu}{2}\|z\|^2_{\mathbf{H}^1(\Omega)}, \tag{32}$$

for all $z \in \mathbf{H}^1(\Omega)$ and all w_s satisfying (29). In particular, for all w_s satisfying (29), we have

$$\big((-A_{w_s} + \omega_0 I)z, z\big)_{\mathbf{L}^2(\Omega)} \ge \frac{\nu}{2}\|z\|^2_{\mathbf{H}^1(\Omega)}, \quad \forall z \in \mathcal{D}(A_{w_s}). \tag{33}$$

THEOREM 3.1. *For all w_s satisfying (29), the operator $(A_{w_s}, \mathcal{D}(A_{w_s}))$ is the infinitesimal generator of an analytic semigroup on $V_n^0(\Omega)$. There exists a sector $\{\omega_0\} + \mathbb{S}_{\pi/2+\delta}$, independent of w_s, such that (7) is satisfied for all w_s satisfying (29).*

We set

$$(A, \mathcal{D}(A)) = (A_{v_s}, \mathcal{D}(A_{v_s})) \quad \text{and} \quad B = P\chi_{\mathcal{O}}. \tag{34}$$

When $\kappa = 0$ and $y_0 \in V_n^0(\Omega)$, the solutions y to system (3), over a time interval $(0,T)$, can be defined as variational solutions in $W(0,T;V_0^1(\Omega),V^{-1}(\Omega)) = L^2(0,T;V_0^1(\Omega))\cap H^1(0,T;V^{-1}(\Omega))$, where $V^{-1}(\Omega)$ is the dual of $V_0^1(\Omega)$ with $V_n^0(\Omega)$ as pivot space. We can easily prove that, when $\kappa = 0$ and $y_0 \in V_n^0(\Omega)$, system (3) admits a unique solution in $W(0,T;V_0^1(\Omega),V^{-1}(\Omega))$, for all $T > 0$. Moreover this solution obeys $Py(t) = y(t)$ for all $t \ge 0$. Using the Oseen operator $(A,\mathcal{D}(A))$, system (3) can be written in the form

$$y' = Ay + Bu, \quad y(0) = y_0, \tag{35}$$

in the sense that the variational solution y to system (3) coincides with the mild solution to equation (35).

3.2. **The pseudo-compressible Oseen system**

We assume that

$$\|v_s^\varepsilon - v_s\|_{\mathbf{H}^1(\Omega)} \le C_s\,\varepsilon, \quad \text{for all } \varepsilon \in (0,1). \tag{36}$$

Let us notice that such an estimate is consistent with the one for the approximation of the Stokes system by the pseudo-compressibility method stated in [10, (1.5)].

We set $\varepsilon_0 = 1/C_s$. Thus, for all $\varepsilon \in (0,\varepsilon_0)$, $w_s = v_s^\varepsilon$ satisfies (29).

To rewrite system (4) as an evolution equation, we introduce the penalized Oseen operator A_ε (an unbounded operator in $\mathbf{L}^2(\Omega)$) defined by

$$\begin{aligned} &\mathcal{D}(A_\varepsilon) = \mathbf{H}^2(\Omega) \cap \mathbf{H}^1_0(\Omega), \\ &A_\varepsilon v = \nu\Delta v - (v\cdot\nabla)v_s^\varepsilon - (v_s^\varepsilon\cdot\nabla)v + \tfrac{1}{\varepsilon}\nabla(\operatorname{div} v). \end{aligned} \tag{37}$$

The control operator $B_\varepsilon \in \mathcal{L}(U,H)$ is defined by

$$B_\varepsilon = \chi_{\mathcal{O}}. \tag{38}$$

System (4) can be rewritten in the form

$$y_\varepsilon' = A_\varepsilon y_\varepsilon + B_\varepsilon u, \quad y_\varepsilon(0) = y_0. \tag{39}$$

According to (32), we have

$$\big((-A_\varepsilon + \omega_0 I)z, z\big)_{\mathbf{L}^2(\Omega)} \ge \frac{\nu}{2}\|z\|^2_{\mathbf{H}^1(\Omega)},$$

for all $z \in \mathcal{D}(A_\varepsilon)$, all $\varepsilon \in (0,\varepsilon_0)$.

Proposition 3.2. *We choose $\lambda_0 > \omega_0$. There exists $\widetilde{\varepsilon}_0 \in (0,\varepsilon_0)$ such that the following results hold.*

1. *A_ε is the infinitesimal generator of an analytic semigroup on $\mathbf{L}^2(\Omega)$, uniformly in $\varepsilon \in (0,\widetilde{\varepsilon}_0)$.*

2. *For all $\varepsilon_0 \in (0,\widetilde{\varepsilon}_0)$, $(A_\varepsilon^*, \mathcal{D}(A_\varepsilon^*))$, the adjoint of $(A_\varepsilon, \mathcal{D}(A_\varepsilon))$ in $Z_\varepsilon = \mathbf{L}^2(\Omega)$, is defined by*

$$\begin{aligned} &\mathcal{D}(A_\varepsilon^*) = \mathbf{H}^2(\Omega) \cap \mathbf{H}^1_0(\Omega), \\ &A_\varepsilon^*\phi = \nu\Delta\phi_\varepsilon - (\nabla v_s^\varepsilon)^T\phi_\varepsilon + (v_s^\varepsilon\cdot\nabla)\phi_\varepsilon + \tfrac{1}{\varepsilon}\nabla(\operatorname{div}\phi) + \operatorname{div}(v_s^\varepsilon)\,\phi. \end{aligned} \tag{40}$$

3. *The following bounds hold, uniformly in $\varepsilon \in (0,\widetilde{\varepsilon}_0)$:*

$$\|z\|_{\mathbf{H}^2(\Omega)} + \tfrac{1}{\varepsilon}\|\operatorname{div} z\|_{H^1(\Omega)} \le C\|(\lambda_0 I - A_\varepsilon)z\|_{\mathbf{L}^2(\Omega)}, \quad \textit{for all } z \in \mathcal{D}(A_\varepsilon),$$

$$\|\phi\|_{\mathbf{H}^2(\Omega)} + \tfrac{1}{\varepsilon}\|\operatorname{div}\phi\|_{H^1(\Omega)} \le C\|(\lambda_0 I - A_\varepsilon^*)\phi\|_{\mathbf{L}^2(\Omega)}, \quad \textit{for all } \phi \in \mathcal{D}(A_\varepsilon^*).$$

4. *The following approximation property holds:*

$$\|(\lambda_0 I - A)^{-1}P - (\lambda_0 I - A_\varepsilon)^{-1}\|_{\mathcal{L}(\mathbf{L}^2(\Omega))} \le C\varepsilon, \quad \forall\varepsilon \in (0,\widetilde{\varepsilon}_0). \tag{41}$$

5. *The control operators B and B_ε satisfy the following approximation property*

$$\|(\lambda_0 I - A)^{-1}B - (\lambda_0 I - A_\varepsilon)^{-1}B_\varepsilon\|_{\mathcal{L}(\mathbf{L}^2(\Omega))} \leq C\varepsilon, \quad \forall \varepsilon \in (0, \widetilde{\varepsilon}_0). \tag{42}$$

Proof. Proof of 1, 2 and 3. The first three points are proved in [10] when A is the Stokes operator (that is when $v_s = 0$), and extended to the Oseen operator in [1, Theorem 5].

Proof of 4 and 5. For $f \in \mathbf{L}^2(\Omega)$, we consider the solution v of the equation $(\lambda_0 I - A_{v_s^\varepsilon})v = f$, or equivalently

$$\begin{aligned} &\lambda_0 v - \nu\Delta v + (v\cdot\nabla)v_s^\varepsilon + (v_s^\varepsilon\cdot\nabla)v + \nabla q = f \ \text{ in } \Omega, \\ &\operatorname{div} v = 0 \text{ in } \Omega, \quad v = 0 \ \text{ on } \Gamma, \end{aligned} \tag{43}$$

and the solution w of the equation $(\lambda_0 I - A_{v_s})w = f$, that is the solution to

$$\begin{aligned} &\lambda_0 w - \nu\Delta w + (v\cdot\nabla)v_s + (v_s\cdot\nabla)w + \nabla p = f \ \text{ in } \Omega, \\ &\operatorname{div} w = 0 \text{ in } \Omega, \quad w = 0 \ \text{ on } \Gamma. \end{aligned} \tag{44}$$

We also consider the solution v_ε of the equation $(\lambda_0 I - A_\varepsilon)v_\varepsilon = f$, or equivalently

$$\begin{aligned} &\lambda_0 v_\varepsilon - \nu\Delta v_\varepsilon + (v_\varepsilon\cdot\nabla)v_s^\varepsilon + (v_s^\varepsilon\cdot\nabla)v_\varepsilon + \nabla q_\varepsilon = f \ \text{ in } \Omega, \\ &\operatorname{div} v_\varepsilon + \varepsilon q_\varepsilon = 0 \text{ in } \Omega, \quad v_\varepsilon = 0 \ \text{ on } \Gamma. \end{aligned} \tag{45}$$

We need to estimate $\|w - v_\varepsilon\|_{\mathbf{L}^2(\Omega)}$. From [3, Proposition 3.6, (3.19)] and (36), it follows that

$$\|w - v\|_{\mathbf{L}^2(\Omega)} \leq C\|v_s - v_s^\varepsilon\|_{\mathbf{H}^1(\Omega)}\,\|f\|_{\mathbf{L}^2(\Omega)} \leq C\,\varepsilon\,\|f\|_{\mathbf{L}^2(\Omega)}.$$

Let us now estimate $\|v - v_\varepsilon\|_{\mathbf{L}^2(\Omega)}$. For that, we introduce the solution Φ_ε of the adjoint equation $(\lambda_0 I - A_\varepsilon^*)\Phi_\varepsilon = v_\varepsilon - v$, or equivalently

$$\begin{aligned} &\lambda_0\Phi_\varepsilon - \nu\Delta\Phi_\varepsilon + (\nabla v_s^\varepsilon)^T\Phi_\varepsilon - (v_s^\varepsilon\cdot\nabla)\Phi_\varepsilon + \nabla\psi_\varepsilon - \operatorname{div}(v_s^\varepsilon)\Phi_\varepsilon \\ &\qquad\qquad = v_\varepsilon - v \ \text{ in } \Omega, \\ &\operatorname{div}\Phi_\varepsilon + \varepsilon\psi_\varepsilon = 0 \text{ in } \Omega, \quad \Phi_\varepsilon = 0 \ \text{ on } \Gamma. \end{aligned} \tag{46}$$

Next, we write the system satisfied by $v_\varepsilon - v$ by substracting (43) from (45), and we multiply by Φ_ε. With a Green formula, we obtain

$$\int_\Omega |v_\varepsilon - v|^2 \mathrm{d}x = \varepsilon\int_\Omega q\psi_\varepsilon \mathrm{d}x \leq \varepsilon\|q\|_{L^2(\Omega)}\|\psi_\varepsilon\|_{H^1(\Omega)}. \tag{47}$$

From [3, Proposition 3.5], it follows that the pressure $q_{v_s^\varepsilon} = q$ in equation (43) satisfies

$$\|q_{v_s^\varepsilon}\|_{H^1(\Omega)} \leq C\,\|f\|_{\mathbf{L}^2(\Omega)},$$

for all $\varepsilon \in (0, \varepsilon_0)$, because in that case (36) leads to $\|v_s^\varepsilon\|_{\mathbf{H}^1(\Omega)} \leq \|v_s\|_{\mathbf{H}^1(\Omega)} + 1$.

Due to Proposition 3.2(2), $\psi_\varepsilon = -\frac{1}{\varepsilon}\mathrm{div}\,\Phi_\varepsilon$ satisfies

$$\|\psi_\varepsilon\|_{H^1(\Omega)} \le C\|v_\varepsilon - v\|_{\mathbf{L}^2(\Omega)}. \tag{48}$$

This completes the proof of point 4.

To prove point 5, it is sufficient to notice that

$$(\lambda_0 I - A)^{-1}Bu - (\lambda_0 I - A_\varepsilon)^{-1}B_\varepsilon u = \left((\lambda_0 I - A)^{-1}P - (\lambda_0 I - A_\varepsilon)^{-1}\right)\chi_{\mathcal{O}}\, u,$$

and to apply point 4. □

Let (A, B) be the pair of operators introduced in (34). Let $(A_\varepsilon, B_\varepsilon)$ be the pair of operators introduced in (37) and (38).

With [5, Theorem A.3] and Assumption (28), it follows that the pair (A, B) is stabilizable in Z.

Let $\mathcal{C} \in \mathcal{L}(\mathbf{L}^2(\Omega))$ be an operator such that $(19)_2$ is satisfied. Let Π be the solution of (20) and Π_ε be the solution of (22). Let us consider the following closed-loop Oseen system

$$\begin{aligned} &\frac{\partial y}{\partial t} - \nu\Delta y + (y\cdot\nabla)v_s + (v_s\cdot\nabla)y + \nabla p = -\chi_{\mathcal{O}}\Pi y \ \text{ in } Q,\\ &\mathrm{div}\, y = 0 \text{ in } Q, \qquad y = 0 \ \text{ on } \Sigma,\\ &Py(0) = Py_0 \ \text{ in } \Omega, \end{aligned} \tag{49}$$

and the approximate closed-loop system

$$\begin{aligned} &\frac{\partial y_\varepsilon}{\partial t} - \nu\Delta y_\varepsilon + (y_\varepsilon\cdot\nabla)v_s^\varepsilon + (v_s^\varepsilon\cdot\nabla)y_\varepsilon + \nabla p_\varepsilon = -\chi_{\mathcal{O}}\,\Pi_\varepsilon y_\varepsilon \ \text{ in } Q,\\ &\mathrm{div}\, y_\varepsilon + \varepsilon p_\varepsilon = 0 \text{ in } Q, \qquad y_\varepsilon = 0 \ \text{ on } \Sigma,\\ &y_\varepsilon(0) = y_0 \ \text{ in } \Omega. \end{aligned} \tag{50}$$

Theorem 3.3. *For all $y_0 \in V_n^0(\Omega)$, the closed-loop system* (49) *admits a unique solution y, the closed-loop system* (50) *admits a unique solution y_ε. There exist $\omega_\Pi > 0$, $\varrho > 0$, and $\varepsilon_0 \in (0,1)$, such that, for all $t \in (0,\infty)$, we have*

$$\begin{aligned} &\|y_\varepsilon(t)\|_{\mathbf{L}^2(\Omega)} \le Ce^{(-\omega_\Pi + \varrho\varepsilon|\ln\varepsilon|)t}\|y_0\|_{\mathbf{L}^2(\Omega)}, \quad \forall\varepsilon\in(0,\varepsilon_0),\\ &\|y_\varepsilon(t) - y(t)\|_{\mathbf{L}^2(\Omega)} \le C\,\frac{e^{(-\omega_\Pi+\varrho\varepsilon|\ln\varepsilon|)t}}{t}\,\varepsilon|\ln\varepsilon|\|y_0\|_{\mathbf{L}^2(\Omega)}, \quad \forall\varepsilon\in(0,\varepsilon_0). \end{aligned} \tag{51}$$

We assume that ε_0 is chosen small enough so that $\varrho\varepsilon|\ln\varepsilon| < \omega_\Pi/2$ for all $\varepsilon\in(0,\varepsilon_0)$.

Proof. The assumptions $(H_1)-(H_4)$ are satisfied by the triplet (A, B, P), with $H = \mathbf{L}^2(\Omega)$, $U = \mathbf{L}^2(\Omega)$, and $Z = V_n^0(\Omega)$, and by the triplet $(A_\varepsilon, B_\varepsilon, P_\varepsilon)$, with $Z_\varepsilon = \mathbf{L}^2(\Omega)$, $P_\varepsilon = I_H$, and with $s = r = 1$. Thus, from (26) in Theorem 2.2, it follows that there exists $\varepsilon_0 > 0$ such that

$$\|\chi_{\mathcal{O}}\, \Pi\, P - \chi_{\mathcal{O}}\, \Pi_\varepsilon\|_{\mathcal{L}(H,U)} \le C\, \varepsilon |\ln \varepsilon|, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

Thus, (51) is a direct consequence of (13) and (14) in Theorem 2.1(i), with $\sigma(\varepsilon) = \varepsilon|\ln \varepsilon|$. □

COROLLARY 3.4. *Let $\varepsilon_0 \in (0,1)$ be the parameter introduced in Theorem 3.3. For all $\varepsilon \in (0, \varepsilon_0)$, the solution y to the closed-loop system* (49) *and the solution y_ε to the closed-loop system* (50) *satisfy*

$$\|y_\varepsilon - y\|_{L^p(0,\infty,\mathbf{L}^2(\Omega))} \le C_p\, \varepsilon^{1/p} |\ln \varepsilon|^{1/p} \|y_0\|_{\mathbf{L}^2(\Omega)}, \quad \forall p \in (1, \infty).$$

Proof. With (51), if $1 < p < \infty$, by a direct calculation we obtain

$$\begin{aligned}
&\|y_\varepsilon(t) - y(t)\|_{L^p(0,\infty;\mathbf{L}^2(\Omega))} \\
&\le C\|y_0\|_{\mathbf{L}^2(\Omega)} \left(\int_0^{\varepsilon|\ln(\varepsilon)|} \mathrm{d}t + \varepsilon^p |\ln(\varepsilon)|^p \int_{\varepsilon|\ln(\varepsilon)|}^1 \frac{\mathrm{d}t}{t^p} \right. \\
&\qquad \left. + \varepsilon^p |\ln(\varepsilon)|^p \int_1^\infty e^{(-\omega_\Pi/2)pt} \mathrm{d}t \right)^{1/p} \\
&\le C\|y_0\|_{\mathbf{L}^2(\Omega)} (\varepsilon^{1/p} |\ln(\varepsilon)|^{1/p} + \varepsilon|\ln(\varepsilon)|) \le C\|y_0\|_{\mathbf{L}^2(\Omega)} \varepsilon^{1/p} |\ln(\varepsilon)|^{1/p}.
\end{aligned} \tag{52}$$

Therefore, for all $\varepsilon \in (0, \varepsilon_0)$, we have

$$\|y_\varepsilon - y\|_{L^p(0,\infty,\mathbf{L}^2(\Omega))} \le C_p\, \varepsilon^{1/p} |\ln \varepsilon|^{1/p} \|y_0\|_{\mathbf{L}^2(\Omega)}, \quad \forall p \in (1, \infty).$$

The proof is complete. □

It is also interesting to consider the following closed-loop system

$$\begin{aligned}
&\frac{\partial y^\varepsilon}{\partial t} - \nu \Delta y^\varepsilon + (y^\varepsilon \cdot \nabla) v_s + (v_s \cdot \nabla) y^\varepsilon + \nabla p = -\chi_{\mathcal{O}} \Pi_\varepsilon y^\varepsilon \ \text{ in } Q, \\
&\operatorname{div} y^\varepsilon = 0 \text{ in } Q, \qquad y^\varepsilon = 0 \ \text{ on } \Sigma, \\
&P y^\varepsilon(0) = P y_0 \ \text{ in } \Omega.
\end{aligned} \tag{53}$$

We notice that the solution y^ε to this system can be expressed as follows:

$$y^\varepsilon(t) = e^{(A - B\Pi_\varepsilon)t} P y_0, \quad \forall t \ge 0.$$

In the above definition of y^ε, we used the equality $BB^*\Pi_\varepsilon = B\Pi_\varepsilon$, because $B = \chi_{\mathcal{O}}$. The difference $\|y_\varepsilon(t) - y^\varepsilon(t)\|_{\mathbf{L}^2(\Omega)}$ can be estimated as we did above, by using Theorem 2.1(ii). More precisely we have the following corollary.

COROLLARY 3.5. *Let $\varepsilon_0 \in (0,1)$ be the parameter introduced in Theorem 3.3. There exists $\widetilde{\varepsilon}_0 \in (0, \varepsilon_0)$, such that*

- *the family* $((e^{(A-B\Pi_\varepsilon)t})_{t\geq 0})_{0<\varepsilon<\widetilde{\varepsilon}_0}$*, of strongly continuous semigroups on* Z_ε*, is exponentially stable on* Z_ε*, uniformly with respect to* $\varepsilon \in (0, \widetilde{\varepsilon}_0)$*,*

- *the solution* y^ε *to the closed-loop system* (53) *and the solution* y_ε *to the closed-loop system* (50) *satisfy*

$$\|y_\varepsilon - y^\varepsilon\|_{L^p(0,\infty,\mathbf{L}^2(\Omega))} \leq C_p\, \varepsilon^{1/p} \|y_0\|_{\mathbf{L}^2(\Omega)}, \quad \forall p \in (1, \infty).$$

Proof. We first obtain a result similar to that of Theorem 3.3 to estimate $\|y_\varepsilon(t) - y^\varepsilon(t)\|_{\mathbf{L}^2(\Omega)}$, by using Theorem 2.1(ii) with $\sigma \equiv 0$. Indeed, from Theorem 3.3, it follows that

$$(e^{A_{\varepsilon,\Pi_\varepsilon} + \omega_\Pi/4t})_{t\geq 0} \quad \text{is exponentially stable on } Z_\varepsilon, \text{ uniformly in } \varepsilon \in (0, \varepsilon_0).$$

Thus, we can apply Theorem 2.1(ii) with $\sigma \equiv 0$, and there exists $\widetilde{\varrho} > 0$ and $\widetilde{\varepsilon}_0 \in (0, \varepsilon_0)$ such that the family $((e^{(A-B\Pi_\varepsilon)t})_{t\geq 0})_{0<\varepsilon<\widetilde{\varepsilon}_0}$, of strongly continuous semigroups on Z, is exponentially stable on Z, uniformly with respect to $\varepsilon \in (0, \widetilde{\varepsilon}_0)$, and such that we have

$$\|y_\varepsilon(t) - y^\varepsilon(t)\|_{\mathbf{L}^2(\Omega)} \leq C\, \frac{e^{(-\omega_\Pi/4 + \widetilde{\varrho}\varepsilon)t}}{t}\, \varepsilon \|y_0\|_{\mathbf{L}^2(\Omega)}, \quad \forall \varepsilon \in (0, \widetilde{\varepsilon}_0), \quad \forall t \in (0, \infty).$$

We can conclude with the same calculations as in the proof of Corollary 3.4. □

4. THE BOUSSINESQ SYSTEM AND THE PSEUDO-COMPRESSIBLE BOUSSINESQ SYSTEM

In this section, Ω satisfies the assumptions of Section 3, and $\mathcal{O}$ is a nonempty open domain in Ω.

4.1. The Boussinesq system

We consider the stationary Boussinesq system

$$\begin{aligned}
&(v_s \cdot \nabla) v_s - \nu \Delta v_s + \nabla q_s = f_s + \vec{\beta}\, \tau_s, \quad \operatorname{div} v_s = 0 \quad \text{in} \quad \Omega, \\
&v_s = h_s \quad \text{on} \quad \Gamma, \\
&-\mu \Delta \tau_s + v_s \cdot \nabla \tau_s = g_s, \quad \text{in} \quad \Omega, \quad \tau_s = k_s \quad \text{on} \quad \Gamma,
\end{aligned} \tag{54}$$

with $\nu > 0$, $\mu > 0$, and $\vec{\beta}$ is a vector in $\mathbb{R}^d$. We assume that system (54) admits a variational solution $(v_s, q_s, \tau_s) \in \mathbf{H}^1(\Omega) \times L^2(\Omega) \times H^1(\Omega)$, and that in addition

$$v_s \in \mathbf{H}^1(\Omega) \cap \mathbf{L}^\infty(\Omega) \quad \text{and} \quad \tau_s \in H^1(\Omega) \cap L^\infty(\Omega). \tag{55}$$

We consider the control Boussinesq system

$$
(56)\quad
\begin{aligned}
&\frac{\partial v}{\partial t} + (v\cdot\nabla)v - \nu\Delta v + \nabla q = f_s + \vec{\beta}\,\tau + \chi_{\mathcal{O}}\, f_c, \quad \text{in } Q,\\
&\operatorname{div} v = 0 \quad \text{in } Q, \quad v = h_s \quad \text{on } \Sigma,\\
&v(0) = v_0 = v_s + y_0 \quad \text{on} \quad \Omega,\\
&\frac{\partial \tau}{\partial t} - \mu\Delta\tau + v\cdot\nabla\tau = g_s + \chi_{\mathcal{O}}\, g_c, \quad \text{in } Q,\\
&\tau = k_s \quad \text{on} \quad \Gamma,\\
&\tau(0) = \tau_0 = \tau_s + \theta_0 \quad \text{on} \quad \Omega.
\end{aligned}
$$

In the above system $f_c \in L^2(0,\infty;\mathbf{L}^2(\Omega))$ and $g_c \in L^2(0,\infty;L^2(\Omega))$ are control variables. We will set $u = \begin{pmatrix} f_c \\ g_c \end{pmatrix}$.

The nonlinear system satisfied by $(y,p,\theta) = (v,q,\tau) - (v_s,q_s,\tau_s)$ is

(57)

$$
\begin{aligned}
&\frac{\partial y}{\partial t} + (y\cdot\nabla)v_s + (v_s\cdot\nabla)y + \kappa\,(y\cdot\nabla)y - \nu\Delta y + \nabla p = \vec{\beta}\,\theta + \chi_{\mathcal{O}}\, f_c \ \text{ in } Q,\\
&\operatorname{div} y = 0 \text{ in } Q, \quad y(0) = y_0 \ \text{ in } \Omega,\\
&y = 0 \ \text{ on } \Sigma,\\
&\frac{\partial \theta}{\partial t} - \mu\Delta\theta + y\cdot\nabla\tau_s + v_s\cdot\nabla\theta + \kappa\, y\cdot\nabla\theta = \chi_{\mathcal{O}}\, g_c, \quad \text{in } Q,\\
&\theta = 0 \quad \text{on} \quad \Sigma,\\
&\theta(0) = \theta_0 \quad \text{on} \quad \Omega,
\end{aligned}
$$

with $\kappa = 1$. The Boussinesq system linearized around (v_s, τ_s) corresponds to system (57) with $\kappa = 0$.

We set $Z = V^0_n(\Omega) \times L^2(\Omega)$, $H = \mathbf{L}^2(\Omega) \times L^2(\Omega)$, $U = \mathbf{L}^2(\Omega) \times L^2(\Omega)$, and

$$
(58)\qquad B = \begin{pmatrix} P_L\chi_{\mathcal{O}} \\ \chi_{\mathcal{O}} \end{pmatrix}, \quad P = \begin{pmatrix} P_L \\ I \end{pmatrix},
$$

where P_L is the Leray projector introduced in Section 3.

For all (w_s, ϑ_s) satisfying

$$
(59)\qquad \|w_s\|_{\mathbf{H}^1(\Omega)} + \|\vartheta_s\|_{H^1(\Omega)} \le \|v_s\|_{\mathbf{H}^1(\Omega)} + \|\tau_s\|_{H^1(\Omega)} + 1,
$$

we set

$$
(60)\qquad
\begin{aligned}
&A_{(w_s,\vartheta_s)} \begin{pmatrix} y \\ \theta \end{pmatrix} = \begin{pmatrix} P_L\left(\nu\Delta y - (y\cdot\nabla)w_s - (w_s\cdot\nabla)y + \vec{\beta}\,\theta\right) \\ \mu\Delta\theta - y\cdot\nabla\vartheta_s - w_s\cdot\nabla\theta \end{pmatrix}\\
&\text{and} \quad \mathcal{D}(A_{(w_s,\vartheta_s)}) = (V^2(\Omega)\cap V^1_0(\Omega)) \times (H^2(\Omega)\cap H^1_0(\Omega)).
\end{aligned}
$$

For any $(w_s, \vartheta_s) \in \mathbf{H}^1(\Omega) \times H^1(\Omega)$, we introduce the continuous bilinear form $a_{(w_s,\vartheta_s)}$ on $\big(\mathbf{H}^1(\Omega) \times H^1(\Omega)\big) \times \big(\mathbf{H}^1(\Omega) \times H^1(\Omega)\big)$ defined by

$$\begin{aligned} & a_{(w_s,\vartheta_s)}(z,\zeta) = a_{(w_s,\vartheta_s)}((y,\theta),(\phi,\xi)) \\ & = \int_\Omega \Big(\nu\nabla y : \nabla\phi + (w_s\cdot\nabla)y\cdot\phi + (y\cdot\nabla)w_s\cdot\phi + (\vec{\beta}\cdot\phi)\theta\Big)\,\mathrm{d}x \\ & \quad + \int_\Omega (\mu\nabla\theta\cdot\nabla\xi + (w_s\cdot\nabla\theta)\xi + (v\cdot\nabla\vartheta_s)\xi)\,\mathrm{d}x, \end{aligned} \tag{61}$$

with $z = (y,\theta)$ and $\zeta = (\phi,\xi)$. As in Section 3, we choose $\omega_0 > 0$ such that

$$\omega_0\|z\|^2_{\mathbf{L}^2(\Omega)\times L^2(\Omega)} + a_{(w_s,\vartheta_s)}(z,z) \geq \frac{\min(\nu,\mu)}{2}\|z\|^2_{\mathbf{H}^1(\Omega)\times H^1(\Omega)}, \tag{62}$$

for all $z \in \mathbf{H}^1(\Omega) \times H^1(\Omega)$ and all (w_s,ϑ_s) satisfying (59). In particular, for all (w_s,ϑ_s) satisfying (59), we have

$$((-A_{(w_s,\vartheta_s)} + \omega_0 I)z, z) \geq \frac{\min(\nu,\mu)}{2}\|z\|^2_{\mathbf{H}^1(\Omega)\times H^1(\Omega)}, \quad \forall z \in \mathcal{D}(A_{(w_s,\vartheta_s)}). \tag{63}$$

THEOREM 4.1. *For all (w_s,ϑ_s) satisfying (59), the operator $(A_{(w_s,\vartheta_s)}, \mathcal{D}(A_{(w_s,\vartheta_s)}))$ is the infinitesimal generator of an analytic semigroup on Z. There exists a sector $\{\omega_0\} + \mathbb{S}_{\pi/2+\delta}$, independent of (w_s,ϑ_s), such that (7) is satisfied for all (w_s,ϑ_s) satisfying (59).*

By setting $z = \begin{pmatrix} y \\ \theta \end{pmatrix}$ and $z_0 = \begin{pmatrix} y_0 \\ \theta_0 \end{pmatrix}$, system (57) with $\kappa = 0$ can be written in the form

$$z' = Az + Bu, \quad z(0) = z_0, \tag{64}$$

where $(A,\mathcal{D}(A)) = (A_{(v_s,\tau_s)}, \mathcal{D}(A_{(v_s,\tau_s)}))$, and B is defined in (58).

The operator $(A^*,\mathcal{D}(A^*))$, the adjoint of $(A,\mathcal{D}(A))$, is needed to study the stabilizability of the pair (A,B). Following [9], we can verify that $(A^*,\mathcal{D}(A^*))$ is defined by

$$\begin{aligned} & \mathcal{D}(A^*) = V^2(\Omega) \cap V^1_0(\Omega) \times (H^2(\Omega) \cap H^1_0(\Omega)), \\ & A^*\begin{pmatrix} \phi \\ \xi \end{pmatrix} = \begin{pmatrix} P_L\left(\nu\Delta\phi - (\nabla v_s)^T\phi + (v_s\cdot\nabla)\phi - \nabla\tau_s\,\xi\right) \\ \mu\Delta\xi + v_s\cdot\nabla\xi + \vec{\beta}\cdot\phi \end{pmatrix}. \end{aligned} \tag{65}$$

4.2. The pseudo-compressible Boussinesq system

We choose $(v_s^\varepsilon, \tau_s^\varepsilon) \in \mathbf{H}^1(\Omega) \times H^1(\Omega)$ such that

$$\|v_s^\varepsilon - v_s\|_{\mathbf{H}^1(\Omega)} + \|\tau_s^\varepsilon - \tau_s\|_{H^1(\Omega)} \leq C_s\,\varepsilon, \quad \text{for all } \varepsilon \in (0,1). \tag{66}$$

We set $\varepsilon_0 = 1/C_s$. Thus, for all $\varepsilon \in (0,\varepsilon_0)$, $(w_s,\vartheta_s) = (v_s^\varepsilon, \tau_s^\varepsilon)$ satisfies (59).

We approximate system (57) with $\kappa = 0$ by the pseudo-compressibility method, which consists of replacing the divergence condition $\operatorname{div} y = 0$ by $\operatorname{div} y + \varepsilon p = 0$, with $\varepsilon \in (0,1)$:

$$
\begin{aligned}
&\frac{\partial y}{\partial t} + (y\cdot\nabla)v_s^\varepsilon + (v_s^\varepsilon\cdot\nabla)y - \nu\Delta y + \nabla p = \vec{\beta}\theta + \chi_{\mathcal{O}}\, f_c \ \text{ in } Q,\\
&\operatorname{div} y + \varepsilon p = 0 \text{ in } Q, \qquad y = 0 \ \text{ on } \Sigma,\\
&y(0) = y_0 \ \text{ in } \Omega,\\
&\frac{\partial\theta}{\partial t} - \mu\Delta\theta + y\cdot\nabla\tau_s^\varepsilon + v_s^\varepsilon\cdot\nabla\theta = \chi_{\mathcal{O}}\, g_c, \quad \text{in } Q,\\
&\theta = 0 \quad \text{on} \quad \Sigma,\\
&\theta(0) = \theta_0 \quad \text{on} \quad \Omega.
\end{aligned}
\tag{67}
$$

We set $B_\varepsilon = \begin{pmatrix}\chi_{\mathcal{O}}\\ \chi_{\mathcal{O}}\end{pmatrix}$, $Z_\varepsilon = H = \mathbf{L}^2(\Omega)\times L^2(\Omega)$, and $P_\varepsilon = I_H$, where I_H is the identity in H.

To rewrite System (67) as an evolution equation, we introduce the penalized Boussinesq operator A_ε (an unbounded operator in $H = \mathbf{L}^2(\Omega)\times L^2(\Omega)$) defined by

$$
\begin{aligned}
&\mathcal{D}(A_\varepsilon) = (\mathbf{H}^2(\Omega)\cap \mathbf{H}^1_0(\Omega))\times(H^2(\Omega)\cap H^1_0(\Omega)),\\
&A_\varepsilon\begin{pmatrix} y\\ \theta\end{pmatrix} = \begin{pmatrix}\nu\Delta y - (y\cdot\nabla)v_s^\varepsilon - (v_s^\varepsilon\cdot\nabla)y) + \frac{1}{\varepsilon}\nabla(\operatorname{div} y) + \vec{\beta}\theta\\ \mu\Delta\theta - y\cdot\nabla\tau_s^\varepsilon - v_s^\varepsilon\cdot\nabla\theta\end{pmatrix}.
\end{aligned}
\tag{68}
$$

System (67) can be rewritten in the form

$$
\begin{pmatrix} y_\varepsilon'\\ \theta_\varepsilon'\end{pmatrix} = A_\varepsilon\begin{pmatrix} y_\varepsilon\\ \theta_\varepsilon\end{pmatrix} + B_\varepsilon u, \quad \begin{pmatrix} y_\varepsilon(0)\\ \theta_\varepsilon(0)\end{pmatrix} = \begin{pmatrix} y_0\\ \theta_0\end{pmatrix}.
\tag{69}
$$

Proposition 4.2. *We choose $\lambda_0 > \omega_0$. There exists $\widetilde{\varepsilon}_0 \in (0,\varepsilon_0)$ such that the following results hold.*

1. A_ε is the infinitesimal generator of an analytic semigroup on H, uniformly in $\varepsilon\in(0,\varepsilon_0)$, and analytic estimate (8) is satisfied for all $\varepsilon\in(0,\varepsilon_0)$.

2. For all $\varepsilon\in(0,\widetilde{\varepsilon}_0)$, $(A_\varepsilon^*, \mathcal{D}(A_\varepsilon^*))$, the adjoint of $(A_\varepsilon,\mathcal{D}(A_\varepsilon))$, is defined by

$$
\begin{aligned}
&\mathcal{D}(A_\varepsilon^*) = (\mathbf{H}^2(\Omega)\cap \mathbf{H}^1_0(\Omega))\times(H^2(\Omega)\cap H^1_0(\Omega)),\\
&A_\varepsilon^*\begin{pmatrix}\phi\\ \xi\end{pmatrix} = \begin{pmatrix}\nu\Delta\phi - (\nabla v_s^\varepsilon)^T\phi + (v_s^\varepsilon\cdot\nabla)\phi + \frac{1}{\varepsilon}\nabla(\operatorname{div}\phi) - \nabla\tau_s\,\xi + \operatorname{div}(\nabla v_s^\varepsilon)\phi\\ \mu\Delta\xi + v_s^\varepsilon\cdot\nabla\xi + \vec{\beta}\cdot\phi\end{pmatrix}.
\end{aligned}
\tag{70}
$$

3. *The following bound holds, uniformly in* $\varepsilon \in (0, \widetilde{\varepsilon}_0)$:
$\|y\|_{\mathbf{H}^2(\Omega)} + \frac{1}{\varepsilon}\|\mathrm{div}\, y\|_{H^1(\Omega)} + \|\theta\|_{\mathbf{H}^2(\Omega)} \le C\|(\lambda_0 I - A_\varepsilon)z\|_{\mathbf{L}^2(\Omega)}$,
for all $z = (y, \theta) \in \mathcal{D}(A_\varepsilon)$,
$\|\phi\|_{\mathbf{H}^2(\Omega)} + \frac{1}{\varepsilon}\|\mathrm{div}\, \phi\|_{H^1(\Omega)} + \|\xi\|_{H^2(\Omega)} \le C\|(\lambda_0 I - A_\varepsilon^*)\zeta\|_{\mathbf{L}^2(\Omega)}$,
for all $\zeta = (\phi, \xi) \in \mathcal{D}(A_\varepsilon^*)$.
4. *The following approximation inequality holds:*

$$\|(\lambda_0 I - A)^{-1}P - (\lambda_0 I - A_\varepsilon)^{-1}\|_{\mathcal{L}(H)} \le C\varepsilon, \quad \forall \varepsilon \in (0, \varepsilon_0). \tag{71}$$

5. *The control operators* B *and* B_ε *satisfy the rate of convergence*

$$\|(\lambda_0 I - A)^{-1}B - (\lambda_0 I - A_\varepsilon)^{-1}B_\varepsilon\|_{\mathcal{L}(U,H)} \le C\varepsilon, \quad \forall \varepsilon \in (0, \varepsilon_0). \tag{72}$$

Proof. The proof is mainly based on Proposition 3.2. □

4.3. **Closed-loop linearized Boussinesq system and its approximation**

Let (A, B) be the pair of operators introduced in Section 4.1. Let $(A_\varepsilon, B_\varepsilon)$ be the pair of operators introduced in Section 4.2.

PROPOSITION 4.3. *The pair* (A, B) *is stabilizable in* Z.

Proof. According to [5], it is enough to prove that any solution to the system

$$(\phi, \xi) \in \mathcal{D}(A^*),\ \lambda \in \mathbb{C}, \quad A^* \begin{pmatrix} \phi \\ \xi \end{pmatrix} = \lambda \begin{pmatrix} \phi \\ \xi \end{pmatrix},$$

such that

$$\phi = 0 \quad \text{and} \quad \xi = 0, \quad \text{in } \mathcal{O},$$

obeys

$$\phi = 0 \quad \text{and} \quad \xi = 0, \quad \text{in } \Omega.$$

Such a unique continuation result is proved in [9, Appendix A.1]. However, in [9, Appendix A.1], it is assumed that $(v_s, \tau_s) \in \mathbf{H}^{2+\alpha_0}(\Omega) \times H^{2+\alpha_0}(\Omega)$, for some $\alpha_0 > 0$. We want to obtain the same result but with assumption (55).

For that, we can verify that $(A^*, \mathcal{D}(A^*))$ defined in (65) can be written in the form

$$A^* \begin{pmatrix} \phi \\ \xi \end{pmatrix} = \begin{pmatrix} P_L\left(\nu\Delta\phi + (\nabla\phi)^T v_s + (v_s\cdot\nabla)\phi + \tau_s\nabla\xi\right) \\ \mu\Delta\xi + v_s\cdot\nabla\xi + \vec{\beta}\cdot\phi \end{pmatrix}.$$

It is the kind of transformation used in [5] to prove a unique continuation result for the Oseen system. Next using the Carleman estimate stated in [5, Theorem A.3] for the fluid equation, combined with the Carleman estimate for the stationary convection diffusion equation stated in [6, Theorem 1.2] (as we did in [9, Appendix A.1]), we can obtained a combined Carleman estimate leading to the desired unique continuation result. □

Let $\mathcal{C} \in \mathcal{L}(H)$ be an operator such that $(19)_2$ is satisfied. Let $\Pi = \begin{pmatrix} \Pi_y \\ \Pi_\theta \end{pmatrix}$ be the solution of (20) and $\Pi_\varepsilon = \begin{pmatrix} \Pi_{\varepsilon,y} \\ \Pi_{\varepsilon,\theta} \end{pmatrix}$ be the solution of (22).

Let us consider the following closed-loop linearized Boussinesq system

$$
(73)\qquad
\begin{aligned}
&\frac{\partial y}{\partial t} + (y\cdot\nabla)v_s + (v_s\cdot\nabla)y - \nu\Delta y + \nabla p = \vec{\beta}\,\theta - \chi_{\mathcal{O}}\,\Pi_y(y,\theta) \ \text{ in } Q,\\
&\operatorname{div} y = 0 \text{ in } Q, \qquad y(0) = y_0 \ \text{ in } \Omega,\\
&y = 0 \ \text{ on } \Sigma,\\
&\frac{\partial \theta}{\partial t} - \mu\Delta\theta + y\cdot\nabla\tau_s + v_s\cdot\nabla\theta + \kappa\, y\cdot\nabla\theta = -\chi_{\mathcal{O}}\,\Pi_\theta(y,\theta) \quad \text{in } Q,\\
&\theta = 0 \quad \text{on} \quad \Sigma,\\
&\theta(0) = \theta_0 \quad \text{on} \quad \Omega,
\end{aligned}
$$

and the approximate closed-loop system

$$
(74)\qquad
\begin{aligned}
&\frac{\partial y_\varepsilon}{\partial t} + (y_\varepsilon\cdot\nabla)v_s^\varepsilon + (v_s^\varepsilon\cdot\nabla)y_\varepsilon - \nu\Delta y_\varepsilon + \nabla p = \vec{\beta}\theta - \chi_{\mathcal{O}}\,\Pi_{\varepsilon,y}(y_\varepsilon,\theta_\varepsilon) \ \text{ in } Q,\\
&\operatorname{div} y_\varepsilon + \varepsilon p_\varepsilon = 0 \text{ in } Q, \qquad y_\varepsilon = 0 \ \text{ on } \Sigma,\\
&y_\varepsilon(0) = y_0 \ \text{ in } \Omega,\\
&\frac{\partial \theta_\varepsilon}{\partial t} - \mu\Delta\theta + y_\varepsilon\cdot\nabla\tau_s^\varepsilon + v_s^\varepsilon\cdot\nabla\theta_\varepsilon = -\chi_{\mathcal{O}}\,\Pi_{\varepsilon,\theta}(y_\varepsilon,\theta_\varepsilon) \quad \text{in } Q,\\
&\theta = 0 \quad \text{on} \quad \Sigma,\\
&\theta_\varepsilon(0) = \theta_0 \quad \text{on} \quad \Omega.
\end{aligned}
$$

THEOREM 4.4. *For all $z_0 = (y_0,\theta_0) \in V_n^0(\Omega)\times L^2(\Omega)$, the closed-loop system (49) admits a unique solution, the closed-loop system (50) admits a unique solution. There exist $\omega_\Pi > 0$, $\varrho > 0$, and $\widetilde{\varepsilon}_0 \in (0,1)$ such that these solutions satisfy*

$$
(75)\qquad
\begin{aligned}
&\|z_\varepsilon(t)\|_{\mathbf{L}^2(\Omega)} \le C e^{(-\omega_\Pi + \varrho\varepsilon|\ln\varepsilon|)t}\|z_0\|_H, \quad \forall \varepsilon \in (0,\widetilde{\varepsilon}_0),\\
&\|z_\varepsilon(t) - z(t)\|_{\mathbf{L}^2(\Omega)} \le C\, \frac{e^{(-\omega_\Pi + \varrho\varepsilon|\ln\varepsilon|)t}}{t}\, \varepsilon|\ln\varepsilon|\|z_0\|_H, \quad \forall \varepsilon \in (0,\widetilde{\varepsilon}_0).
\end{aligned}
$$

We assume that $\widetilde{\varepsilon}_0$ is chosen small enough so that $\varrho\varepsilon|\ln\varepsilon| < \omega_\Pi/2$ for all $\varepsilon \in (0,\widetilde{\varepsilon}_0)$.

Proof. The proof is similar to that of Theorem 3.3. $\square$

As at the end of Section 3, we can prove that, for all $\varepsilon \in (0, \widetilde{\varepsilon}_0)$, we have

$$\|z_\varepsilon - z\|_{L^p(0,\infty,H)} \leq C_p\, \varepsilon^{1/p} |\ln \varepsilon|^{1/p} \|z_0\|_H, \quad \forall p \in (1, \infty).$$

REFERENCES

[1] M. Badra, J.-M. Buchot, and L. Thevenet, *Méthode de pénalisation pour le contrôle frontière des équations de Navier-Stokes*. Journal Européen des Systèmes Automatisés, J.E.S.A. **45** (2011), 1–34.

[2] M. Badra and J.-P. Raymond, *Nonconforming approximation of feedback gains stabilizing parabolic equations*. Preprint, 2021.

[3] M. Badra and J.-P. Raymond, *Numerical approximation of feedback gains stabilizing the Oseen system*. Preprint, 2021.

[4] M. Badra and T. Takahashi, *Stabilization of parabolic nonlinear systems with finite dimensional feedback or dynamical controllers. application to the Navier-Stokes system*. SIAM J. Control Optim. **49** (2011), 420–463.

[5] M. Badra and T. Takahashi, *On the Fattorini criterion for approximate controllability and stabilizability of parabolic systems*. ESAIM Control Optim. Calc. Var. **20** (2014), 924–956.

[6] O. Y. Imanuvilov and J.-P. Puel, *Global Carleman estimates for weak solutions of elliptic nonhomogeneous Dirichlet problems*. C. R. Math. Acad. Sci. Paris **335** (2002), 33–38.

[7] K. Ito, *Strong convergence and convergence rates of approximating solutions for algebraic Riccati equations in Hilbert spaces*. ICASE Report 87–31, 1987.

[8] K. Le Balc'h and M. Tucsnak, *A penalty approach to the infinite horizon LQR optimal control problem for the linearized Boussinesq system*. ESAIM Control Optim. Calc. Var. **27** (2021), Paper No. 17.

[9] M. Ramaswamy, J.-P. Raymond, and A. Roy, *Boundary feedback stabilization of the Boussinesq system with mixed boundary conditions* J. Differential Equations **266** (2019), 4268–4304.

[10] J. Shen, *On error estimates of the penalty method for unsteady Navier-Stokes equations*. SIAM J. Numer. Anal. **32** (1995), 386–403.

Université Paul Sabatier
Institut de Mathématiques de Toulouse & CNRS, 31062 Toulouse Cedex, France
mehdi.badra@math.univ-toulouse.fr,
jean-pierre.raymond@math.univ-toulouse.fr