Dedicated to Marius Tucsnak on the occasion of his 60^{th} anniversary

A UNIFIED STRATEGY FOR OBSERVABILITY OF WAVES IN AN ANNULUS WITH VARIOUS BOUNDARY CONDITIONS

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In this work, we develop a unified strategy which allows to derive observability results for waves in an annulus when the observation is done on the external boundary and under various boundary conditions at the internal boundary. Our approach is based on the fact that the observability of a linear abstract conservative system is equivalent to a suitable resolvent estimate. We develop a full machinery to derive suitable resolvent estimates under some weak assumptions on the boundary condition on the internal sphere, which are shown to be close to sharp. In fact, our strategy allows to exclude the existence of trapped rays close to the internal sphere when the boundary conditions satisfy a suitable assumption. As an application, we apply our results to several wave models.

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1. INTRODUCTION

The goal of this article is to provide a robust strategy to address the observability of the wave equation in a bounded domain with various boundary conditions. In particular, we would like to exhibit sufficient conditions on the boundary conditions which guarantee the observability of the equation.

In order to describe our strategy and to avoid difficulties linked with the geometry, we consider the case of an annulus of $\mathbb{R}^2$, *i.e.* $A(R_0, R_1) = B(R_1) \setminus \overline{B(R_0)}$, where $0 < R_0 < R_1$ and $B(R)$ denotes the open ball of $\mathbb{R}^2$ with center at the origin and radius $R > 0$, which is observed through the whole external boundary $S(R_1)$ ($S(R)$ denotes the sphere of $\mathbb{R}^2$ of radius R).

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We will assume that the boundary conditions, at the external boundary $S(R_1)$, are homogeneous Dirichlet boundary conditions, since this does not create any specific difficulty as the observation is done there.

Our goal is to give a functional setting allowing different types of boundary conditions on the internal boundary $S(R_0)$, including in particular those given below, and to obtain observability results at once for all those cases.

1.1. **Examples**

Let us mention below several examples of interest on which our method will apply. We emphasize that, in each case, the domain in which the wave equation takes place is the annulus $A(R_0, R_1) = B(R_1) \setminus \overline{B(R_0)}$, where $0 < R_0 < R_1$, and that the observation is performed on the external boundary $S(R_1)$.

Fourier Boundary Conditions. Let $\alpha \geqslant 0$ and consider the wave equation with Fourier boundary conditions on $S(R_0)$:

$$(1.1) \qquad \begin{cases} \partial_{tt} y(t,x) - \Delta y(t,x) = 0, & \text{in } (0,T) \times A(R_0, R_1), \\ y(t,x) = 0, & \text{on } (0,T) \times S(R_1), \\ \partial_\nu y(t,x) + \alpha y(t,x) = 0, & \text{on } (0,T) \times S(R_0), \\ (y(0,\cdot), \partial_t y(0,\cdot)) = (y^0, y^1), & \text{in } A(R_0, R_1), \end{cases}$$

where ∂_ν denotes the normal derivative in the outward direction, *i.e.* $\partial_\nu = -\partial_r$ on $S(R_0)$, and $\partial_\nu = \partial_r$ on $S(R_1)$.

The question of observability for (1.1) through the boundary $S(R_1)$ we shall address asks the existence of a time $T > 0$ and a constant $C > 0$ such that for all $(y^0, y^1) \in H^1(A(R_0, R_1)) \times L^2(A(R_0, R_1))$ with $y^0 = 0$ on $S(R_1)$, the solution y of (1.1) satisfies

$$(1.2) \qquad \left\| (y^0, y^1) \right\|_{H^1(A(R_0,R_1)) \times L^2(A(R_0,R_1))} \leqslant C \| \partial_\nu y \|_{L^2(0,T;L^2(S(R_1)))}.$$

For system (1.1), the observability inequality (1.2) is known to hold for T large enough, namely $T > 2\sqrt{R_1^2 - R_0^2}$, and to be false for $T < 2\sqrt{R_1^2 - R_0^2}$ (see [3]).

Note that, in this example and in the following ones, the observation is always done through the external boundary $S(R_1)$, where the boundary condition is the homogeneous Dirichlet condition, meaning that we observe $\partial_\nu y$ in $L^2(0,T; L^2(S(R_1)))$.

Waves on the boundary. Likewise, we shall consider the case in which a wave equation also takes place on the internal boundary $S(R_0)$, also known as dynamic Wentzell boundary conditions. Namely, for $\alpha > 0$ and $\beta > 0$, we consider the equation

$$
\begin{cases}
\partial_{tt} y(t,x) - \Delta y(t,x) = 0, & \text{in } (0,T) \times A(R_0,R_1), \\
y(t,x) = 0, & \text{on } (0,T) \times S(R_1), \\
z(t,x) = y(t,x), & \text{on } (0,T) \times S(R_0), \\
\partial_\nu y(t,x) + \alpha \partial_{tt} z(t,x) - \beta \Delta_{S(R_0)} z = 0, & \text{on } (0,T) \times S(R_0), \\
(y(0,\cdot), \partial_t y(0,\cdot)) = (y^0, y^1), & \text{in } A(R_0,R_1), \\
(z(0,\cdot), \partial_t z(0,\cdot)) = (z^0, z^1), & \text{on } S(R_0).
\end{cases} \tag{1.3}
$$

Here, the operator $\Delta_{S(R_0)}$ simply coincides with $\partial_{\theta\theta}/R_0^2$.

The relevant observability inequality for (1.3) through $S(R_1)$ then asks the existence of a time $T > 0$ and a constant $C > 0$ such that for all $(y^0, y^1, z^0, z^1) \in H^1(A(R_0,R_1)) \times L^2(A(R_0,R_1)) \times H^1(S(R_0)) \times L^2(S(R_0))$ with $y^0|_{S(R_0)} = z^0$ and $y^0|_{S(R_1)} = 0$, the solution (y,z) of (1.3) satisfies

$$
\begin{aligned}
\left\|(y^0, y^1, z^0, z^1)\right\|_{H^1(A(R_0,R_1)) \times L^2(A(R_0,R_1)) \times H^1(S(R_0)) \times L^2(S(R_0))} \\
\leqslant C \|\partial_\nu y\|_{L^2(0,T;L^2(S(R_1)))}.
\end{aligned} \tag{1.4}
$$

Up to our knowlegde, the results we will obtain on the observability properties of (1.3) are new (see Section 2.2). We refer to [5, 13] for controllability results on closely related models.

Fractional Laplacian on the boundary. We will also study the case in which the internal boundary condition is given through a non-local boundary condition expressed in terms of powers of the Laplacian on the sphere, given for $s \in (0,1]$ as follows:

$$
\begin{cases}
\partial_{tt} y(t,x) - \Delta y(t,x) = 0, & \text{in } (0,T) \times A(R_0,R_1), \\
y(t,x) = 0, & \text{on } (0,T) \times S(R_1), \\
\partial_\nu y(t,x) + (-\Delta_{S(R_0)})^s y(t,x) = 0, & \text{on } (0,T) \times S(R_0), \\
(y(0,\cdot), \partial_t y(0,\cdot)) = (y^0, y^1), & \text{in } A(R_0,R_1),
\end{cases} \tag{1.5}
$$

Note in particular that, for $s = 1/2$, using the classical interpretation of the square root of the Laplace operator (see e.g. [8]) $(-\Delta_{S(R_0)})^{1/2}$ corresponds to the Dirichlet to Neumann map for the Laplacian in the ball $B(0,R_0)$, see also Remark 2.5 for more details.

We shall then investigate the existence of a time $T > 0$ and a constant $C > 0$ such that for all $(y^0, y^1) \in H^1(A(R_0,R_1)) \times L^2(A(R_0,R_1))$ with $y^0 = 0$ on $S(R_1)$, the corresponding solution of (1.5) satisfies:

$$
\left\|(y^0, y^1)\right\|_{H^1(A(R_0,R_1)) \times L^2(A(R_0,R_1))} \leqslant C \|\partial_\nu y\|_{L^2(0,T;L^2(S(R_1)))}. \tag{1.6}
$$

We are not aware of any result on that system in the literature.

A simplified fluid structure model. One might also consider non-local operators given as follows:

$$
(1.7)\quad \begin{cases}
\partial_{tt} y(t,x) - \Delta y(t,x) = 0, & \text{in } (0,T)\times A(R_0,R_1),\\
y(t,x) = 0, & \text{on } (0,T)\times S(R_1),\\
\partial_\nu y(t,x) = s'(t)\cdot \vec{\nu}, & \text{on } (0,T)\times S(R_0),\\
s''(t) + s(t) = -\displaystyle\int_{S(R_0)} \partial_t y(t,x)\,\vec{\nu}\,\mathrm{d}\sigma, & \text{in } (0,T),\\
(y(0,\cdot),\partial_t y(0,\cdot), s(0), s'(0)) = (y^0,y^1,s^0,s^1), & \text{in } A(R_0,R_1)\times\mathbb{R}^2\times\mathbb{R}^2,
\end{cases}
$$

where $\vec{\nu}$ is the outward normal to $A(R_0, R_1)$. This models a fluid-structure interaction problem, see [9], for which y is the velocity potential of the fluid, and the function s corresponds to the displacement of an oscillator located in the ball $B(0, R_0)$.

Here, the relevant observability inequality reads as follows: there exist a time $T > 0$ and a constant $C > 0$ such that for all $(y^0, y^1, s^0, s^1) \in H^1(A(R_0, R_1)) \times L^2(A(R_0, R_1)) \times \mathbb{R}^2 \times \mathbb{R}^2$ with $y^0 = 0$ on $S(R_1)$, the corresponding solution of (1.7) satisfies:

$$
(1.8)\quad \left\|(y^0,y^1,s^0,s^1)\right\|_{H^1(A(R_0,R_1))\times L^2(A(R_0,R_1))\times\mathbb{R}^2\times\mathbb{R}^2} \leqslant C\|\partial_\nu y\|_{L^2(0,T;L^2(S(R_1)))}.
$$

We refer to [19, 20, 11] for observability results on that model. In particular, it has been proved in [11] that this model is observable in any time $T > 2\sqrt{R_1^2 - R_0^2}$, and is not observable in any time $T < 2\sqrt{R_1^2 - R_0^2}$.

1.2. **Related results**

There are many works on the observability of wave type equations, mainly triggered by the fact that this is dual to the controllability of such systems, see e.g. [17, 16], and of course we cannot give here a complete account of the theory, but only a rapid overview.

Among the several methods employed for the study of observability for the wave systems, the first one was probably the multiplier technique, due to [14] and popularized by [16], which was used to study wave equations with homogeneous Dirichlet boundary conditions in many situations, proving in particular observability in sufficiently large time when the observation set is a part of the boundary in the shadow of a light point, *i.e.* there exists $x_0 \in \mathbb{R}^d$ such that the observation set contains $\{x \in \partial\Omega,\ (x - x_0)\cdot\nu > 0\}$. Such condition is usually called the multiplier condition or the Γ-condition of Lions. Note that this condition is satisfied in our setting, since the boundary $S(R_1)$ is in the shadow from the light point $x_0 = 0$. Still, the multiplier condition is quite restrictive on the boundary conditions required at the sphere $S(R_0)$, and it does not allow to keep track precisely of what happens on the illuminated boundary $S(R_0)$. In particular, it is not clear at all how it can be applied with the generality we are considering here.

For instance, when considering observability issues for the wave equation with Neumann boundary conditions on the internal sphere $S(R_0)$, *i.e.* (1.1) when $\alpha = 0$, the classical multiplier approach, which consists in multiplying the equation (1.1) by $x \cdot \nabla y + y/2$ and doing integration by parts, fails to prove (1.2). Indeed, these computations would yield to

$$\begin{aligned}
&\frac{1}{2}\int_0^T \int_\Omega \left(|\partial_t y(t,x)|^2 + |\nabla y(t,x)|^2\right) \mathrm{d}t\mathrm{d}x \\
&+ \int_\Omega \partial_t y(\cdot,x) \left(x \cdot \nabla y(\cdot,x) + \frac{1}{2} y(\cdot,x)\right) \mathrm{d}x \Big|_0^T \\
&+ \frac{1}{2}\int_0^T \int_{S(R_0)} |x\cdot\nu| \left(|\partial_t y(t,x)|^2 - |\nabla_\tau y(t,x)|^2\right) \mathrm{d}\sigma\mathrm{d}t \\
&\qquad = \frac{1}{2}\int_0^T \int_{S(R_1)} x\cdot\nu |\partial_\nu y(t,x)|^2 \mathrm{d}\sigma\mathrm{d}t,
\end{aligned}$$

where ∇_τ denotes the tangential derivative on $S(R_0)$, *i.e.* $\nabla_\tau y = \nabla y - (\nabla y \cdot \nu)\nu$. It turns out that this will not allow to conclude (1.2) since the third term in the left hand side

$$\begin{aligned}
&\int_0^T \int_{S(R_0)} |x\cdot\nu| \left(|\partial_t y(t,x)|^2 - |\nabla_\tau y(t,x)|^2\right) \mathrm{d}\sigma\mathrm{d}t \\
&= R_0 \int_0^T \int_{S(R_0)} \left(|\partial_t y(t,x)|^2 - |\nabla_\tau y(t,x)|^2\right) \mathrm{d}\sigma\mathrm{d}t
\end{aligned} \tag{1.9}$$

is not signed. We refer to the textbook [16, Chapitre III Section 1] for more details on this case when considering the multiplier technique.

Another approach was then developed based on the analysis of the propagation of singularities and on microlocal analysis, yielding to the works [2, 3, 6], and giving a sharp geometric condition for the observation and control of waves, now known under the name of geometric control condition, and asserting that all the rays of geometric optics meet the boundary at a non-diffractive point. However, here again, the analysis close to the part of the boundary on which the control does not act strongly depends on the boundary conditions at hand, and the literature is mainly developed only in the case of Dirichlet or Neumann boundary conditions. In particular, one should exclude the possibility of getting rays captured within the internal sphere $S(R_0)$, *i.e.* solutions whose energy is localized in a neighborhood of $S(R_0)$, see also [10].

The goal of our work is then to develop an approach which presents a detailed analysis of the phenomena appearing on the internal boundary. In order to do so with sufficient generality so that our results apply to all the models presented above, we will write the boundary condition on the internal sphere as some specific instances of microlocal operators.

1.3. **Main results**

To address observability issues for all the models presented above, our strategy relies on the fact that all the above examples can be written under an abstract form as

$$Y'(t) = AY(t), \quad t \geqslant 0, \qquad Y(0) = Y^0, \tag{1.10}$$

where A is a skew-adjoint operator defined on a Hilbert space H with domain $\mathscr{D}(A)$, and each one of the corresponding observability inequalities reads as follows: There exist a constant $C > 0$ and a time $T > 0$ such that for all $Y^0 \in \mathscr{D}(A)$, the solution Y of (1.10) satisfies

$$\|Y^0\|_H \leqslant C \,\|BY\|_{L^2(0,T;U)}\,, \tag{1.11}$$

where BY will correspond to $\partial_\nu y|_{S(R_1)}$, and $U = L^2(S(R_1))$.

In all the above cases, as we will justify later, the observation operator B is admissible (in the sense of [21, Section 4.3]) and satisfies $B \in \mathcal{L}(\mathscr{D}(A), L^2(S(R_1)))$. Note that the standard notation for observation operators is C, while B is rather used in general for control operators, see for instance [21]. Here, we will keep the notation B for the observation operator and C will denote generic constants in the following.

We shall then use a classical result (see for instance [7, 18]) which states that, under this setting, the observability inequality (1.11) is equivalent to the following resolvent estimate, also known as the Hautus test:

Theorem 1.1 ([7, 18]). *Let A be an unbounded skew-adjoint operator on a Hilbert space H with domain $\mathscr{D}(A)$, and $B \in \mathcal{L}(\mathscr{D}(A), U)$ be an admissible observation operator (in the sense of [21, Section 4.3]) for the group $(e^{tA})_{t\in\mathbb{R}}$.*

Then there exist a time $T > 0$ and a constant $C > 0$ such that the observability inequality (1.11) *holds for all solutions Y of* (1.10) *with initial datum in $\mathscr{D}(A)$ if and only if there exist $M > 0$ and $m > 0$ such that*

$$\|Y\|_H^2 \leqslant M^2 \,\|(A - i\omega)Y\|_H^2 + m^2 \,\|BY\|_U^2\,, \tag{1.12}$$

for all $\omega \in \mathbb{R}$, and for all $Y \in \mathscr{D}(A)$.

Moreover, when (1.12) *holds, the abstract system* (1.10) *is observable in the sense of* (1.11) *for any time $T > M\pi$.*

In fact, our main results below give sufficient conditions on the boundary conditions to ensure that the resolvent estimate (1.12) holds.

In order to express those conditions, due to the radial symmetry of the problems we are considering, it is interesting to decompose Y using spherical harmonics, indexed by $k \in \mathbb{Z}$:

$$Y_k(r) = \frac{1}{2\pi} \int_0^{2\pi} Y(r,\theta) e^{-ik\theta} \,\mathrm{d}\theta.$$

Then, as it will be made precise in Section 2, the resolvent condition (1.12) can be reduced to show estimates on the function y_k solution of

$$\begin{cases} \omega^2 y_k + \dfrac{1}{r}\partial_r(r\partial_r y_k) - \dfrac{k^2}{r^2} y_k = i\omega f_{1,k} + f_{2,k}, & \text{in } (R_0, R_1), \\ y_k(R_1) = 0, \\ R_0 \partial_r y_k(R_0) = \rho(\omega,k) y_k(R_0) + g_k, \end{cases} \tag{1.13}$$

where

$$\rho = \rho(\omega,k) : \mathbb{R} \times \mathbb{Z} \to \mathbb{R}, \tag{1.14}$$

is a kernel attached to the boundary conditions under considerations, and $f_{1,k} \in H^1(R_0,R_1)$ with $f_{1,k}(R_1) = 0$, $f_{2,k} \in L^2(R_0,R_1)$ and $g_k \in \mathbb{C}$.

We will show in Section 2 that, in the examples listed in Section 1.1, the kernel ρ used within the corresponding resolvent estimates can be given explicitly:

- For $(\omega,k) \in \mathbb{R}\times\mathbb{Z}$, $\rho(\omega,k) = R_0\alpha$ for the wave equation (1.1) with Fourier boundary conditions;
- For $(\omega,k) \in \mathbb{R}\times\mathbb{Z}$, $\rho(\omega,k) = R_0\left(-\alpha\omega^2 + \beta\frac{k^2}{R_0^2}\right)$ for the wave equation (1.3) with waves on the boundary;
- For $(\omega,k) \in \mathbb{R}\times\mathbb{Z}$, $\rho(\omega,k) = R_0^{1-2s}|k|^{2s}$ for the wave equation (1.5) with fractional Laplacian on the boundary;
- For $(\omega,k) \in \mathbb{R}\times\mathbb{Z}\setminus\{(\pm1,\pm1),(\pm1,\mp1)\}$, $\rho(\omega,k) = \dfrac{\pi R_0\omega^2}{\omega^2-1} 1_{|k|=1}$ for the simplified fluid structure model (1.7). (Note that, in this last example, ρ is singular when $|k| = |\omega| = 1$.)

We are now in position to state our main result :

Theorem 1.2. *Assume that there exists $M \geqslant 0$ such that*

[A1] *$\rho = \rho(\omega,k)$ is well-defined for $(\omega,k) \in \mathbb{R}\times\mathbb{Z}$ satisfying $\omega^2 R_0^2 + k^2 \geqslant M^2$ and $\omega^2 R_0^2 \leqslant k^2$ with values in $\mathbb{R}$;*

[A2] *there exist* $\varepsilon \in (0,1)$, $\mathbf{r} > 0$, *and* $\delta > 0$ *such that for all* $(\omega, k) \in \mathbb{R} \times \mathbb{Z}$ *with* $\omega^2 R_0^2 + k^2 \geqslant M^2$ *and*

$$\log\left(\frac{|k|}{|\omega| R_0}\right) \geqslant \frac{\mathbf{r}}{|k|^{2/3}}, \tag{1.15}$$

we have

$$I_\varepsilon(\omega,k)^2 \geqslant \delta\left(\left(k^2 - \omega^2 R_0^2\right) - \rho(\omega,k)^2 - \rho(\omega,k)\right), \tag{1.16}$$

where $I_\varepsilon(\omega,k)$ *is defined as*

$$I_\varepsilon(\omega,k) = \inf_{c\in[1-\varepsilon,1]} \left|\rho(\omega,k) + c\sqrt{k^2 - \omega^2 R_0^2}\right|. \tag{1.17}$$

Then there exists a constant $C > 0$ *such that for all* $(\omega, k) \in \mathbb{R} \times \mathbb{Z}$ *and any solution* y_k *of*

$$\begin{cases} \omega^2 y_k + \dfrac{1}{r}\partial_r(r\partial_r y_k) - \dfrac{k^2}{r^2} y_k = i\omega f_{1,k} + f_{2,k}, & \text{in } (R_0,R_1), \\ y_k(R_1) = 0, & \\ R_0 \partial_r y_k(R_0) = \rho(\omega,k) y_k(R_0) + g_k, & \text{if } \omega^2 R_0^2 \leqslant k^2 \\ & \text{with } \omega^2 R_0^2 + k^2 \geqslant M^2, \end{cases} \tag{1.18}$$

with $f_{1,k} \in H^1(R_0,R_1)$ *satisfying* $f_{1,k}(R_1) = 0$, $f_{2,k} \in L^2(R_0,R_1)$ *and* $g_k \in \mathbb{C}$, *it holds*

$$\begin{aligned} &\|\partial_r y_k\|_{L^2(R_0,R_1;r\mathrm{d}r)} + |k|\,\|y_k/r\|_{L^2(R_0,R_1;r\mathrm{d}r)} + |\omega|\,\|y_k\|_{L^2(R_0,R_1;r\mathrm{d}r)} \\ &\leqslant C\Big(\|\partial_r f_{1,k}\|_{L^2(R_0,R_1;r\mathrm{d}r)} + |k|\,\|f_{1,k}/r\|_{L^2(R_0,R_1;r\mathrm{d}r)} + \|f_{2,k}\|_{L^2(R_0,R_1;r\mathrm{d}r)} \\ &\qquad + |g_k| 1_{\substack{\omega^2R_0^2+k^2\geqslant M^2\\ \omega^2R_0^2\leqslant k^2}} + |\partial_r y_k(R_1)|\Big). \end{aligned} \tag{1.19}$$

Furthermore, there exist constants $C > 0$ *and* $A > 0$ *such that*

$$\begin{aligned} &1_{\substack{\omega^2R_0^2+k^2\geqslant M^2\\ \omega^2R_0^2\geqslant k^2}}\left(|\partial_r y_k(R_0)|^2 + \max\left\{\sqrt{k^2+\omega^2R_0^2}, \omega^2R_0^2 - k^2\right\}|y_k(R_0)|^2\right) \\ &+ 1_{\substack{\omega^2R_0^2+k^2\geqslant M^2\\ \omega^2R_0^2\leqslant k^2}}\Bigg(I_\varepsilon(\omega,k)^2 1_{\log\left(\frac{|k|}{|\omega|R_0}\right)\geqslant \frac{A}{|k|^{2/3}}} \\ &\qquad + \max\{|k|^{4/3}, I_1(\omega,k)^2\} 1_{\log\left(\frac{|k|}{|\omega|R_0}\right)\leqslant \frac{A}{|k|^{2/3}}}\Bigg)|y_k(R_0)|^2 \\ &+ 1_{\omega^2R_0^2+k^2\leqslant M^2}\left(|\partial_r y_k(R_0)|^2 + |y_k(R_0)|^2\right) \\ &\qquad \leqslant C\Big(\|\partial_r f_{1,k}\|_{L^2(R_0,R_1;r\mathrm{d}r)} + |k|\,\|f_{1,k}/r\|_{L^2(R_0,R_1;r\mathrm{d}r)} \\ &\qquad\quad + \|f_{2,k}\|_{L^2(R_0,R_1;r\mathrm{d}r)} + |g_k| 1_{\substack{\omega^2R_0^2+k^2\geqslant M^2\\ \omega^2R_0^2\leqslant k^2}} + |\partial_r y_k(R_1)|\Big)^2, \end{aligned} \tag{1.20}$$

where, by analogy with (1.17), $I_1 = I_1(\omega, k)$ *is defined for* $(\omega, k) \in \mathbb{R} \times \mathbb{Z}$ *with* $\omega^2 R_0^2 \leqslant k^2$ *by*

$$I_1(\omega, k) = \inf_{c \in [0,1]} \left| \rho(\omega, k) + c\sqrt{k^2 - \omega^2 R_0^2} \right|. \tag{1.21}$$

Condition (1.16) is a sufficient condition to get estimate (1.19), and of course it can be relaxed into a slightly more explicit formulation, proved in Appendix A:

COROLLARY 1.3. *With the same notations as in Theorem 1.2, all solutions* y_k *of* (1.18) *with* $f_{1,k} \in H^1(R_0, R_1)$ *satisfying* $f_{1,k}(R_1) = 0$, $f_{2,k} \in L^2(R_0, R_1)$ *and* $g_k \in \mathbb{C}$ *satisfy the estimates* (1.19) *and* (1.20) *if* ρ *satisfies* **[A1]** *and there exist* $\gamma > 0$, $r > 0$ *and* $M \geqslant 0$ *such that for all* $(\omega, k) \in \mathbb{R} \times \mathbb{Z}$ *satisfying* $\omega^2 R_0^2 + k^2 \geqslant M^2$ *and* (1.15), $\rho(\omega, k)$ *satisfies*

$$\rho(\omega, k) \leqslant -\sqrt{k^2 - \omega^2 R_0^2} - \frac{1+\gamma}{2} \quad \text{or} \quad \rho(\omega, k) \geqslant (-1+\gamma)\sqrt{k^2 - \omega^2 R_0^2}. \tag{1.22}$$

In particular, if $\rho(\omega, k) \geqslant 0$ *for all* $(\omega, k) \in \mathbb{R} \times \mathbb{Z}$ *satisfying* (1.15), *estimates* (1.19) *and* (1.20) *hold for solutions of* (1.18).

The condition (1.16) is very likely not optimal, but we believe that it is close to be, in the sense that Corollary 1.3 indicates that the resolvent estimate (1.19) holds as soon as we can guarantee that $\rho(\omega, k)$ is far from $-\sqrt{k^2 - \omega^2 R_0^2}$ for all $(\omega, k) \in \mathbb{R} \times \mathbb{Z}$ with $\omega^2 R_0^2 + k^2$ large and $\omega^2 R_0^2 \leqslant k^2$.

In fact, we state below the following necessary conditions for (3.3), proved in Section 4, which underlines that $|\rho(\omega, k) + \sqrt{k^2 - \omega^2 R_0^2}|$ should be sufficiently large for (1.19) to hold:

THEOREM 1.4. *Assume that there exists* $C > 0$ *such that for all* $(\omega, k) \in \mathbb{R} \times \mathbb{Z}$ *satisfying* (1.15), *any solution* y_k *of* (1.18) *with* $f_{1,k} \in H^1(R_0, R_1)$ *satisfying* $f_{1,k}(R_1) = 0$, $f_{2,k} \in L^2(R_0, R_1)$ *and* $g_k \in \mathbb{C}$ *satisfies the resolvent estimate* (1.19).

Then, for all $C_0 > 0$, *there exists* $C > 0$ *such that for any sequence* $(\omega_n, k_n)_{n \in \mathbb{N}}$ *of elements in* $\mathbb{R} \times \mathbb{Z}$ *such that*

$$\lim_{n \to \infty} \left(\omega_n^2 R_0^2 + k_n^2\right) = \infty, \qquad \lim_{n \to \infty} \frac{|k_n|^{2/3}}{\log^{2/3}(|k_n|)} \log\left(\frac{|k_n|}{|\omega_n| R_0}\right) = \infty, \tag{1.23}$$

and

$$\forall n \in \mathbb{N}, \qquad \log\left(\frac{|k_n|}{|\omega_n| R_0}\right) \leqslant C_0, \tag{1.24}$$

we have

$$\frac{k_n^2}{\sqrt{k_n^2 - \omega_n^2 R_0^2}} \leqslant C \left(\rho(\omega_n, k_n) + \sqrt{k_n^2 - \omega_n^2 R_0^2}\right)^2. \tag{1.25}$$

Remark 1.5. Although it is not completely obvious at first glance, tedious computations show that, as expected, condition (1.16) implies that the sufficient condition of Theorem 1.4 is satisfied.

To finish the presentation of our main results, we also add the following comments.

Our approach does not produce a good estimate of the time needed for observability. Therefore, in that sense, our approach gives worse results than the ones provided by multiplier techniques [16] or microlocal techniques [2, 3, 6] when available, which give explicit estimates on the time of observability for the waves. We refer to Section 5.2 for additional comments.

Despite this fact, by producing a detailed analysis of resolvent estimates for a family of 1d equations, Theorems 1.2 and 1.4 really emphasize that resolvent estimates could behave badly only when $\rho(\omega, k)$ can get close to $-\sqrt{k^2 - \omega^2 R_0^2}$ for large $|(\omega, k)|$.

Interestingly, Theorems 1.2 and 1.4, and their proofs in Sections 3 and 4, also underline the strong role of the quantity $k^2 - \omega^2 R_0^2$ and show that different estimates can be obtained whether it is positive or negative (recall for instance (1.20)). Note that this is what is expected when considering the boundary term (1.9) appearing when using the multiplier technique for the wave equation (1.1) with Neumann boundary conditions on the internal sphere $S(R_0)$ (corresponding to $\alpha = 0$).

Let us finally mention the two following remarks:

- Our approach can also be developed on the d-dimensional case. This is, when the set Ω is of the form $B(R_1) \setminus \overline{B(R_0)}$ where $0 < R_0 < R_1$ and $B(R)$ is the d-dimensional ball of radius R. The proofs are identical by introducing the spherical harmonics on the sphere of dimension $d - 1$.

- Since the observations are done on the whole set $S(R_1)$, the boundary conditions imposed there are not really important, and we only need that the observation operator is suitably adapted to the particular boundary conditions imposed on $S(R_1)$. For instance, if one considers Neumann homogeneous boundary conditions on $S(R_1)$, then the corresponding observation should be $\|y|_{S(R_1)}\|_{L^2(0,T;H^1(S(R_1))}$ instead of $\|\partial_\nu y\|_{L^2(0,T;L^2(S(R_1))}$.

1.4. **Outline**

In Section 2, we explain how observability results for the wave models (1.1), (1.3), (1.5), (1.7) can be deduced from Theorem 1.1, Theorem 1.2, Corollary 1.3 and Theorem 1.4. Section 3 then focuses on the proof of Theorem 1.2.

Theorem 1.4 is then proved in Section 4. Section 5 provides further comments. Finally, the proof of Corollary 1.3 is given in Appendix A.

2. OBSERVABILITY AND NON-OBSERVABILITY OF SEVERAL WAVE MODELS

The goals of this section are to explain the functional setting allowing to recast each of the wave equations (1.1), (1.3), (1.5), (1.7) into the abstract form (1.10), prove then that the corresponding resolvent estimate (1.12) reduces to the proof of estimates of the form (1.19) for a suitable kernel function ρ, and deduce from them some observability and non-observability results for each one of the wave models presented in (1.1), (1.3), (1.5), (1.7).

2.1. The wave equation (1.1) with Fourier boundary conditions

In this subsection, α is a fixed non-negative real number.

Abstract form. The wave equation (1.1) with Fourier boundary conditions takes the form (1.10) by setting

$$Y(t) = \begin{pmatrix} y(t) \\ \partial_t y(t) \end{pmatrix}, \quad A = \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix} \tag{2.1}$$

where A_0 is the Laplace operator defined on $L^2(\Omega)$ with domain $\mathscr{D}(A_0) = \{y \in H^2(A(R_0, R_1)),\ \partial_\nu y + \alpha y = 0 \text{ on } S(R_0),\ y = 0 \text{ on } S(R_1)\}$ by $A_0 y = -\Delta y$, and A is defined on

$$H = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \text{ with } y_1 \in H^1(A(R_0, R_1)), y_2 \in L^2(A(R_0, R_1)) \right.$$
$$\left. \text{and } y_1 = 0 \text{ on } S(R_1) \right\},$$

with domain

$$\mathscr{D}(A) = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \text{ with } y_1 \in \mathscr{D}(A_0),\ y_2 \in H^1(A(R_0, R_1)) \right.$$
$$\left. \text{and } y_2 = 0 \text{ on } S(R_1) \right\}.$$

One then easily checks that A is skew adjoint on H provided H is endowed with the scalar product

$$\left\langle \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix} \right\rangle_H = \int_{A(R_0,R_1)} (\nabla y_1 \cdot \overline{\nabla \tilde{y}_1} + y_2\, \overline{\tilde{y}_2})\, \mathrm{d}x + \alpha \int_{S(R_0)} y_1\, \overline{\tilde{y}_1}\, \mathrm{d}\sigma.$$

We then define B as follows:

$$B \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \partial_\nu y_1|_{S(R_1)}. \tag{2.2}$$

With this choice, we clearly have that $B \in \mathcal{L}(\mathscr{D}(A), U)$, with $U = L^2(S(R_1))$.

The fact that B is an admissible observation operator in the sense of [21, Section 4.3] follows immediately from classical multiplier arguments as in [16, Section I.4.1].

Resolvent estimate. In view of the above setting, the resolvent estimate (1.12) reads as follows. There exists a constant $C > 0$ (corresponding to $\max\{m, M\}$ in (1.12)) such that if (y_1, y_2) solves

$$\begin{cases} -i\omega y_1 + y_2 = f_1, & \text{in } A(R_0, R_1), \\ \Delta y_1 - i\omega y_2 = f_2, & \text{in } A(R_0, R_1), \\ y_1 = y_2 = 0, & \text{in } S(R_1), \\ \partial_r y_1 = \alpha y_1, & \text{on } S(R_0), \end{cases} \tag{2.3}$$

for some $(f_1, f_2)^{\rm tr} \in H$ and $\omega \in \mathbb{R}$, then

$$\int_{A(R_0,R_1)} (|\nabla y_1|^2 + |y_2|^2)\,\mathrm{d}x + \alpha \int_{S(R_0)} |y_1|^2\,\mathrm{d}\sigma \tag{2.4}$$
$$\leqslant C^2 \left(\int_{A(R_0,R_1)} (|\nabla f_1|^2 + |f_2|^2)\,\mathrm{d}x + \alpha \int_{S(R_0)} |f_1|^2\,\mathrm{d}\sigma \right) + C^2 \left\| \partial_r y_1 \right\|^2_{L^2(S(R_1))}.$$

Note that for $(f_1, f_2)^{\rm tr} \in H$, using the boundary condition $f_1 = 0$ on $S(R_1)$, Poincaré's estimate gives that

$$\int_{A(R_0,R_1)} |f_1|^2\,\mathrm{d}x \leqslant C \int_{A(R_0,R_1)} |\nabla f_1|^2\,\mathrm{d}x.$$

Accordingly, the above resolvent estimate is equivalent to show the existence of a constant $C > 0$ such that if y (corresponding to y_1) solves

$$\begin{cases} \omega^2 y + \Delta y = i\omega f_1 + f_2, & \text{in } A(R_0, R_1), \\ y = 0, & \text{in } S(R_1), \\ \partial_r y = \alpha y, & \text{on } S(R_0), \end{cases} \tag{2.5}$$

for some $(f_1, f_2)^{\rm tr} \in H$ and $\omega \in \mathbb{R}$, then

$$\int_{A(R_0,R_1)} (|\nabla y|^2 + |\omega|^2 |y|^2)\,\mathrm{d}x + \alpha \int_{S(R_0)} |y|^2\,\mathrm{d}\sigma \tag{2.6}$$
$$\leqslant C^2 \left(\int_{A(R_0,R_1)} (|\nabla f_1|^2 + |f_2|^2)\,\mathrm{d}x + \alpha \int_{S(R_0)} |f_1|^2\,\mathrm{d}\sigma \right) + C^2 \left\| \partial_r y \right\|^2_{L^2(S(R_1))}.$$

Writing the equations (2.5) using the spherical harmonics decomposition, we immediately have that the above resolvent estimate is equivalent to show that there exists a constant $C > 0$ such that for all $k \in \mathbb{Z}$, if y_k solves

$$\text{(2.7)} \qquad \begin{cases} \omega^2 y_k + \frac{1}{r}\partial_r(r\partial_r y_k) - \frac{k^2}{r^2} y_k = i\omega f_{1,k} + f_{2,k}, & \text{in } (R_0, R_1), \\ y_k(R_1) = 0, \\ \partial_r y_k(R_0) = \alpha y_k(R_0), \end{cases}$$

for some $(f_{1,k}, f_{2,k}) \in H^1(R_0, R_1) \times L^2(R_0, R_1)$ with $f_{1,k}(R_1) = 0$ and $\omega \in \mathbb{R}$, then

$$\text{(2.8)} \quad \int_{R_0}^{R_1} \left(|\partial_r y_k|^2 + \left(\frac{k^2}{r^2} + |\omega|^2\right) |y_k|^2 \right) r\mathrm{d}r + \alpha R_0 |y_k(R_0)|^2$$
$$\leqslant C^2 \Big(\int_{R_0}^{R_1} \left(|\partial_r f_1|^2 + \frac{k^2}{r^2}|f_1|^2 + |f_2|^2 \right) r\mathrm{d}r + \alpha R_0 |f_{1,k}(R_0)|^2 \Big) + C^2 |\partial_r y_k(R_1)|^2.$$

The equation satisfied by y_k in (2.7) thus corresponds to the equation (1.13) with the choice $\rho(\omega, k) = R_0\alpha$ as claimed in Section 1.3.

Application of Theorem 1.2. Since we assumed $\alpha \geqslant 0$, the function ρ is non-negative and thus the condition (1.16) is automatically satisfied. In particular, taking $g_k = 0$ in (1.13), Theorem 1.2 implies that there exists a constant $C > 0$ such that for all $(\omega, k) \in \mathbb{R} \times \mathbb{Z}$ and any solution y_k of (2.7) with $f_{1,k} \in H^1(R_0, R_1)$ satisfying $f_{1,k}(R_1) = 0$ and $f_{2,k} \in L^2(R_0, R_1)$,

$$\text{(2.9)} \quad \|\partial_r y_k\|_{L^2(R_0,R_1;r\mathrm{d}r)} + |k|\, \|y_k/r\|_{L^2(R_0,R_1;r\mathrm{d}r)} + |\omega|\, \|y_k\|_{L^2(R_0,R_1;r\mathrm{d}r)}$$
$$\leqslant C \left(\|\partial_r f_{1,k}\|_{L^2(R_0,R_1;r\mathrm{d}r)} + |k|\, \|f_{1,k}/r\|_{L^2(R_0,R_1;r\mathrm{d}r)} + \|f_{2,k}\|_{L^2(R_0,R_1;r\mathrm{d}r)} \right)$$
$$+ C|\partial_r y_k(R_1)|.$$

Poincaré estimates then easily implies the resolvent estimate (2.8). Accordingly, as a corollary of Theorem 1.2 and Corollary 1.3, we get the following result:

THEOREM 2.1. *Let $\alpha \geqslant 0$. Then the wave equation* (1.1) *with Fourier boundary conditions is observable in some time $T > 0$, i.e. there exists a time $T > 0$ such that the observability inequality* (1.2) *is true for solutions of* (1.1).

2.2. The wave equation (1.3) with waves on the boundary

In this subsection, α and β are fixed positive numbers.

Abstract form. Let us consider the functional space

$$H = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ z_1 \\ z_2 \end{pmatrix}, \, y_1 \in H_1, \, y_2 \in L^2(A(R_0,R_1)), \, z_1 \in H^1(S(R_0)), \right.$$

$$\left. z_2 \in L^2(S(R_0)), \, y_1|_{S(R_0)} = z_1 \right\},$$

where

$$H_1 = \left\{ y \in H^1(A(R_0,R_1)), \, y|_{S(R_1)} = 0 \right\},$$

endowed with the scalar product

$$\left\langle \begin{pmatrix} y_1 \\ y_2 \\ z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} \right\rangle_H = \int_{A(R_0,R_1)} (\nabla y_1 \cdot \overline{\nabla \tilde{y}_1} + y_2 \, \overline{\tilde{y}_2}) \, \mathrm{d}x$$

$$+ \beta \int_{S(R_0)} \nabla_{S(R_0)} z_1 \cdot \overline{\nabla_{S(R_0)} \tilde{z}_1} \mathrm{d}\sigma + \alpha \int_{S(R_0)} z_2 \, \overline{\tilde{z}_2} \, \mathrm{d}\sigma,$$

where $\nabla_{S(R_0)} = \vec{e}_\theta \dfrac{\partial_\theta}{R_0}$ is the gradient operator on the sphere $S(R_0)$. One easily checks that H is a Hilbert space.

We then set the operator

$$A : \begin{pmatrix} y_1 \\ y_2 \\ z_1 \\ z_2 \end{pmatrix} \in \mathscr{D}(A) \subset H \mapsto \begin{pmatrix} y_2 \\ \Delta y_1 \\ z_2 \\ \frac{\beta}{\alpha} \Delta_{S(R_0)} z_1 - \frac{1}{\alpha} \partial_\nu y_1 \end{pmatrix} \in H,$$

with domain $\mathscr{D}(A)$ corresponding to the set

$$\left\{ \begin{pmatrix} y_1 \\ y_2 \\ z_1 \\ z_2 \end{pmatrix} \in H, \Delta y_1 \in L^2(A(R_0,R_1)), \, y_2 \in H_1, z_1 \in H^2(S(R_0)), \right.$$

$$\left. z_2 \in H^1(S(R_0)), \, y_2|_{S(R_0)} = z_2 \right\}.$$

Note that for $(y_1, y_2, z_1, z_2)^{\mathrm{tr}} \in \mathscr{D}(A)$, as y_1 belongs to $H^1(A(R_0,R_1))$, Δy_1 is in $L^2(A(R_0,R_1))$, and $y_1 = z_1 \in H^2(S(R_0))$, by elliptic regularity $\partial_\nu y_1$ belongs to $L^2(S(R_0))$, hence A is well-defined. Furthermore, A is skew-adjoint.

Then, if we consider the unknown

$$Y(t) = \begin{pmatrix} y(t) \\ \partial_t y(t) \\ z(t) \\ \partial_t z(t) \end{pmatrix},$$

Problem (1.3) rewrites in the abstract form $Y' = AY$, that is in the form of (1.10).

We can then define B as follows:

$$B \begin{pmatrix} y_1 \\ y_2 \\ z_1 \\ z_2 \end{pmatrix} = \partial_\nu y_1|_{S(R_1)}.$$

With this choice, we clearly have that $B \in \mathcal{L}(\mathscr{D}(A), U)$, with $U = L^2(S(R_1))$, and again, B is an admissible observation operator in the sense of [21, Section 4.3] from classical multiplier arguments (see [16, Section I.4.1]).

Resolvent estimate. In our context, the resolvent estimate (1.12) reads as follows. There exists a constant $C > 0$ such that if $(y_1, y_2, z_1, z_2)^{\mathrm{tr}} \in \mathscr{D}(A)$ verifies

$$\text{(2.10)} \qquad \begin{cases} -i\omega y_1 + y_2 = f_1 & \text{in } A(R_0, R_1), \\ \Delta y_1 - i\omega y_2 = f_2 & \text{in } A(R_0, R_1), \\ -i\omega z_1 + z_2 = g_1 & \text{on } S(R_0), \\ \dfrac{\beta}{\alpha}\Delta_{S(R_0)} z_1 - \dfrac{1}{\alpha}\partial_\nu y_1 - i\omega z_2 = g_2 & \text{on } S(R_0), \end{cases}$$

with $(f_1, f_2, g_1, g_2)^{\mathrm{tr}} \in H$ and $\omega \in \mathbb{R}$, one has

$$\int_{A(R_0,R_1)} \left(|\nabla y_1|^2 + |y_2|^2\right) \mathrm{d}x + \beta \int_{S(R_0)} |\nabla_{S(R_0)} z_1|^2 \mathrm{d}\sigma + \alpha \int_{S(R_0)} |z_2|^2 \mathrm{d}\sigma$$
$$\leqslant C^2 \left(\int_{A(R_0,R_1)} \left(|\nabla f_1|^2 + |f_2|^2\right) \mathrm{d}x + \beta \int_{S(R_0)} |\nabla_{S(R_0)} g_1|^2 \mathrm{d}\sigma + \alpha \int_{S(R_0)} |g_2|^2 \mathrm{d}\sigma \right)$$
$$+ C^2 \int_{S(R_1)} |\partial_\nu y_1|^2 \mathrm{d}\sigma.$$

Now, using that by definition of H, $y_1 = z_1$ on $S(R_0)$, system (2.10) reduces to, with $y \in H_1$ (corresponding to y_1),

$$\text{(2.11)} \qquad \begin{cases} \Delta y + \omega^2 y = f_2 + i\omega f_1 & \text{in } A(R_0, R_1), \\ \partial_\nu y - \beta \Delta_{S(R_0)} y - \alpha\omega^2 y = \alpha(g_2 + i\omega g_1) & \text{on } S(R_0), \end{cases}$$

and the desired resolvent estimate is equivalent to the existence of a constant $C > 0$ such that

$$\int_{A(R_0,R_1)} \left(|\nabla y|^2 + |\omega|^2 |y|^2\right) \mathrm{d}x + \beta \int_{S(R_0)} |\nabla_{S(R_0)} y|^2 \mathrm{d}\sigma + \alpha \int_{S(R_0)} |\omega|^2 |y|^2 \mathrm{d}\sigma$$
$$\leqslant C^2 \left(\int_{A(R_0,R_1)} \left(|\nabla f_1|^2 + |f_2|^2\right) \mathrm{d}x + \beta \int_{S(R_0)} |\nabla_{S(R_0)} g_1|^2 \mathrm{d}\sigma + \alpha \int_{S(R_0)} |g_2|^2 \mathrm{d}\sigma \right)$$
$$+ C^2 \int_{S(R_1)} |\partial_\nu y|^2 \mathrm{d}\sigma.$$

Decomposing our new problem on spherical harmonics leads to the following system, with $f_{1,k} \in H^1(R_0, R_1)$ such that $f_{1,k}(R_1) = 0$, and $f_{2,k} \in L^2(R_0, R_1)$:
(2.12)

$$\begin{cases} \omega^2 y_k + \dfrac{1}{r} \partial_r (r \partial_r y_k) - \dfrac{k^2}{r^2} y_k = f_{2,k} + i\omega f_{1,k} & \text{in } (R_0, R_1), \\ y_k(R_1) = 0 \\ \partial_r y_k(R_0) = \left(\beta \dfrac{k^2}{R_0^2} - \alpha \omega^2 \right) y_k(R_0) + \alpha (g_{2,k} + i\omega g_{1,k}), \end{cases}$$

that is equation (1.13) with the choice

$$\rho(\omega, k) = R_0 \left(\beta \frac{k^2}{R_0^2} - \alpha \omega^2 \right)$$

as claimed in Section 1.3. Hence, our goal is to find a constant $C > 0$ such that for all $k \in \mathbb{Z}$, all $\omega \in \mathbb{R}$, each solution of (2.12) satisfies the estimate

$$\text{(2.13)} \quad \int_{R_0}^{R_1} \left(|\partial_r y_k|^2 + \left(\frac{k^2}{r^2} + \omega^2 \right) |y_k|^2 \right) r \, \mathrm{d}r + \left(\beta \, \frac{k^2}{R_0^2} + \alpha \, \omega^2 \right) |y_k(R_0)|^2$$
$$\leqslant C^2 \left(\int_{R_0}^{R_1} \left(|\partial_r f_{1,k}|^2 + \frac{k^2}{r^2} |f_{1,k}|^2 + |f_{2,k}|^2 \right) r \mathrm{d}r + \beta \frac{k^2}{R_0^2} |g_{1,k}|^2 + \alpha |g_{2,k}|^2 \right)$$
$$+ C^2 |\partial_r y_k(R_1)|^2.$$

Application of Theorem 1.2: The case $0 < \alpha < \beta$. As an application of Theorem 1.2 and Corollary 1.3, we get the following positive result:

THEOREM 2.2. *Let α and β be positive constants with $\alpha < \beta$. Then the wave equation* (1.3) *satisfies the observability inequality* (1.4) *in some time $T > 0$.*

Proof. For $(\omega, k) \in \mathbb{R} \times \mathbb{Z}$ with $\omega^2 R_0^2 \leqslant k^2$ we have

$$\text{(2.14)} \qquad \rho(\omega, k) = \frac{1}{R_0} (\beta k^2 - \alpha \, \omega^2 \, R_0^2) \geqslant \frac{\beta \, k^2}{R_0} \left(1 - \frac{\alpha}{\beta} \right),$$

Therefore, for $\alpha < \beta$, for all $(\omega, k) \in \mathbb{R} \times \mathbb{Z}$ with $\omega^2 R_0^2 + k^2 \geqslant M^2$ and $\omega^2 R_0^2 \leqslant k^2$, $\rho(\omega, k) \geqslant 0$. Corollary 1.3 then applies and the resolvent estimates

(1.19)–(1.20) hold, which imply in particular the existence of $C > 0$ such that for all $(\omega, k) \in \mathbb{R} \times \mathbb{Z}$,

$$
\begin{aligned}
(2.15) \quad & \int_{R_0}^{R_1} \left(|\partial_r y_k|^2 + \left(\frac{k^2}{r^2} + \omega^2 \right) |y_k|^2 \right) r \, \mathrm{d}r \\
& \leqslant C^2 \left(\int_{R_0}^{R_1} \left(|\partial_r f_{1,k}|^2 + \frac{k^2}{r^2} |f_{1,k}|^2 + |f_{2,k}|^2 \right) r \mathrm{d}r + \beta \frac{k^2}{R_0^2} |g_{1,k}|^2 + \alpha |g_{2,k}|^2 \right) \\
& \qquad + C^2 |\partial_r y_k(R_1)|^2.
\end{aligned}
$$

It remains to check that the boundary estimates (1.20) indeed yields that the boundary term

$$
\left(\beta \frac{k^2}{R_0^2} + \alpha \, \omega^2 \right) |y_k(R_0)|^2 \tag{2.16}
$$

in the left hand-side of (2.13) is bounded by the right hand-side of (2.13), which is of course delicate only for $\omega^2 R_0^2 + k^2$ large enough.

For $(\omega, k) \in \mathbb{R} \times \mathbb{Z}$ with $\omega^2 R_0^2 \leqslant k^2$, due to (2.14),

$$
I_\varepsilon(\omega, k)^2 \geqslant I_1(\omega, k)^2 \geqslant \frac{\beta^2 k^4}{R_0^2} \left(1 - \frac{\alpha}{\beta} \right)^2 .
$$

On the other hand, for $(\omega, k) \in \mathbb{R} \times \mathbb{Z}$ with $\omega^2 R_0^2 \geqslant k^2$ and $\omega^2 R_0^2 \geqslant 1$, taking into account the boundary condition on R_0 of system (2.12), we have

$$
\left(\beta \frac{k^2}{R_0^2} - \alpha \omega^2 \right)^2 |y_k(R_0)|^2 \leqslant 2 |\partial_r y_k(R_0)|^2 + 2 |\alpha (g_{2,k} + i \omega g_{1,k})|^2,
$$

so that

$$
\frac{k^2}{\omega^2} \left(\beta \frac{k^2}{R_0^2} - \alpha \omega^2 \right)^2 |y_k(R_0)|^2 \leqslant C |\partial_r y_k(R_0)|^2 + C |g_{2,k}|^2 + C k^2 |g_{1,k}|^2,
$$

Accordingly, the estimate on the boundary terms at R_0 in (1.20) corresponding to the case $\omega^2 R_0^2 \geqslant k^2$ and $\omega^2 R_0^2 + k^2 \geqslant 2 R_0^2$ (implying $\omega^2 \geqslant 1$) allows to estimate

$$
\begin{aligned}
& \left(\frac{k^2}{\omega^2} \left(\beta \frac{k^2}{R_0^2} - \alpha \omega^2 \right)^2 + \max \left\{ \sqrt{k^2 + \omega^2 R_0^2}, \omega^2 R_0^2 - k^2 \right\} \right) |y_k(R_0)|^2 \\
& \quad \geqslant \omega^2 R_0^2 \left(\alpha^2 \frac{k^2}{\omega^2 R_0^2} \left(1 - \frac{\beta}{\alpha} \frac{k^2}{\omega^2 R_0^2} \right)^2 + 1 - \frac{k^2}{\omega^2 R_0^2} \right) |y_k(R_0)|^2 \\
& \quad \geqslant \omega^2 R_0^2 \inf_{\tau \in [0,1]} \left\{ \alpha^2 \tau^2 \left(1 - \frac{\beta}{\alpha} \tau^2 \right)^2 + 1 - \tau^2 \right\} |y_k(R_0)|^2 \geqslant c_* \omega^2 R_0^2 |y_k(R_0)|^2,
\end{aligned}
$$

for some $c_* > 0$.

The previous inequalities and the boundary estimates (1.20) show that the boundary term (2.16) is bounded by the right hand-side of (2.13) with a constant independent of (ω, k). □

Remark 2.3. When $\alpha = \beta$, Theorem 1.2 still applies and the above arguments immediately yield (2.15). However, the boundary estimates (1.20) fail to give estimate on (2.16) by the right hand-side of (2.13), in particular in the range $\omega^2 R_0^2 \simeq k^2$.

Application of Theorem 1.4: The case $0 < \beta < \alpha$. As a consequence of Theorem 1.4, we show the following result:

THEOREM 2.4. *Let α and β be positive constants with $\alpha > \beta$. Then the wave equation* (1.3) *does not satisfy the observability inequality* (1.4) *in any time $T > 0$.*

Proof. We use Theorem 1.4 and construct a sequence $(\omega_n, k_n) \in \mathbb{R} \times \mathbb{Z}$ satisfying (1.23)–(1.24) and for which (1.25) cannot hold.

We set $A = \beta/R_0$ and $\gamma = \alpha/\beta$, which satisfies $\gamma > 1$ from the assumption $\alpha > \beta$. With these notations ρ writes $\rho(\omega, k) = A(k^2 - \gamma\omega^2 R_0^2)$.

Then, for $n \in \mathbb{N}$, we set

$$k_n = n, \quad \omega_n = \frac{1}{R_0}\sqrt{n^2 - \frac{1}{4\gamma^2 A^2}\left(\sqrt{1 + 4n^2\gamma(\gamma - 1)A^2} - 1\right)^2}.$$

This choice is done to guarantee that

$$\forall n \in \mathbb{N}, \qquad \rho(\omega_n, k_n) + \sqrt{k_n^2 - \omega_n^2 R_0^2} = 0. \tag{2.17}$$

It is also easy to check that

$$k_n^2 - \omega_n^2 R_0^2 = \frac{1}{4\gamma^2 A^2}\left(\sqrt{1 + 4n^2\gamma(\gamma - 1)A^2} - 1\right)^2 \leqslant \frac{\gamma - 1}{\gamma} k_n^2,$$

hence that

$$\forall n \in \mathbb{N}, \quad 1 \leqslant \frac{|k_n|}{|\omega_n| R_0} \leqslant \sqrt{\gamma}, \quad \lim_{n\to\infty} \log\left(\frac{|k_n|}{|\omega_n| R_0}\right) = \frac{1}{2}\log(\gamma) > 0,$$

$$\text{and} \quad \lim_{n\to\infty} \frac{\sqrt{k_n^2 - \omega_n^2 R_0^2}}{|k_n|} = \sqrt{1 - \frac{1}{\gamma}}.$$

Accordingly, we easily get that the sequence $(\omega_n, k_n)_{n\in\mathbb{N}}$ satisfies (1.23)–(1.24), while the relation (2.17) disproves estimate (1.25). This concludes the proof of Theorem 2.4. □

2.3. The wave equation (1.5) with a fractional Laplacian

In this subsection, s denotes a positive constant in $(0,1]$.

Abstract form. Let $f \in L^2(S(R_0))$. We denote f_k its k-th Fourier coefficient, that is

$$f_k = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, e^{-ik\theta} d\theta,$$

which implies the expansion $f(\theta) = \sum_{k\in\mathbb{Z}} f_k e^{ik\theta}$,

For $s \in (0,1]$, we define the usual fractional Laplace-Beltrami operator

$$(-\Delta_{S(R_0)})^s : f \in \mathscr{D}((-\Delta_{S(R_0)})^s) \subset L^2(S(R_0)) \mapsto (-\Delta_{S(R_0)})^s f \in L^2(S(R_0)),$$

by

$$\mathscr{D}((-\Delta_{S(R_0)})^s) = \left\{ f \in L^2(S(R_0)),\ \sum_{k\in\mathbb{Z}} |k|^{4s} |f_k|^2 < \infty \right\},$$

and,

$$((-\Delta_{S(R_0)})^s f)(\theta) = \frac{1}{R_0^{2s}} \sum_{k\in\mathbb{Z}} |k|^{2s} f_k\, e^{ik\theta}.$$

We then define

$$H = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},\ y_1 \in H^1(A(R_0,R_1)),\ y_1|_{S(R_0)} \in \mathscr{D}((-\Delta_{S(R_0)})^{s/2}),\ y_1(R_1) = 0, \right.$$
$$\left. y_2 \in L^2(A(R_0,R_1)) \right\},$$

which is an Hilbert space when equipped with the scalar product

$$\left\langle \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix} \right\rangle_H = \int_\Omega (\nabla y_1 \cdot \overline{\nabla \tilde{y}_1} + y_2\, \overline{\tilde{y}_2})\, \mathrm{d}x + 2\pi R_0^{1-2s} \sum_{k\in\mathbb{Z}} |k|^{2s} y_1(R_0)_k\, \overline{\tilde{y}_1(R_0)_k}.$$

We also define

$$A : \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathscr{D}(A) \subset H \mapsto \begin{pmatrix} y_2 \\ \Delta y_1 \end{pmatrix} \in H,$$

with

$$\mathscr{D}(A) := \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in H,\ \Delta y_1 \in L^2(A(R_0,R_1)),\ \partial_\nu y_1 + (-\Delta_{S(R_0)})^s (y_{1|S(R_0)}) = 0 \right.$$
$$\left. \text{on } S(R_0),\ y_2|_{S(R_0)} \in \mathscr{D}((-\Delta_{S(R_0)})^{s/2}),\ y_2 \in H^1(A(R_0,R_1)),\ y_2(R_1) = 0 \right\}.$$

It is not difficult to check that the operator A is skew-adjoint on H. Therefore, if we consider the unknown $Y(t) = \begin{pmatrix} y(t) \\ \partial_t y(t) \end{pmatrix}$, Problem (1.5) rewrites in the abstract form $Y' = AY$, that is in the form of (1.10).

The observation operator B can then be deduced on H by the same formula as in (2.2), and again, $B \in \mathcal{L}(\mathscr{D}(A), U)$, with $U = L^2(S(R_1))$, and B is an admissible observation operator in the sense of [21, Section 4.3] due to classical multiplier arguments as in [16, Section I.4.1].

Resolvent estimate. In view of the above setting, the resolvent estimate (1.12) reads as follows. There exists a constant $C > 0$ such that if (y_1, y_2) solves

$$\text{(2.18)} \qquad \begin{cases} -i\omega y_1 + y_2 = f_1, & \text{in } A(R_0, R_1), \\ \Delta y_1 - i\omega y_2 = f_2, & \text{in } A(R_0, R_1), \\ y_1 = y_2 = 0, & \text{in } S(R_1), \\ \partial_r y_1 = -(-\Delta_{S(R_0)})^s (y_1|_{S(R_0)}), & \text{on } S(R_0), \end{cases}$$

for some $(f_1, f_2)^{\text{tr}} \in H$ and $\omega \in \mathbb{R}$, then

$$\text{(2.19)} \qquad \int_{A(R_0,R_1)} (|\nabla y_1|^2 + |y_2|^2)\,\mathrm{d}x + 2\,\pi R_0^{1-2s} \sum_{k\in\mathbb{Z}} |k|^{2\,s} |y_1(R_0)_k|^2$$
$$\leqslant C^2 \left(\int_{A(R_0,R_1)} (|\nabla f_1|^2 + |f_2|^2)\,\mathrm{d}x + 2\,\pi\, R_0^{1-2s} \sum_{k\in\mathbb{Z}} |k|^{2\,s} |f_1(R_0)_k|^2 \right)$$
$$+ C^2 \left\| \partial_r y_1 \right\|^2_{L^2(S(R_1))}.$$

Eliminating y_2, and writing y for y_1 leads to the equivalent simplified system

$$\begin{cases} \Delta y + \omega^2 y = f_2 + i\omega f_1, & \text{in } A(R_0, R_1), \\ y = 0, & \text{in } S(R_1), \\ \partial_r y = -(-\Delta_{S(R_0)})^s (y|_{S(R_0)}), & \text{on } S(R_0), \end{cases}$$

for which we aim to prove the following equivalent estimate

$$\text{(2.20)} \qquad \int_{A(R_0,R_1)} (|\nabla y|^2 + |\omega|^2 |y|^2)\,\mathrm{d}x + 2\,\pi R_0^{1-2s} \sum_{k\in\mathbb{Z}} |k|^{2\,s} |y(R_0)_k|^2$$
$$\leqslant C^2 \left(\int_{A(R_0,R_1)} (|\nabla f_1|^2 + |f_2|^2)\,\mathrm{d}x + 2\,\pi\, R_0^{1-2s} \sum_{k\in\mathbb{Z}} |k|^{2\,s} |f_1(R_0)_k|^2 \right)$$
$$+ C^2 \left\| \partial_r y \right\|^2_{L^2(S(R_1))}.$$

As usual, we now decompose our problem on spherical harmonics. This leads to the following problem

$$\text{(2.21)} \qquad \begin{cases} \omega^2 y_k + \dfrac{1}{r}\partial_r(r\partial_r y_k) - \dfrac{k^2}{r^2} y_k = f_{2,k} + i\omega f_{1,k} & \text{in } (R_0, R_1), \\ y_k(R_1) = 0, \\ \partial_r y_k(R_0) = \dfrac{|k|^{2s}}{R_0^{2s}} y_k(R_0), \end{cases}$$

with $f_{1,k} \in H^1(R_0, R_1)$, $f_{1,k}(R_1) = 0$, $f_{2,k} \in L^2(R_0, R_1)$, for which we aim to prove the existence of $C > 0$ independent of k and ω such that

$$\begin{aligned}(2.22)\quad &\int_{R_0}^{R_1} \left(|\partial_r y_k|^2 + \left(\frac{k^2}{r^2} + \omega^2 \right) |y_k|^2 \right) r \,\mathrm{d}r + |k|^{2s} |y_k(R_0)|^2 \\ &\leqslant C^2 \left(\int_{R_0}^{R_1} \left(|\partial_r f_{1,k}|^2 + \frac{k^2}{r^2} |f_{1,k}|^2 + |f_{2,k}|^2 \right) r \mathrm{d}r + |k|^{2s} |f_{1,k}(R_0)|^2 \right) \\ &\qquad + C^2 |\partial_r y_k(R_1)|^2.\end{aligned}$$

In other words, we are back to the study of (1.13) with

$$\rho(\omega, k) = R_0^{1-2s} |k|^{2s}.$$

Remark 2.5 (The case $s = 1/2$). The operator $(-\Delta_{S(R_0)})^{1/2}$ actually corresponds to a Dirichlet to Neumann map for the Laplacian in $B(0, R_0)$. Namely, if $h \in \mathscr{D}((-\Delta_{S(R_0)})^{1/2})$, the solution y of

$$\Delta y = 0 \text{ in } B(0, R_0), \text{ with } y = h \text{ on } S(R_0),$$

corresponding to $h(\theta) = \sum_k h_k e^{ik\theta}$, is explicitly given by

$$y(r, \theta) = \sum_{k \in \mathbb{Z}} h_k \left(\frac{r}{R_0} \right)^{|k|} e^{ik\theta}, \qquad \text{in } B(0, R_0),$$

so that $\partial_r y|_{S(R_0)} = (-\Delta_{S(R_0)})^{1/2} h$.

Application of Theorem 1.2.

THEOREM 2.6. *Let $s \in (0, 1]$. The wave equation* (1.5) *is observable in some time $T > 0$.*

Proof. Since $\rho(\omega, k) \geqslant 0$ for all $(\omega, k) \in \mathbb{R} \times \mathbb{Z}$, we can apply Corollary 1.3 and deduce the resolvent estimates (1.19) and (1.20) for solutions of (2.21).

Therefore, the only thing to check is that the strength of the boundary terms in (1.20) indeed allows to estimate $|k|^{2s} |y_k(R_0)|^2$ by the right hand-side of (2.22). As before, this is delicate only when $\omega^2 R_0^2 + k^2$ is large enough. Then, for $\omega^2 R_0^2 \geqslant k^2$, we simply use that

$$|\partial_r y_k(R_0)|^2 = R_0^{-4s} |k|^{4s} |y_k(R_0)|^2.$$

For $\omega^2 R_0^2 \leqslant k^2$, we use that, since $\rho(\omega, k) = R_0^{1-2s} |k|^{2s} \geqslant 0$,

$$I_\varepsilon(\omega, k)^2 \geqslant I_1(\omega, k)^2 \geqslant R_0^{2-4s} |k|^{4s}.$$

According to the two above estimates, we obviously get the estimate on $|k|^{2s} |y_k(R_0)|^2$ by the right hand-side of (2.22). $\square$

2.4. **The simplified fluid structure model (1.7)**

Abstract form. We consider the functional space

$$H = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ s_1 \\ s_2 \end{pmatrix}, s_1 \in \mathbb{C}^2,\ s_2 \in \mathbb{C}^2, y_1 \in H^1(A(R_0,R_1)), y_1|_{S(R_1)} = 0, \right.$$
$$\left. y_2 \in L^2(A(R_0,R_2)) \right\},$$

endowed with the scalar product

$$\left\langle \begin{pmatrix} y_1 \\ y_2 \\ s_1 \\ s_2 \end{pmatrix}, \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \tilde{s}_1 \\ \tilde{s}_2 \end{pmatrix} \right\rangle_H = \int_{A(R_0,R_1)} (\nabla y_1 \cdot \overline{\nabla \tilde{y}_1} + y_2\, \overline{\tilde{y}_2})\,\mathrm{d}x + s_1 \cdot \overline{\tilde{s}_1} + s_2 \cdot \overline{\tilde{s}_2},$$

which obviously is a Hilbert space.

We then set the operator

$$A : \begin{pmatrix} y_1 \\ y_2 \\ s_1 \\ s_2 \end{pmatrix} \in \mathscr{D}(A) \subset H \mapsto \left(y_2, \Delta y_1, s_2, -\left(\int_{S(R_0)} y_2\, \nu \,\mathrm{d}\sigma \right) - s_1 \right) \in H,$$

with domain $\mathscr{D}(A)$ corresponding to the set

$$\left\{ \begin{pmatrix} y_1 \\ y_2 \\ s_1 \\ s_2 \end{pmatrix} \in H, \Delta y_1 \in L^2(A(R_0,R_1)),\ y_2 \in H^1(A(R_0,R_1)),\ y_2|_{S(R_1)} = 0, \right.$$
$$\left. \partial_\nu y_1 = s_2 \cdot \nu \text{ on } S(R_0) \right\}.$$

Then, if we consider the unknown $Y(t) = (y(t), \partial_t y(t), s(t), s'(t))^{\mathrm{tr}}$, problem (1.7) rewrites in the abstract form $Y' = AY$, that is in the form of (1.10), A being a skew-adjoint operator on H.

Similarly as before, we define the operator

$$B \begin{pmatrix} y_1 \\ y_2 \\ s_1 \\ s_2 \end{pmatrix} = \partial_\nu y_1|_{S(R_1)}.$$

With this choice, we clearly have that $B \in \mathcal{L}(\mathscr{D}(A), U)$, with $U = L^2(S(R_1))$, and that B is an admissible observation operator in the sense of [21, Section 4.3] according to classical multiplier arguments as in [16, Section I.4.1].

Resolvent estimate. In our context, the resolvent estimate (1.12) reads as follows. There exists a constant $C > 0$ such that if $(y_1, y_2, s_1, s_2)^{\mathrm{tr}}$ in $\mathscr{D}(A)$ verifies

$$\begin{cases} -i\omega y_1 + y_2 = f_1 & \text{in } A(R_0, R_1), \\ \Delta y_1 - i\omega y_2 = f_2 & \text{in } A(R_0, R_1), \\ -i\omega s_1 + s_2 = g_1 & \text{on } S(R_0), \\ -s_1 - \left(\int_{S(R_0)} y_2\, \nu \, \mathrm{d}\sigma \right) - i\omega s_2 = g_2 & \text{on } S(R_0), \end{cases} \tag{2.23}$$

with $(f_1, f_2, g_1, g_2)^{\mathrm{tr}}$ in H and ω in $\mathbb{R}$, one has

$$\int_{A(R_0,R_1)} \left(|\nabla y_1|^2 + |y_2|^2 \right) \mathrm{d}x + |s_1|^2 + |s_2|^2 \leqslant C^2 \int_{S(R_1)} |\partial_\nu y_1|^2 \mathrm{d}\sigma$$
$$+ C^2 \left(\int_{A(R_0,R_1)} \left(|\nabla f_1|^2 + |f_2|^2 \right) \mathrm{d}x + |g_1|^2 + |g_2|^2 \right).$$

We already noted that the condition $\partial_\nu y_1 = s_2 \cdot \nu$ on $S(R_0)$, when projected on the cylindrical harmonics, immediately gives that $\partial_r y_{1,k}|_{S(R_0)} = 0$ for all k but -1 and 1. Conversely, it is not difficult to see that if $w \in L^2(S(R_0))$ verifies $w_k = 0$ for all k but -1 and 1, then there exists $s \in \mathbb{C}^2$ such that $w = s \cdot \nu$. Accordingly $|s_2|$ is of the order of the $L^2(S(R_0))$-norm of $\partial_\nu y_1|_{S(R_0)}$.

Similarly, the term

$$\left| \int_{S(R_0)} y_2 \, \nu \, \mathrm{d}\sigma \right|$$

is of the order of $|y_{2,1}(R_0)| + |y_{2,-1}(R_0)|$. From the third and fourth lines of (2.23), we can thus get

$$|s_1| \leqslant |s_2| + |g_1| \quad \text{if } |\omega| \geqslant 1,$$
$$|s_1| \leqslant |s_2| + |g_2| + C(|y_{2,-1}(R_0)| + |y_{2,1}(R_0)|) \quad \text{if } |\omega| \leqslant 1.$$

Eliminating all unknowns but $y = y_1$ in system (2.23), we obtain the following reduced system

$$\begin{cases} \Delta y + \omega^2 y = f_2 + i\omega f_1 & \text{in } A(R_0, R_1), \\ (1-\omega^2)\, \partial_\nu y - \omega^2 \left(\int_{S(R_0)} y \, \nu \, \mathrm{d}\sigma \right) \cdot \nu & \\ \qquad = (g_1 - i\omega g_2) \cdot \nu - i\omega \left(\int_{S(R_0)} f_1 \, \nu \mathrm{d}\sigma \right) \cdot \nu & \text{on } S(R_0). \end{cases} \tag{2.24}$$

and the desired resolvent estimate is thus equivalent to the existence of a constant $C > 0$ independent of ω such that

$$\int_{A(R_0,R_1)} \left(|\nabla y|^2 + |\omega|^2 |y|^2\right) \mathrm{d}x + \int_{S(R_0)} |\partial_\nu y|^2 \mathrm{d}\sigma + |y_1(R_0)|^2 + |y_{-1}(R_0)|^2 \tag{2.25}$$
$$\leqslant C^2 \int_{S(R_1)} |\partial_\nu y|^2 \mathrm{d}\sigma + C^2 \Big(\int_{A(R_0,R_1)} \left(|\nabla f_1|^2 + |f_2|^2\right) \mathrm{d}x + |g_1|^2 + |g_2|^2 \Big).$$

Decomposing our new problem on spherical harmonics leads to the following system, with $f_{1,k} \in H^1(R_0, R_1)$ such that $f_{1,k}(R_1) = 0$, and $f_{2,k} \in L^2(R_0, R_1)$:

$$\begin{cases} \omega^2 y_k + \dfrac{1}{r}\partial_r(r\partial_r y_k) - \dfrac{k^2}{r^2} y_k = f_{2,k} + i\omega f_{1,k} & \text{in } (R_0, R_1), \\ y_k(R_1) = 0, \quad \partial_r y_k(R_0) = 0 & \text{if } k \text{ in } \mathbb{Z} \setminus \{-1, 1\}, \\ (\omega^2 - 1)\,\partial_r y_1(R_0) - \pi\, R_0\, \omega^2\, y_1(R_0) & \\ \qquad = \frac{1}{2}(i\omega g_2 - g_1) \cdot \begin{pmatrix} 1 \\ -i \end{pmatrix} - \pi\, i\, R_0 f_{1,1}(R_0), & \text{if } k = 1, \\ (\omega^2 - 1)\,\partial_r y_{-1}(R_0) - \pi\, R_0\, \omega^2\, y_{-1}(R_0) & \\ \qquad = \frac{1}{2}(i\omega g_2 - g_1) \cdot \begin{pmatrix} 1 \\ i \end{pmatrix} - \pi\, i\, R_0 f_{1,-1}(R_0), & \text{if } k = -1, \end{cases} \tag{2.26}$$

that is system equation (1.13) with the choice

$$\rho(\omega, k) = \pi R_0 \frac{\omega^2}{\omega^2 - 1} \delta_{k=\pm 1}$$

as claimed in Section 1.3. Hence, our goal is to find a constant $C > 0$ such that for all $\omega \in \mathbb{R}$ and all $k \in \mathbb{Z}$, each solution of (2.26) satisfies the estimate

$$\int_{R_0}^{R_1} \left(|\partial_r y_k|^2 + \left(\frac{k^2}{r^2} + \omega^2\right) |y_k|^2 \right) r\, \mathrm{d}r + 1_{k=\pm 1} \left(|\partial_r y_k(R_0)|^2 + |y_k(R_0)|^2\right) \tag{2.27}$$
$$\leqslant C^2 \left(\int_{R_0}^{R_1} \left(|\partial_r f_{1,k}|^2 + \frac{k^2}{r^2} |f_{1,k}|^2 + |f_{2,k}|^2 \right) r \mathrm{d}r + |g_1|^2 + |g_2|^2 \right) + C^2 |\partial_r y_k(R_1)|^2.$$

Application of Theorem 1.2. In this section, we prove the following result:

THEOREM 2.7. *There exists a time T such that the equation* (1.7) *is observable in time $T > 0$.*

Proof. For $(\omega, k) \in \mathbb{R} \times \mathbb{Z}$ with $\omega^2 R_0^2 + k^2 \geqslant 2(R_0^2 + 1)$, we necessarily have $\rho(\omega, k) \geqslant 0$ and that $\rho(\omega, k)$ is bounded, since either $|k| > 1$ or $|\omega| \geqslant \sqrt{(2R_0^2 + 1)/R_0} > 1$. Using Corollary 1.3, Theorem 1.2 applies. Estimate (1.19) immediately yields the estimates on the integrated terms, and it thus remains to check that the estimates (1.20) yields the accurate estimates on the boundary terms. This concerns only the case $|k| = 1$. Thus, the only delicate case is when ω is large, for which estimate (1.20) immediately yields the result. This concludes the proof of the resolvent estimate (2.27). $\square$

3. PROOF OF THEOREM 1.2

Theorem 1.2 relies on several strategies depending on the value of the resolvent parameter $\omega \in \mathbb{R}$ and the frequency parameter $k \in \mathbb{Z}$.

To study the equation (1.13), we first remove the metric from the Laplacian, which can be done by the change of variable $x = \log(r)$, so that $z_k(x) = y_k(r)$ and $F_{1,k}(x) = r^2 f_{1,k}(r)$, $F_{2,k}(x) = r^2 f_{2,k}(r)$, for which (1.13) can be rewritten as follows:

$$\left\{ \begin{array}{ll} \partial_x^2 z_k - (k^2 - \omega^2 e^{2x}) z_k = i\omega F_{1,k} + F_{2,k}, & \text{in } (a_0, a_1). \\ z_k(a_1) = 0, & \\ \partial_x z_k(a_0) = \rho(\omega, k) z_k(a_0) + g_k, & \end{array} \right. \tag{3.1}$$

where we have set $a_0 = \log(R_0)$, $a_1 = \log(R_1)$ and where

$$F_{1,k} \in H^1(a_0, a_1), \text{ with } F_{1,k}(a_1) = 0, \qquad \text{and} \qquad F_{2,k} \in L^2(a_0, a_1). \tag{3.2}$$

The resolvent estimate we aim at proving, corresponding to (1.19), then reads as follows:

$$\begin{aligned} &\|\partial_x z_k\|_{L^2(a_0,a_1)} + |k| \, \|z_k\|_{L^2(a_0,a_1)} + |\omega| \, \|z_k e^x\|_{L^2(a_0,a_1)} \\ &\quad \leqslant C \Big(\|\partial_x F_{1,k}\|_{L^2(a_0,a_1)} + |k| \, \|F_{1,k}\|_{L^2(a_0,a_1)} + \|F_{2,k} e^x\|_{L^2(a_0,a_1)} \\ &\qquad\qquad + |g_k| 1_{\substack{\omega^2 R_0^2 + k^2 \geqslant M^2 \\ \omega^2 R_0^2 \leqslant k^2}} + |\partial_x z_k(a_1)| \Big), \end{aligned} \tag{3.3}$$

and, corresponding to (1.20),

$$\begin{aligned} &1_{|\omega|^2 R_0^2 + |k|^2 \leqslant M^2} \left(|\partial_x z_k(a_0)|^2 + |z_k(a_0)|^2 \right) \\ &+ 1_{\substack{|\omega|^2 R_0^2 + |k|^2 \geqslant M^2 \\ \omega^2 R_0^2 \geqslant |k|^2}} \left(|\partial_x z_k(a_0)|^2 + \max\left\{ \sqrt{k^2 + \omega^2 R_0^2}, \omega^2 R_0^2 - k^2 \right\} |z_k(a_0)|^2 \right) \\ &+ 1_{\substack{|\omega|^2 R_0^2 + |k|^2 \geqslant M^2 \\ \omega^2 R_0^2 \leqslant |k|^2}} \Bigg(I_\varepsilon(\omega, k)^2 1_{\log\left(\frac{|k|}{|\omega| R_0}\right) \geqslant \frac{A}{|k|^{2/3}}} |z_k(a_0)|^2 \\ &\qquad + \max\{|k|^{4/3}, I_1(\omega, k)^2\} 1_{\log\left(\frac{|k|}{|\omega| R_0}\right) \leqslant \frac{A}{|k|^{2/3}}} |z_k(a_0)|^2 \Bigg) \\ &\leqslant C \Big(\|\partial_x F_{1,k}\|^2_{L^2(a_0,a_1)} + |k|^2 \, \|F_{1,k}\|^2_{L^2(a_0,a_1)} + \|F_{2,k}\|^2_{L^2(a_0,a_1)} \\ &+ |g_k|^2 1_{\substack{\omega^2 R_0^2 + k^2 \geqslant M^2 \\ \omega^2 R_0^2 \leqslant k^2}} + |\partial_x z_k(a_1)|^2 \Big). \end{aligned} \tag{3.4}$$

We will now focus on the proof of the resolvent estimate (3.3) for solutions of (3.1). In order to do that, we will distinguish several cases:

- The case of bounded frequency parameters $\omega^2 R_0^2 + k^2 \leqslant M^2$, see Section 3.1 and Lemma 3.2, which can be handled using classical Carleman estimates.
- The case $|\omega| \ggg |k|$, see Section 3.2, which we deal with using multiplier techniques.
- The case $|\omega| \lll |k|$, see Section 3.3, in which we use a factorization technique.

Actually, we will do a slightly more subtle analysis, that allows to consider $|\omega|$ and $|k|$ of the same order in Section 3.2 and in Section 3.3, and to conclude the resolvent estimate (3.3) for any $(\omega, k) \in \mathbb{R} \times \mathbb{Z}$. To be more precise, for $(\omega, k) \in \mathbb{R} \times \mathbb{Z}$, we introduce the function

$$H_{\omega,k}(x) := k^2 - \omega^2 e^{2x}, \quad x \in \mathbb{R}. \tag{3.5}$$

and we denote by $x_0^*(\omega, k)$ its zero:

$$x_0^*(\omega, k) = \log\left(\frac{|k|}{|\omega|}\right). \tag{3.6}$$

Since $H_{\omega,k}$ is a decreasing function of x and $H_{\omega,k}(x_0^*(\omega, k)) = 0$, it is positive for $x < x_0^*(\omega, k)$ and negative for $x > x_0^*(\omega, k)$. We shall thus do as follows:

- When $x \in [\max\{a_0, x_0^*(\omega, k)\}, a_1]$, corresponding to $|\omega| e^x \geqslant |k|$, we use a multiplier method, see Section 3.2 and Corollary 3.6.
- When $x \in [a_0, \min\{x_0^*(\omega, k), a_1\}]$, corresponding to $|\omega| e^x \leqslant |k|$, we use a factorization argument to get an estimate on $z_k(a_0)$, see Section 3.3 and Lemma 3.11.
- And actually, when $x_0^*(\omega, k) > a_0$, we also apply the multiplier method in the whole interval (a_0, a_1) to conclude the resolvent estimate (3.3) more directly, see Theorem 3.8.

Remark 3.1. Neglecting the boundary, the symbol of the operator appearing in the right hand-side of equation (3.1) is of the form $p(x, \omega, k, \xi) = -\xi^2 - k^2 + \omega^2 e^{2x}$, where $x \in \mathbb{R}$, $\omega \in \mathbb{R}$, $k \in \mathbb{Z}$ and $\xi \in \mathbb{R}$, where ξ is the dual variable of $x \in \mathbb{R}$, and the above mentioned cases can be clearly understood at this level:

- When $x \geqslant x_0^*(\omega, k)$, that is $|\omega| e^x \geqslant |k|$, the polynomial $\xi \mapsto p(x, \omega, k, \xi)$ has two distinct real roots. The equation is thus hyperbolic there.

- When $x \leqslant x_0^*(\omega, k)$, that is $|\omega| e^x \leqslant |k|$, the polynomial $\xi \mapsto p(x, \omega, k, \xi)$ has two distinct imaginary roots. The equation is thus elliptic there.

It then appears clearly that our model is in fact closely related to the Friedlander model (see [12]) analyzed in details with homogeneous Dirichlet boundary conditions, and indeed shares some of its features.

3.1. **Bounded frequency parameters**

The first estimate we give here is rather classical and focuses on the case where $|\omega| + |k|$ is bounded. It is based on the following result:

LEMMA 3.2. *Let $M > 0$. There exists C such that for all $q \in L^\infty(a_0, a_1)$ with $\|q\|_{L^\infty} \leqslant M$ and for all z in $H^2(a_0, a_1)$ satisfying $z(a_1) = 0$,*

$$\begin{aligned} \|\partial_x z\|_{L^2(a_0,a_1)} + \|z\|_{L^2(a_0,a_1)} + |\partial_x z(a_0)| + |z(a_0)| \\ \leqslant C \left\| -\partial_x^2 z + qz \right\|_{L^2(a_0,a_1)} + C|\partial_x z(a_1)|. \end{aligned} \tag{3.7}$$

In particular, for all $M > 0$, estimates (3.3)–(3.4), for $z \in H^2(a_0, a_1)$ solution of (3.1), holds uniformly in the ball

$$\left\{(\omega, k) \in \mathbb{R} \times \mathbb{Z}, \, |k|^2 + |\omega|^2 R_0^2 \leqslant M^2\right\}.$$

Proof. We first deal with the case $q = 0$. For any $s > 0$ and $z \in H^2(a_0, a_1)$ with $z(a_1) = 0$, we set $w = e^{sx} z$, and we consider the conjugate operator

$$P_s w := e^{sx} \partial_x^2 (e^{-sx} w) = (\partial_x - s)^2 w, \tag{3.8}$$

so that $P_s w = e^{sx} \partial_x^2 z$.

Setting

$$v := (\partial_x - s) w, \tag{3.9}$$

we have

$$P_s w := (\partial_x - s) v. \tag{3.10}$$

Noting that by (3.9) one has

$$|v|^2 = |\partial_x w|^2 + s^2 |w|^2 - s\, \partial_x(|w|^2),$$

and integrating over (a_0, a_1), we get

$$s|w(a_0)|^2 + \int_{a_0}^{a_1} (|\partial_x w|^2 + s^2 |w|^2) \mathrm{d}x = \int_{a_0}^{a_1} |v|^2 \mathrm{d}x + s|w(a_1)|^2. \tag{3.11}$$

From $|(\partial_x - s) v|^2 = |\partial_x v|^2 + s^2 |v|^2 - s \partial_x(|v|^2)$, we get similarly

$$s|v(a_0)|^2 + s^2 \int_{a_0}^{a_1} |v|^2 \mathrm{d}x \leqslant \int_{a_0}^{a_1} |P_s w|^2 \mathrm{d}x + s|v(a_1)|^2. \tag{3.12}$$

It stems from these estimates that

$$s^3|w(a_0)|^2 + s|v(a_0)|^2 + s^4\|w\|^2_{L^2(a_0,a_1)} + s^2\|\partial_x w\|^2_{L^2(a_0,a_1)} \leqslant \|P_s w\|^2_{L^2(a_0,a_1)} + s|v(a_1)|^2 + s^3|w(a_1)|^2.$$

Since $w = e^{sx}z$, recalling $z(a_1) = 0$, we have

$$w(a_1) = 0 \text{ and } v(a_1) = (\partial_x - s)(e^{sx}z)(a_1) = e^{sa_1}\partial_x z(a_1).$$

Similarly, $w(a_0) = z(a_0)e^{sa_0}$, and $v(a_0) = (\partial_x z(a_0) - sz(a_0))e^{sa_0}$. Accordingly, recalling also that $P_s w = e^{sx}\partial_x^2 z$, there exists a constant $C > 0$ such that for all $s \geqslant 1$,

$$e^{2sa_0}\left(s^3|z(a_0)|^2 + s|\partial_x z(a_0)|^2\right) + s^4\|e^{sx}z\|^2_{L^2(a_0,a_1)} + s^2\|e^{sx}\partial_x z\|^2_{L^2(a_0,a_1)} \leqslant C^2\left(\left\|e^{sx}\partial_x^2 z\right\|^2_{L^2(a_0,a_1)} + se^{2sa_1}|\partial_x z(a_1)|^2\right). \tag{3.13}$$

Now, since we assume $\|q\|_{L^\infty} \leqslant M$, taking $s \geqslant 2\sqrt{CM}$ in (3.13) we get

$$e^{2sa_0}\left(s^3|z(a_0)|^2 + s|\partial_x z(a_0)|^2\right) + s^4\|e^{sx}z\|^2_{L^2(a_0,a_1)} + s^2\|e^{sx}\partial_x z\|^2_{L^2(a_0,a_1)} \leqslant 4C^2\left(\left\|e^{sx}(-\partial_x^2 + q)z\right\|^2_{L^2(a_0,a_1)} + se^{2sa_1}|\partial_x z(a_1)|^2\right). \tag{3.14}$$

Finally, taking lower and upper bounds for the weights e^{sx} and fixing s, we easily deduce Lemma 3.2. □

3.2. High-frequency estimate $|\omega| \ggg |k|$: the multiplier method

As we said previously, here, $|\omega| \ggg |k|$ means that $x_0^*(\omega, k)$ defined in (3.6) is such that $x_0^*(\omega, k) < a_1$. As we shall see, our arguments below will be used mainly on the interval $[\max\{a_0, x_0^*(\omega, k)\}, a_1]$.

LEMMA 3.3. *For any $x_0 \in [a_0, a_1)$, for any $(\omega, k) \in \mathbb{R} \times \mathbb{Z}$, any solution z_k of equation* (3.1) *satisfies*

$$|\partial_x z_k(x_0)|^2 + \Re\left(\bar{z}_k(x_0)\partial_x z_k(x_0)\right) + \int_{x_0}^{a_1}\left[|\partial_x z_k|^2 + \left(k^2 + \omega^2 e^{2x}\right)|z_k|^2\right] \mathrm{d}x = |\partial_x z_k(a_1)|^2 - \Re\left(\int_{x_0}^{a_1}(2\partial_x \bar{z}_k + \bar{z}_k)(i\omega F_{1,k} + F_{2,k})\,\mathrm{d}x\right) + H_{\omega,k}(x_0)|z_k(x_0)|^2. \tag{3.15}$$

Remark 3.4. Note that, when considering Dirichlet boundary conditions $z_k(a_0) = 0$, Lemma 3.3 applied to $x_0 = a_0$ proves the resolvent estimate at once, provided we suitably estimate the terms involving $F_{1,k}$ and $F_{2,k}$, which can of course be done (see for instance the proof of estimate (3.25) afterwards, which can be easily adapted to the case $z_k(a_0) = 0$). In fact, this is the usual multiplier method.

Proof. Multiplying equation (3.1) by $(2\partial_x \bar{z}_k + \bar{z}_k)$ and integrating on (x_0, a_1) easily leads to

$$\begin{aligned}|\partial_x z_k(x_0)|^2 - \left(k^2 - \omega^2 e^{2x}\right)|z_k(x_0)|^2 + \Re\left(\bar{z}_k(x_0)\partial_x z_k(x_0)\right)\\ + \int_{x_0}^{a_1} \left(|\partial_x z_k|^2 + \left(k^2 + \omega^2 e^{2x}\right)|z_k|^2\right) \mathrm{d}x\\ = |\partial_x z_k(a_1)|^2 - \Re\left(\int_{x_0}^{a_1} (2\partial_x \bar{z}_k + \bar{z}_k)(i\omega F_{1,k} + F_{2,k})\,\mathrm{d}x\right)\end{aligned}$$

since $z_k(a_1) = 0$. Of course, this coincides with the multiplier identity (3.15). □

Remark 3.5. Another proof of Lemma 3.3 can be obtained by considering the quantity

$$E_k(x) = |\partial_x z_k(x)|^2 - \left(k^2 - \omega^2 e^{2x}\right)|z_k(x)|^2 + \Re\left(\bar{z}_k(x)\partial_x z_k(x)\right), \quad (x \in [a_0, a_1]),$$

and compute

$$\begin{aligned}\frac{d}{\mathrm{d}x}E_k(x) + |\partial_x z_k(x)|^2 + \left(k^2 + \omega^2 e^{2x}\right)|z_k(x)|^2\\ = \Re\left(\left(2\partial_x \bar{z}_k(x) + \bar{z}_k(x)\right)\left(i\omega F_{1,k}(x) + F_{2,k}(x)\right)\right).\end{aligned}$$

Of course, this approach yields the same identity as in (3.15), but the above computation can be interpreted as a version of the so-called lateral propagation of the energy in 1-space dimension for the resolvent equation corresponding to the wave equation.

An immediate corollary of Lemma 3.3 is the following one:

Corollary 3.6. *Let $M \geqslant 2$. There exists a constant $C > 0$ such that for any z_k satisfying* (3.1) *for $(\omega, k) \in \mathbb{R} \times \mathbb{Z}$ with $\omega^2 R_0^2 + k^2 \geqslant M^2$ with $x_0^*(\omega, k) < a_1$ and $(F_{1,k}, F_{2,k})$ as in* (3.2)*, for all $x_0 \in [\max\{a_0, x_0^*(\omega, k)\}, a_1]$,*

$$\begin{aligned}|\partial_x z_k(x_0)|^2 + \max\left\{\sqrt{k^2 + \omega^2 e^{2x_0}}, \omega^2 e^{2x_0} - k^2\right\}|z_k(x_0)|^2\\ + \int_{x_0}^{a_1} \left[|\partial_x z_k|^2 + \left(k^2 + \omega^2 e^{2x}\right)|z_k|^2\right] \mathrm{d}x\\ \leqslant C|\partial_x z_k(a_1)|^2 + C\int_{x_0}^{a_1} \left(|\partial_x F_{1,k}|^2 + k^2|F_{1,k}|^2 + |F_{2,k}|^2\right)\mathrm{d}x.\end{aligned} \tag{3.16}$$

In particular, if $\omega^2 R_0^2 + k^2 \geqslant M^2$ and $x_0^*(\omega, k) \leqslant a_0$ (i.e. $\omega^2 R_0^2 \geqslant k^2$), taking $x_0 = a_0$ in the above estimate yields the resolvent estimate

$$
\begin{aligned}
(3.17)\quad & |\partial_x z_k(a_0)|^2 + \max\left\{\sqrt{k^2+\omega^2 R_0^2}, \omega^2 R_0^2 - k^2\right\} |z_k(a_0)|^2 \\
& + \int_{a_0}^{a_1} \left[|\partial_x z_k|^2 + \left(k^2 + \omega^2 e^{2\,x}\right) |z_k|^2 \right] \mathrm{d}x \\
& \leqslant C |\partial_x z_k(a_1)|^2 + C \int_{a_0}^{a_1} (|\partial_x F_{1,k}|^2 + k^2 |F_{1,k}|^2 + |F_{2,k}|^2) \,\mathrm{d}x,
\end{aligned}
$$

that is estimates (3.3)–(3.4).

Proof. We start from (3.15), and we notice that for $x_0 \geqslant x_0^*(\omega, k)$, $H_{\omega,k}(x_0) = k^2 - \omega^2 e^{2x_0} \leqslant 0$. Thus, we deduce

$$
\begin{aligned}
(3.18)\quad & |\partial_x z_k(x_0)|^2 + \Re(\overline{z}_k(x_0)\partial_x z_k(x_0)) + (\omega^2 e^{2x_0} - k^2)|z_k(x_0)|^2 \\
& + \int_{x_0}^{a_1} \left[|\partial_x z_k|^2 + \left(k^2 + \omega^2 e^{2\,x}\right) |z_k|^2 \right] \mathrm{d}x \\
& = |\partial_x z_k(a_1)|^2 + 2\Re\left(\int_{x_0}^{a_1} \left(\partial_x \overline{z}_k + \frac{\overline{z}_k}{2} \right) (i\omega F_{1,k} + F_{2,k}) \,\mathrm{d}x \right).
\end{aligned}
$$

Now, using $z_k(a_1) = 0$, we get

$$
\begin{aligned}
(3.19)\quad \sqrt{k^2+\omega^2 e^{2x_0}} |z_k(x_0)|^2 &= -2\sqrt{k^2+\omega^2 e^{2x_0}}\, \Re\left(\int_{x_0}^{a_1} (\partial_x z_k)\overline{z}_k \,\mathrm{d}x \right) \\
& \leqslant 2 \left\| \partial_x z_k \right\|_{L^2(x_0,a_1)} \left\| \sqrt{k^2+\omega^2 e^{2x_0}}\, z_k \right\|_{L^2(a_0,a_1)} \\
& \leqslant \int_{x_0}^{a_1} \left[|\partial_x z_k|^2 + \left(k^2 + \omega^2 e^{2\,x}\right) |z_k|^2 \right] \mathrm{d}x,
\end{aligned}
$$

which immediately implies that

(3.20)

$$
\begin{aligned}
& |\partial_x z_k(x_0)|^2 + \Re(\overline{z}_k(x_0)\partial_x z_k(x_0)) + ((\omega^2 e^{2x_0} - k^2) + \frac{1}{2}\sqrt{k^2+\omega^2 e^{2x_0}})|z_k(x_0)|^2 \\
& + \frac{1}{2} \int_{x_0}^{a_1} \left[|\partial_x z_k|^2 + \left(k^2 + \omega^2 e^{2\,x}\right) |z_k|^2 \right] \mathrm{d}x \\
& \leqslant |\partial_x z_k(a_1)|^2 + 2 \left| \int_{x_0}^{a_1} \left(\partial_x \overline{z}_k + \frac{\overline{z}_k}{2} \right) (i\omega F_{1,k} + F_{2,k}) \,\mathrm{d}x \right|.
\end{aligned}
$$

Then, from $\omega^2 R_0^2 + k^2 \geqslant 4$, *i.e.* $\sqrt{k^2+\omega^2 e^{2x_0}} \geqslant 2$, we obtain

$$
\begin{aligned}
|\overline{z}_k(x_0)\partial_x z_k(x_0)| & \leqslant \frac{1}{2}|\partial_x z_k(x_0)|^2 + \frac{1}{2}|z_k(x_0)|^2 \\
& \leqslant \frac{1}{2}|\partial_x z_k(x_0)|^2 + \frac{1}{4}\sqrt{k^2+\omega^2 e^{2x_0}}|z_k(x_0)|^2,
\end{aligned}
$$

and

$$\frac{1}{2}|\partial_x z_k(x_0)|^2 + ((\omega^2 e^{2x_0} - k^2) + \frac{1}{4}\sqrt{k^2+\omega^2 e^{2x_0}})|z_k(x_0)|^2 + \frac{1}{2}\int_{x_0}^{a_1}\left[|\partial_x z_k|^2 + \left(k^2+\omega^2 e^{2\,x}\right)|z_k|^2\right]\mathrm{d}x \leqslant |\partial_x z_k(a_1)|^2 + 2\left|\int_{x_0}^{a_1}\left(\partial_x \overline{z}_k + \frac{\overline{z}_k}{2}\right)(i\omega F_{1,k} + F_{2,k})\,\mathrm{d}x\right|. \tag{3.21}$$

We now write that

$$\int_{x_0}^{a_1}\left(\partial_x \overline{z}_k + \frac{\overline{z}_k}{2}\right)(i\omega F_{1,k} + F_{2,k})\,\mathrm{d}x = \overline{z}_k(a_1) i\omega F_{1,k}(a_1) - \overline{z}_k(x_0) i\omega F_{1,k}(x_0) + \int_{x_0}^{a_1}\left(-i\omega \overline{z}_k \partial_x F_{1,k} + \partial_x \overline{z}_k F_{2,k} + \frac{i\omega \overline{z}_k}{2} F_{1,k} + \frac{\overline{z}_k}{2} F_{2,k}\right)\mathrm{d}x. \tag{3.22}$$

Using $F_{1,k}(a_1) = 0$, and the fact that, similarly as in (3.19),

$$|k||F_{1,k}(x_0)|^2 \leqslant \int_{x_0}^{a_1}(|\partial_x F_{1,k}|^2 + k^2|F_{1,k}|^2)\,\mathrm{d}x, \tag{3.23}$$

and that, for $x_0 \geqslant x_0^*(\omega, k)$, $\omega^2 e^{2x_0} - k^2 \geqslant 0$, the boundary terms in (3.22) can be estimated as follows:

$$\begin{aligned}
&|\overline{z}_k(a_1) i\omega F_{1,k}(a_1) - \overline{z}_k(x_0) i\omega F_{1,k}(x_0)| \leqslant |\omega||z_k(x_0)||F_{1,k}(x_0)| \\
&\leqslant \frac{1}{16}\max\left\{\sqrt{k^2+\omega^2 e^{2x_0}}, \omega^2 e^{2x_0} - k^2\right\}|z_k(x_0)|^2 \\
&\qquad + C\frac{\omega^2}{\max\left\{\sqrt{k^2+\omega^2 e^{2x_0}}, \omega^2 e^{2x_0} - k^2\right\}}|F_{1,k}(x_0)|^2 \\
&\leqslant \frac{1}{16}\max\left\{\sqrt{k^2+\omega^2 e^{2x_0}}, \omega^2 e^{2x_0} - k^2\right\}|z_k(x_0)|^2 + C|k||F_{1,k}(x_0)|^2 \\
&\leqslant \frac{1}{16}\max\left\{\sqrt{k^2+\omega^2 e^{2x_0}}, \omega^2 e^{2x_0} - k^2\right\}|z_k(x_0)|^2 \\
&\qquad + C\int_{x_0}^{a_1}(|\partial_x F_{1,k}|^2 + k^2|F_{1,k}|^2)\,\mathrm{d}x.
\end{aligned}$$

On the other hand, we easily get

$$\begin{aligned}
&\left| \int_{x_0}^{a_1} \left(-i\omega \overline{z}_k \partial_x F_{1,k} + \partial_x \overline{z}_k F_{2,k} + \frac{i\omega \overline{z}_k}{2} F_{1,k} + \frac{\overline{z}_k}{2} F_{2,k} \right) \mathrm{d}x \right| \\
&\leqslant \|\omega z_k\|_{L^2(x_0,a_1)} \|\partial_x F_{1,k}\|_{L^2(x_0,a_1)} + \|\omega z_k\|_{L^2(x_0,a_1)} \|F_{1,k}\|_{L^2(x_0,a_1)} \\
&\qquad + (\|\partial_x z_k\|_{L^2(x_0,a_1)} + \|\omega z_k\|_{L^2(x_0,a_1)}) \|F_{2,k}\|_{L^2(x_0,a_1)} \\
&\leqslant \frac{1}{8} \left(\|\partial_x z_k\|^2_{L^2(x_0,a_1)} + \|\omega e^x z_k\|^2_{L^2(x_0,a_1)} \right) \\
&\qquad + C \int_{x_0}^{a_1} (|\partial_x F_{1,k}|^2 + k^2 |F_{1,k}|^2 + |F_{2,k}|^2) \, \mathrm{d}x.
\end{aligned} \tag{3.24}$$

Combining the last two estimates with (3.22), we obtain

$$\begin{aligned}
&\left| \int_{x_0}^{a_1} \left(\partial_x \overline{z}_k + \frac{\overline{z}_k}{2} \right) (i\omega F_{1,k} + F_{2,k}) \, \mathrm{d}x \right| \\
&\leqslant \frac{1}{16} \max \left\{ \sqrt{k^2 + \omega^2 e^{2x_0}}, \omega^2 e^{2x_0} - k^2 \right\} |z_k(x_0)|^2 \\
&\quad + \frac{1}{8} \left(\|\partial_x z_k\|^2_{L^2(x_0,a_1)} + \|\omega e^x z_k\|^2_{L^2(x_0,a_1)} \right) \\
&\qquad + C \int_{x_0}^{a_1} (|\partial_x F_{1,k}|^2 + k^2 |F_{1,k}|^2 + |F_{2,k}|^2) \, \mathrm{d}x.
\end{aligned} \tag{3.25}$$

for some C independent of (ω, k). Plugging this last estimate in (3.21), we obtain (3.16).

The fact that the resolvent estimate (3.16) yields the resolvent estimate (3.3) when (ω, k) satisfy $\omega^2 R_0^2 + k^2 \geqslant 4$ and $x_0^*(\omega, k) \leqslant a_0$ easily follows. □

Another interesting straightforward corollary of the identity (3.15) is the following one:

Corollary 3.7. *Let $M \geqslant 2$. For any $(\omega, k) \in \mathbb{R} \times \mathbb{Z}$ with $\omega^2 R_0^2 + k^2 \geqslant M^2$ and $\omega^2 R_0^2 \leqslant k^2$, any solution z_k of equation* (3.1) *satisfies, for some C independent of k,*

$$\begin{aligned}
&\sqrt{k^2 + \omega^2 e^{2a_0}} |z_k(a_0)|^2 + \int_{a_0}^{a_1} \left[|\partial_x z_k|^2 + \left(k^2 + \omega^2 e^{2x} \right) |z_k|^2 \right] \mathrm{d}x \\
&\leqslant C |\partial_x z(a_1)|^2 + C \int_{a_0}^{a_1} (|\partial_x F_{1,k}|^2 + k^2 |F_{1,k}|^2 + |F_{2,k}|^2) \, \mathrm{d}x \\
&\qquad + C (H_{\omega,k}(a_0) - \rho(\omega,k)^2 - \rho(\omega,k)) |z_k(a_0)|^2.
\end{aligned} \tag{3.26}$$

Indeed, this corollary is interesting since it implies that if we know how to estimate $(H_{\omega,k}(a_0) - \rho(\omega,k)^2 - \rho(\omega,k)) |z_k(a_0)|^2$ by the squares of the right hand side of (3.3), we immediately derive the resolvent estimate (3.3). This is in fact the strategy we follow hereafter in Section 3.3.

Proof. We start from the fact that, coming from (3.20) applied to $x_0 = a_0$ and using the fact that $\partial_x z_k(a_0) = \rho(\omega, k) z_k(a_0)$, that for any $(\omega, k) \in \mathbb{R} \times \mathbb{Z}$, any solution z_k of equation (3.1) satisfies

$$\begin{aligned} &\left(\rho(\omega,k)^2 + \rho(\omega,k) + (\omega^2 e^{2a_0} - k^2) + \frac{1}{2}\sqrt{k^2 + \omega^2 e^{2a_0}}\right) |z_k(a_0)|^2 \\ &\quad + \frac{1}{2}\int_{a_0}^{a_1} \left[|\partial_x z_k|^2 + \left(k^2 + \omega^2 e^{2x}\right)|z_k|^2\right] \mathrm{d}x \\ &\quad \leqslant |\partial_x z_k(a_1)|^2 + 2\left|\int_{a_0}^{a_1} \left(\partial_x \overline{z}_k + \frac{\overline{z}_k}{2}\right)(i\omega F_{1,k} + F_{2,k})\,\mathrm{d}x\right|. \end{aligned} \tag{3.27}$$

To bound the last term, we use again identity (3.22). Recalling $z_k(a_1) = 0$, the pointwise estimate (3.23) applied to $x_0 = a_0$, and using $\omega^2 R_0^2 \leqslant k^2$, we get, for all $\alpha > 0$, (with $C_\alpha = 1/(4\alpha)$):

$$\begin{aligned} |\overline{z}_k(a_1) i\omega F_{1,k}(a_1) - \overline{z}_k(a_0) i\omega F_{1,k}(a_0)| &\leqslant |\omega||z_k(a_0)||F_{1,k}(a_0)| \\ &\leqslant \alpha \frac{|\omega|^2}{|k|}|z_k(a_0)|^2 + C_\alpha |k||F_{1,k}(a_0)|^2 \\ &\leqslant \alpha \frac{|k|}{R_0}|z_k(a_0)|^2 + C_\alpha \int_{a_0}^{a_1} (|\partial_x F_{1,k}|^2 + k^2|F_{1,k}|^2)\,\mathrm{d}x. \end{aligned} \tag{3.28}$$

Similarly as in (3.24), we also get, for all $\alpha > 0$, the existence of a constant $C_\alpha > 0$ such that

$$\begin{aligned} &\left|\int_{a_0}^{a_1} \left(-i\omega \overline{z}_k \partial_x F_{1,k} + \partial_x \overline{z}_k F_{2,k} + \frac{i\omega \overline{z}_k}{2} F_{1,k} + \frac{\overline{z}_k}{2} F_{2,k}\right) \mathrm{d}x\right| \\ &\quad \leqslant \alpha \left(\|\partial_x z_k\|^2_{L^2(a_0,a_1)} + \|\omega e^x z_k\|^2_{L^2(a_0,a_1)}\right) \\ &\qquad + C_\alpha \int_{a_0}^{a_1} (|\partial_x F_{1,k}|^2 + k^2|F_{1,k}|^2 + |F_{2,k}|^2)\,\mathrm{d}x. \end{aligned} \tag{3.29}$$

Choosing $\alpha > 0$ small enough in the above estimates, we get that there exists $C > 0$ such that

$$\begin{aligned} 2\left|\int_{a_0}^{a_1} \left(\partial_x \overline{z}_k + \frac{\overline{z}_k}{2}\right)(i\omega F_{1,k} + F_{2,k})\,\mathrm{d}x\right| &\leqslant \frac{1}{4}\sqrt{k^2 + \omega^2 e^{2a_0}}|z_k(a_0)|^2 \\ &\quad + \frac{1}{4}\int_{a_0}^{a_1} \left[|\partial_x z_k|^2 + \left(k^2 + \omega^2 e^{2x}\right)|z_k|^2\right] \mathrm{d}x \\ &\quad + C\int_{a_0}^{a_1} (|\partial_x F_{1,k}|^2 + k^2|F_{1,k}|^2 + |F_{2,k}|^2)\,\mathrm{d}x. \end{aligned}$$

Combined with (3.27), we easily deduce (3.26). □

3.3. High-frequency estimate $|\omega| \lll |k|$: a factorization argument

When $x_0^*(\omega, k) > a_0$, *i.e.* when $k^2 > \omega^2 R_0^2$, the multiplier identity (3.15) does not allow to conclude immediately since then $H_{\omega,k}(a_0) > 0$.

In this section, we shall always assume that $(\omega, k) \in \mathbb{R} \times \mathbb{Z}$ satisfy

(3.30)
$$k^2 \geqslant \omega^2 R_0^2, \quad \text{(equivalently, } x_0^*(\omega, k) \geqslant a_0\text{)}, \qquad \text{and} \qquad k^2 + \omega^2 R_0^2 \geqslant M^2,$$

where M is the constant appearing in Theorem 1.2.

The main result of this section is the following:

THEOREM 3.8. *Assume that $\rho = \rho(\omega, k)$ is as in* (1.14) *and satisfies the condition* (1.16)*. Then there exist $C > 0$ and $A > 0$ such that for all $(\omega, k) \in \mathbb{R} \times \mathbb{Z}$ satisfying* (3.30) *and any solution z_k of* (3.1) *with $F_{1,k} \in H^1(a_0, a_1)$ satisfying $F_{1,k}(a_1) = 0$, $F_{2,k} \in L^2(a_0, a_1)$ and $g_k \in \mathbb{C}$,*

(3.31)
$$\begin{aligned}\Big(I_\varepsilon(\omega, k)^2 1_{a_0 \leqslant x_0^*(\omega,k) - A|k|^{-2/3}} \\ + \max\{|k|^{4/3}, I_1(\omega, k)^2\} 1_{a_0 \geqslant x_0^*(\omega,k) - A|k|^{-2/3}}\Big) |z_k(a_0)|^2 \\ + \int_{a_0}^{a_1} \left[|\partial_x z_k|^2 + \left(k^2 + \omega^2 e^{2x}\right) |z_k|^2\right] \mathrm{d}x \\ \leqslant C \int_{a_0}^{a_1} (|\partial_x F_{1,k}|^2 + k^2 |F_{1,k}|^2 + |F_{2,k}|^2) \, \mathrm{d}x \\ + C|g_k|^2 + C|\partial_x z_k(a_1)|^2.\end{aligned}$$

Proof. Below, we design a factorization technique to get the resolvent estimate (3.31), which consists in writing the differential operator in (3.1) under the form

(3.32)
$$\partial_x^2 - H_{\omega,k}(x) = (\partial_x - X_{\omega,k})(\partial_x + X_{\omega,k}),$$

where $X_{\omega,k}$ will be assumed to be a non-negative function of x such that

(3.33)
$$\begin{cases} \dfrac{d}{\mathrm{d}x} X_{\omega,k}(x) = X_{\omega,k}^2(x) - H_{\omega,k}(x), \quad x \in [a_0, x^*], \\ X_{\omega,k}(x^*) = X^*, \end{cases}$$

where $x^* > a_0$ and X^* are real numbers to be chosen later.

Indeed, this will allow us to get an estimate on $|z_k(a_0)|$ according to the following lemma:

LEMMA 3.9. *Let us assume that that there exists $x^* \in (a_0, a_1]$ and $X^* \geqslant 0$ such that the ODE* (3.33) *has a solution $X_{\omega,k}$ on the interval $[a_0, x^*]$ satisfying $X_{\omega,k}(x) \in [0, |k|]$ for all $x \in [a_0, x^*]$.*

Then any solution z_k of equation (3.1) satisfies the following estimate: for all $\alpha > 0$, there exists C independent of (ω, k) such that

$$(3.34)\quad (\rho(\omega,k) + X_{\omega,k}(a_0))^2 |z_k(a_0)|^2 \leqslant \alpha \int_{a_0}^{a_1} \left[|\partial_x z_k|^2 + k^2 |z_k|^2\right] \mathrm{d}x + C|\partial_x z_k(x^*)| + C|X^*|^2 |z_k(x^*)|^2 + C|g_k|^2 + C\int_{a_0}^{a_1} (|\partial_x F_{1,k}|^2 + k^2 |F_{1,k}|^2 + |F_{2,k}|^2)\,\mathrm{d}x.$$

Proof. Setting $w_k = \partial_x z_k + X_{\omega,k} z_k$ on the set (a_0, x^*), where z_k solves (3.1), we get that

$$(3.35)\quad \begin{cases} (\partial_x - X_{\omega,k}) w_k = i\omega F_{1,k} + F_{2,k}, & \text{in } (a_0, x^*) \\ w_k(x^*) = \partial_x z_k(x^*) + X_{\omega,k}(x^*) z_k(x^*) = \partial_x z_k(x^*) + X^* z_k(x^*). \end{cases}$$

Multiplying the equation by $\overline{w}_k$ and integrating between a_0 and x^*, we get

$$\frac{1}{2}\left(|w_k(x^*)|^2 - |w_k(a_0)|^2\right) - \int_{a_0}^{x^*} X_{\omega,k} |w_k|^2 \,\mathrm{d}x = \Re\left(\int_{a_0}^{x^*} (i\omega F_{1,k} + F_{2,k})\overline{w}_k \,\mathrm{d}x\right).$$

Recalling that $w_k = \partial_x z_k + X_{\omega,k} z_k$, $X_{\omega,k}$ is assumed to be non-negative and that $w_k(a_0) = \partial_x z_k(a_0) + X_{\omega,k}(a_0) z_k(a_0) = (\rho(\omega,k) + X_{\omega,k}(a_0)) z_k(a_0) + g_k$, we easily deduce:

$$(3.36)\quad (\rho(\omega,k) + X_{\omega,k}(a_0))^2 |z_k(a_0)|^2 \leqslant C\left|\int_{a_0}^{x^*} (i\omega F_{1,k} + F_{2,k})(\partial_x \overline{z}_k + X_{\omega,k}\overline{z}_k)\,\mathrm{d}x\right| + C|\partial_x z_k(x^*)|^2 + C|X^*|^2 |z_k(x^*)|^2 + C|g_k|^2.$$

Then, similarly as in the proofs of Corollary 3.6 and Corollary 3.7, we estimate

$$(3.37)\quad \int_{a_0}^{x^*} (i\omega F_{1,k} + F_{2,k})(\partial_x \overline{z}_k + X_{\omega,k}\overline{z}_k)\,\mathrm{d}x = i\omega F_{1,k}(x^*)\overline{z}_k(x^*) - i\omega F_{1,k}(a_0)\overline{z}_k(a_0) + \int_{a_0}^{x^*} (-i\omega \partial_x F_{1,k}\overline{z}_k + F_{2,k}\partial_x \overline{z}_k + X_{\omega,k} F_{1,k} i\omega \overline{z}_k + F_{2,k} X_{\omega,k}\overline{z}_k)\,\mathrm{d}x.$$

Since $|\omega| \leqslant |k|/R_0$, we have from (3.23) that, for all $\alpha > 0$, there exists $C_\alpha > 0$ such that

$$\begin{aligned} &|i\omega F_{1,k}(a_0)\overline{z}_k(a_0)| + |i\omega F_{1,k}(x^*)\overline{z}_k(x^*)| \\ &\quad\leqslant C|k|\left(|F_{1,k}(a_0)||z_k(a_0)| + |F_{1,k}(x^*)||z_k(x^*)|\right) \\ &\quad\leqslant C_\alpha \int_{a_0}^{a_1} (|\partial_x F_{1,k}|^2 + k^2|F_{1,k}|^2)\,\mathrm{d}x + \frac{\alpha}{4}|k|(|z_k(a_0)|^2 + |z_k(x^*)|^2) \\ (3.38)\quad &\quad\leqslant C_\alpha \int_{a_0}^{a_1} (|\partial_x F_{1,k}|^2 + k^2|F_{1,k}|^2)\,\mathrm{d}x + \frac{\alpha}{2}\int_{a_0}^{a_1} \left[|\partial_x z_k|^2 + k^2|z_k|^2\right]\mathrm{d}x. \end{aligned}$$

Using then that $|X_{\omega,k}| \leqslant k$, and that $|\omega| \leqslant |k|/R_0$,

$$\left| \int_{a_0}^{x^*} (-i\omega\partial_x F_{1,k}\overline{z}_k + F_{2,k}\partial_x\overline{z}_k + X_{\omega,k}F_{1,k}i\omega\overline{z}_k + F_{2,k}X_{\omega,k}\overline{z}_k)\,\mathrm{d}x \right|$$
$$\leqslant C\|\partial_x F_{1,k}\|_{L^2(a_0,a_1)}\||k|z_k\|_{L^2(a_0,a_1)} + \|F_{2,k}\|_{L^2(a_0,a_1)}\|\partial_x z_k\|_{L^2(a_0,a_1)}$$
$$+ C\||k|F_{1,k}\|_{L^2(a_0,a_1)}\||k|z_k\|_{L^2(a_0,a_1)} + C\|F_{2,k}\|_{L^2(a_0,a_1)}\||k|z_k\|_{L^2(a_0,a_1)}.$$

Thus, for all $\alpha > 0$, there exists $C_\alpha > 0$ such that

$$\left| \int_{a_0}^{x^*} (-i\omega\partial_x F_{1,k}\overline{z}_k + F_{2,k}\partial_x\overline{z}_k + X_{\omega,k}F_{1,k}i\omega\overline{z}_k + F_{2,k}X_{\omega,k}\overline{z}_k)\,\mathrm{d}x \right| \tag{3.39}$$
$$\leqslant C_\alpha \int_{a_0}^{a_1} (|\partial_x F_{1,k}|^2 + k^2|F_{1,k}|^2 + |F_{2,k}|^2)\,\mathrm{d}x + \frac{\alpha}{2}\int_{a_0}^{a_1} \left[|\partial_x z_k|^2 + k^2|z_k|^2\right]\,\mathrm{d}x.$$

Plugging this last estimate and (3.38) in the identity (3.37), we easily deduce (3.34) from the estimate (3.36). □

Lemma 3.9 indicates that, provided we get non-negative solutions $X_{\omega,k}$ of (3.33) on an interval of the form $[a_0, x^*]$, one can get an estimate on $|\rho + X_{\omega,k}(a_0)||z_k(a_0)|$ from an estimate on $|\partial_x z_k(x^*)| + |X^*||z_{\omega,k}(x^*)|$, other terms involving the source terms $F_{1,k}$, $F_{2,k}$ and g_k of (3.1), and a rather weak dependence on z_k in (a_0, a_1), quantified through a parameter $\alpha > 0$ that can be made arbitrarily small.

Our goal now is to combine estimate (3.34) with Corollary 3.6. To do so, we aim to choose x^* in $[x_0^*(\omega,k), a_1]$ and $|X^*|^2 = \theta^2(\omega^2 e^{2x^*} - k^2)$ for some positive constant θ independent of ω and k. We see that X^* becomes larger as x^* is chosen further from $x_0^*(\omega,k)$, and as a consequence the estimate becomes better. However, we can prove the existence of $X_{\omega,k}$ solving (3.33) on $[a_0, x^*]$ only for x^* close enough to $x_0^*(\omega,k)$. Hence a compromise has to be made.

More precisely, for $x^* = x_0^*(\omega,k) + c/|k|^\alpha$ with $c \geqslant 0$ and $\alpha > 0$, it is natural to choose $X_* = \theta\sqrt{2c}|k|^{1-\alpha/2}$ for some positive constant θ. In the following, we make two specific choices, the first one being $x^* = x_0^*(\omega,k)$ and $X^* = 0$ (that is $c = 0$), and the second one $x^* = x_0^*(\omega,k) + 1/(2|k|^{2/3})$ and $X^* = 2|k|^{2/3}$ (that is $c = 1/2$, $\theta = 2$, and $\alpha = 2/3$). The solutions corresponding to these choices enjoy the following properties:

Lemma 3.10. *Let $(\omega,k) \in \mathbb{R}\times\mathbb{Z}$ satisfying (3.30).*
The solution $X_{0,\omega,k}$ of

$$\begin{cases} \dfrac{d}{\mathrm{d}x} X_{0,\omega,k}(x) = X_{0,\omega,k}^2(x) - H_{\omega,k}(x), \quad x \in [a_0, x_0^*(\omega,k)], \\ X_{0,\omega,k}(x_0^*(\omega,k)) = 0, \end{cases} \tag{3.40}$$

is well-defined on $[a_0, x_0^*(\omega, k)]$ *and non-negative there.*

Set

$$x_1^*(\omega, k) = x_0^*(\omega, k) + \frac{1}{2|k|^{2/3}}. \tag{3.41}$$

Then the solution $X_{1,\omega,k}$ *of*

$$\begin{cases} \dfrac{d}{\mathrm{d}x} X_{1,\omega,k}(x) = X_{1,\omega,k}^2(x) - H_{\omega,k}(x), \quad x \in [a_0, x_1^*(\omega, k)], \\ X_{1,\omega,k}(x_1^*(\omega, k)) = 2|k|^{2/3}, \end{cases} \tag{3.42}$$

is well-defined on $[a_0, x_1^*(\omega, k)]$ *and non-negative there.*

Besides, we have the following estimates: For all $\varepsilon > 0$*, there exists* $A_\varepsilon > 0$ *independent of* (ω, k) *such that*

$$\begin{aligned} \forall x < x_0^*(\omega, k) - A_\varepsilon k^{-2/3}, \quad (1-\varepsilon)\sqrt{H_{\omega,k}(x)} \leqslant X_{0,\omega,k}(x) \leqslant X_{1,\omega,k}(x) \\ \leqslant \sqrt{H_{\omega,k}(x)} \leqslant |k|. \end{aligned} \tag{3.43}$$

For all $A > 0$*, there exists* $\delta_A > 0$ *independent of* (ω, k) *such that*

$$\forall x \in [x_0^*(\omega, k) - A|k|^{-2/3}, x_0^*(\omega, k)], \qquad X_{1,\omega,k}(x) - X_{0,\omega,k}(x) \geqslant \delta_A k^{2/3}. \tag{3.44}$$

Lemma 3.10 is one of the delicate points of our analysis, and its proof is postponed to Section 3.4. Using then Lemma 3.9, we can prove the resolvent estimates (3.3)–(3.4) for solutions z_k of (3.1) for (ω, k) satisfying (3.30).

Indeed, using Lemma 3.9 with $X_{0,\omega,k}$ on $[a_0, \tilde{x}_0^*]$ with $\tilde{x}_0^* = \min\{x_0^*(\omega, k), a_1\}$ and $X_{1,\omega,k}$ on $[a_0, \tilde{x}_1^*]$ with $\tilde{x}_1^* = \min\{x_1^*(\omega, k), a_1\}$ and summing the two estimates, we obtain the following: for all $\alpha > 0$, there exists $C > 0$ such that

$$\begin{aligned} &\left((\rho(\omega, k) + X_{0,\omega,k}(a_0))^2 + (\rho(\omega, k) + X_{1,\omega,k}(a_0))^2\right) |z_k(a_0)|^2 \\ &\leqslant \alpha \int_{a_0}^{a_1} \left[|\partial_x z_k|^2 + k^2 |z_k|^2\right] \mathrm{d}x + C|\partial_x z_k(x_0^*)|^2 + C|\partial_x z_k(x_1^*)|^2 + C|k|^{4/3}|z_k(x_1^*)|^2 \\ &\qquad + C|g_k|^2 + C\int_{a_0}^{a_1} (|\partial_x F_{1,k}|^2 + k^2|F_{1,k}|^2 + |F_{2,k}|^2)\,\mathrm{d}x. \end{aligned} \tag{3.45}$$

We shall now bound from below the left hand-side of (3.45):

Lemma 3.11. *Let* $\varepsilon > 0$ *and* $M \geqslant 1$ *and assume the condition* (1.16). *Then there exist a constant* $\delta_0 > 0$ *and* $A_\varepsilon > 0$ *such that such that for all* $(\omega, k) \in \mathbb{R} \times \mathbb{Z}$ *with* $k^2 + \omega^2 R_0^2 \geqslant M^2$ *and* $k^2 \geqslant \omega^2 R_0^2$*, (equivalently,* $x_0^*(\omega, k) \geqslant a_0$*),*

$$\begin{aligned} &\left((\rho(\omega, k) + X_{0,\omega,k}(a_0))^2 + (\rho(\omega, k) + X_{1,\omega,k}(a_0))^2\right) \\ &\geqslant \delta_0 (I_\varepsilon(\omega, k)^2 1_{a_0 \leqslant x_0^*(\omega,k) - A_\varepsilon |k|^{-2/3}} + \max\{|k|^{4/3}, I_1(\omega, k)^2\} 1_{a_0 \geqslant x_0^*(\omega,k) - A_\varepsilon |k|^{-2/3}} \\ &\qquad + H_{\omega,k}(a_0) - \rho(\omega, k)^2 - \rho(\omega, k)). \end{aligned} \tag{3.46}$$

Proof. First note that

$$(3.47)\quad (\rho(\omega,k)+X_{0,\omega,k}(a_0))^2+(\rho(\omega,k)+X_{1,\omega,k}(a_0))^2$$
$$=2\left(\left(\rho(\omega,k)+\frac{X_{0,\omega,k}(a_0)+X_{1,\omega,k}(a_0)}{2}\right)^2+\left(\frac{X_{1,\omega,k}(a_0)-X_{0,\omega,k}(a_0)}{2}\right)^2\right).$$

Now, let $\varepsilon>0$ and $M\geqslant 1$ and assume condition (1.16). Using Lemma 3.9, there exists $A_\varepsilon>\mathbf{r}$ ($\mathbf{r}$ is the constant in (1.15)) such that (3.43) holds for all $x<x_0^*(\omega,k)-A_\varepsilon|k|^{-2/3}$, and there exists $\delta_\varepsilon>0$ such that (3.44) holds for $x\in[x_0^*(\omega,k)-A_\varepsilon|k|^{-2/3},x_0^*(\omega,k)]$. Consequently, we shall discuss how a_0 compares to $x_0^*(\omega,k)-A_\varepsilon|k|^{-2/3}$.

If $a_0<x_0^*(\omega,k)-A_\varepsilon|k|^{-2/3}$, *i.e.* if $\log(|k|/|\omega|R_0)>A_\varepsilon|k|^{-2/3}$, using the condition (1.16) and the fact that

$$(1-\varepsilon)\sqrt{H_{\omega,k}(a_0)}\leqslant X_{0,\omega,k}(a_0)\leqslant X_{1,\omega,k}(a_0)\leqslant\sqrt{H_{\omega,k}(a_0)},$$

hence that there exists $c\in[1-\varepsilon,1]$ such that

$$\frac{X_{0,\omega,k}(a_0)+X_{1,\omega,k}(a_0)}{2}=c\sqrt{H_{\omega,k}(a_0)},$$

we easily deduce by bounding from below the first term of the right hand side of (3.47) that

$$(\rho(\omega,k)+X_{0,\omega,k}(a_0))^2+(\rho(\omega,k)+X_{1,\omega,k}(a_0))^2\geqslant 2\delta(H_{\omega,k}(a_0)-\rho(\omega,k)^2-\rho(\omega,k)).$$

If $a_0\in[x_0^*(\omega,k)-A_\varepsilon|k|^{-2/3},x_0^*(\omega,k)]$, *i.e.* if $\log(|k|/|\omega|R_0)>A_\varepsilon|k|^{-2/3}$, we simply use (3.44) and the second term in the right hand side of (3.47) to deduce that

$$(\rho(\omega,k)+X_{0,\omega,k}(a_0))^2+(\rho(\omega,k)+X_{1,\omega,k}(a_0))^2\geqslant 2\delta_\varepsilon|k|^{4/3}.$$

On the other hand, since $X_{0,\omega,k}(a_0)\in[0,\sqrt{H_{\omega,k}(a_0)}]$, we easily have

$$(\rho(\omega,k)+X_{0,\omega,k}(a_0))^2\geqslant I_1(\omega,k)^2.$$

To end the proof of Lemma 3.11, it suffices to notice that there exists a constant $C>0$ such that for all $a_0\in[x_0^*(\omega,k)-A_\varepsilon|k|^{-2/3},x_0^*(\omega,k)]$,

$$H_{\omega,k}(a_0)-\rho(\omega,k)^2-\rho(\omega,k)\leqslant H_{\omega,k}(x_0^*(\omega,k)-A_\varepsilon|k|^{-2/3})+1\leqslant C|k|^{4/3}.$$

This last estimate can be easily checked by looking at the asymptotic as $|k|\to\infty$ of

$$H_{\omega,k}(x_0^*(\omega,k)-A_\varepsilon|k|^{-2/3})=k^2-\omega^2e^{2x_0^*(\omega,k)-2A_\varepsilon|k|^{-2/3}}$$
$$=k^2\left(1-e^{-2A_\varepsilon|k|^{-2/3}}\right)\underset{|k|\to\infty}{\simeq}2A_\varepsilon k^{4/3}.$$

We then easily conclude Lemma 3.11. ☐

We then show how Lemma 3.11 and estimate (3.45) imply the resolvent estimate (3.3) for solutions z_k of (3.1) when (ω, k) are as in (3.30).

Using Lemma 3.11, we obtain that for all $\alpha > 0$, there exists C independent of (ω, k) satisfying (3.30) such that

$$
\begin{aligned}
(3.48)\quad & \Big(I_\varepsilon(\omega,k)^2 1_{a_0 \leqslant x_0^*(\omega,k) - A_\varepsilon |k|^{-2/3}} \\
&\quad + \max\{|k|^{4/3}, I_1(\omega,k)^2\} 1_{a_0 \geqslant x_0^*(\omega,k) - A_\varepsilon |k|^{-2/3}}\Big) |z_k(a_0)|^2 \\
&\quad + (H_{\omega,k}(a_0) - \rho(\omega,k)^2 - \rho(\omega,k))|z_k(a_0)|^2 \\
&\leqslant \alpha \int_{a_0}^{a_1} \left[|\partial_x z_k|^2 + k^2 |z_k|^2\right] \mathrm{d}x + C|\partial_x z_k(x_0^*)|^2 \\
&\quad + C|\partial_x z_k(x_1^*)|^2 + C|k|^{4/3}|z_k(x_1^*)|^2 + C|g_k|^2 \\
&\quad + C\int_{a_0}^{a_1} (|\partial_x F_{1,k}|^2 + k^2|F_{1,k}|^2 + |F_{2,k}|^2)\,\mathrm{d}x.
\end{aligned}
$$

We then use Corollary 3.7 to deduce

$$
\begin{aligned}
(3.49)\quad & \Big(I_\varepsilon(\omega,k)^2 1_{a_0 \leqslant x_0^*(\omega,k) - A_\varepsilon |k|^{-2/3}} \\
&\quad + \max\{|k|^{4/3}, I_1(\omega,k)^2\} 1_{a_0 \geqslant x_0^*(\omega,k) - A_\varepsilon |k|^{-2/3}}\Big) |z_k(a_0)|^2 \\
&\quad + \int_{a_0}^{a_1} \left[|\partial_x z_k|^2 + \left(k^2 + \omega^2 e^{2x}\right)|z_k|^2\right] \mathrm{d}x \\
&\leqslant \alpha \int_{a_0}^{a_1} \left[|\partial_x z_k|^2 + k^2 |z_k|^2\right] \mathrm{d}x + C|\partial_x z_k(x_0^*)|^2 \\
&\quad + C|\partial_x z_k(x_1^*)|^2 + C|k|^{4/3}|z_k(x_1^*)|^2 + C|g_k|^2 \\
&\quad + C\int_{a_0}^{a_1} (|\partial_x F_{1,k}|^2 + k^2|F_{1,k}|^2 + |F_{2,k}|^2)\,\mathrm{d}x.
\end{aligned}
$$

Now, using Corollary 3.6 at $x_0 = \tilde{x}_0^*(\omega, k)$ and $x_0 = \tilde{x}_1^*(\omega, k)$, and the fact that

$$
\omega^2 e^{2x_1^*(\omega,k)} - k^2 = k^2\left(e^{2(x_1^*(\omega,k) - x_0^*(\omega,k))} - 1\right) \underset{|k|\to\infty}{\simeq} k^{4/3},
$$

we have

$$
\begin{aligned}
&\sqrt{k^2 + \omega^2 e^{2\tilde{x}_0^*(\omega,k)}}|\partial_x z_k(\tilde{x}_0^*(\omega,k)|^2 + |\partial_x z_k(\tilde{x}_1^*(\omega,k)|^2 + k^{4/3}|z_k(\tilde{x}_1^*(\omega,k))|^2 \\
&\qquad \leqslant C|\partial_x z_k(a_1)|^2 + C\int_{\tilde{x}_0^*}^{a_1} (|\partial_x F_{1,k}|^2 + k^2|F_{1,k}|^2 + |F_{2,k}|^2)\,\mathrm{d}x.
\end{aligned}
$$

Estimate (3.49) thus yields:

$$
\begin{aligned}
(3.50)\quad &\Big(I_\varepsilon(\omega,k)^2 1_{a_0 \leqslant x_0^*(\omega,k) - A_\varepsilon |k|^{-2/3}} \\
&\quad + \max\{|k|^{4/3}, I_1(\omega,k)^2\} 1_{a_0 \geqslant x_0^*(\omega,k) - A_\varepsilon |k|^{-2/3}}\Big) |z_k(a_0)|^2 \\
&\quad + \int_{a_0}^{a_1} \left[|\partial_x z_k|^2 + \left(k^2 + \omega^2 e^{2x}\right) |z_k|^2 \right] \mathrm{d}x \\
&\leqslant \alpha \int_{a_0}^{a_1} \left[|\partial_x z_k|^2 + k^2 |z_k|^2 \right] \mathrm{d}x + C |\partial_x z_k(a_1)|^2 + C|g_k|^2 \\
&\quad + C \int_{a_0}^{a_1} (|\partial_x F_{1,k}|^2 + k^2 |F_{1,k}|^2 + |F_{2,k}|^2)\, \mathrm{d}x.
\end{aligned}
$$

Taking $\alpha > 0$ small enough (independent of (ω, k)), we conclude from (3.15) that

$$
\begin{aligned}
&\Big(I_\varepsilon(\omega,k)^2 1_{a_0 \leqslant x_0^*(\omega,k) - A_\varepsilon |k|^{-2/3}} \\
&\quad + \max\{|k|^{4/3}, I_1(\omega,k)^2\} 1_{a_0 \geqslant x_0^*(\omega,k) - A_\varepsilon |k|^{-2/3}}\Big) |z_k(a_0)|^2 \\
&\quad + \int_{a_0}^{a_1} \left[|\partial_x z_k|^2 + \left(k^2 + \omega^2 e^{2x}\right) |z_k|^2 \right] \mathrm{d}x \\
&\leqslant C \int_{a_0}^{a_1} (|\partial_x F_{1,k}|^2 + k^2 |F_{1,k}|^2 + |F_{2,k}|^2)\, \mathrm{d}x \\
&\quad + C|g_k|^2 + C|\partial_x z_k(a_1)|^2.
\end{aligned}
$$

which concludes the proof of Theorem 3.8. □

Remark 3.12. The starting point of the proof of Theorem 3.8 is the factorization of the differential operator $\partial_x^2 - H_{\omega,k}(x)$ in the set where $H_{\omega,k}(x) = k^2 - \omega^2 e^{2x}$ is positive.

Note that a natural factorization strategy would be to write

$$
\partial_x^2 - H_{\omega,k}(x) = \left(\partial_x - \sqrt{H_{\omega,k}(x)}\right)\left(\partial_x + \sqrt{H_{\omega,k}(x)}\right) + \text{ remainder },
$$

but the remainder, which equals to $-\partial_x\left(\sqrt{H_{\omega,k}(x)}\right)$, is too singular close to the zero of $H_{\omega,k}$ to be handled as a remainder.

Our approach thus relies on the exact factorization of the operator (3.32), and avoids the difficulty of handling such remainder terms.

3.4. **Proof of Lemma 3.10**

With the notations of Lemma 3.10, in this section we study the solutions $X_{0,\omega,k}$ of (3.40) and $X_{1,\omega,k}$ of (3.42). In order to do it, we perform the change of variable and unknowns

$$t = |k|^{2/3}(x_0^*(\omega, k) - x), \quad f_{m,k}(t) = |k|^{-2/3} X_{m,\omega,k}(x), \quad m \in \{0, 1\}.$$

Then both $f_{0,k}$ and $f_{1,k}$ satisfy the ODE

$$f_k' = H_k(t) - f_k^2, \tag{3.51}$$

where

$$H_k : t \in \mathbb{R} \mapsto |k|^{2/3}\left(1 - e^{-\frac{2t}{|k|^{2/3}}}\right),$$

plus the initial condition

$$f_{0,k}(0) = 0, \quad f_{1,k}\left(-\frac{1}{2}\right) = 2. \tag{3.52}$$

Note that this equation is now independent of ω, thus justifying the fact that we omit the dependence in ω and simply use the notation $f_{0,k}$, $f_{1,k}$.

Cauchy-Lipschitz theorem implies that equations (3.51) and (3.52) uniquely define $f_{0,k}$ and $f_{1,k}$ on some maximal intervals denoted $I_{0,k}$ and $I_{1,k}$.

Then, Lemma 3.10 is implied by the following result.

LEMMA 3.13. *There exists $k_0 \geqslant 0$ such that for all $|k| \geqslant k_0$, the following statements are true:*

The function $f_{0,k}$ is well-defined and non-negative on $[0, \infty)$. The function $f_{1,k}$ is well-defined and non-negative on $\left[-\frac{1}{2}, \infty\right)$. Furthermore,

- *for all $A > 0$, there exists δ_A independent of k such that for all $t \in [0, A]$,*

$$f_{1,k}(t) - f_{0,k}(t) \geqslant \delta_A.$$

- *for all $\varepsilon > 0$, there exists $A_\varepsilon \geqslant 0$ independent of k such that for all $t \geqslant A_\varepsilon$,*

$$(1 - \varepsilon)\sqrt{H_k(t)} \leqslant f_{0,k}(t) \leqslant f_{1,k}(t) \leqslant \sqrt{H_k(t)} \leqslant |k|^{\frac{2}{3}}.$$

We divide the proof of Lemma 3.13 into several steps. We first concentrate on the function $f_{0,k}$.

LEMMA 3.14. *The function $f_{0,k}$ is well-defined on $[0, \infty)$, and verifies, for all $t > 0$,*

$$0 < f_{0,k}(t) < \sqrt{H_k(t)}.$$

As a consequence, $f_{0,k}$ is increasing on $[0, \infty)$.

Proof. A direct computation shows that $f_{0,k}(0) = f_{0,k}'(0) = 0$ and $f_{0,k}''(0) > 0$, so there exists $\eta_{0,k} > 0$ such that for all $t \in (0, \eta_{0,k})$, $f_{0,k}(t) > 0$. Denote

$$T_{0,k} = \sup\{\eta \in I_{0,k} \cap (0, \infty),\ \forall t \in (0, \eta),\ f_{0,k}(t) > 0\},$$

such that $\eta_{0,k} \leqslant T_{0,k} \leqslant \sup I_{0,k}$. Suppose $T_{0,k} \in I_{0,k}$, then $f_{0,k}(T_{0,k}) = 0$ which implies $f'_{0,k}(T_{0,k}) = H_k(T_{0,k}) > 0$, leading to an immediate contradiction. Hence $T_{0,k} = \sup I_{0,k}$, and f is positive on $I_{0,k} \cap (0,\infty)$.

Consider now $g_{0,k} : t \in I_{0,k} \mapsto f_{0,k}(t)^2 - H_k(t)$. Using (3.51), we easily obtain that for all t in $I_{0,k}$,

$$g'_{0,k}(t) + 2 f_{0,k}(t) g_{0,k}(t) = -H'_k(t) < 0,$$

from which we infer that for all $t \in I_{0,k} \cap (0,\infty)$, denoting $F_{0,k} : t \in I_{0,k} \mapsto \int_0^t f_{0,k}(s)\, ds$,

$$g_{0,k}(t) e^{2 F_{0,k}(t)} < g_{0,k}(0) e^{2 F_{0,k}(0)} = 0,$$

hence $f_{0,k}^2 < H_k$ on $I_{0,k} \cap (0,\infty)$. This in turn implies $\sup I_{0,k} = \infty$, which ends the proof. □

LEMMA 3.15. *There exist $k_0 > 0$ and a positive constant C such that for $|k| \geqslant k_0$,*

- *For all $T \in [2, |k|^{2/3}]$,*

$$\frac{f_{0,k}(T)}{\sqrt{H_k(T)}} \geqslant 1 - \frac{C \log(T)}{T\sqrt{T}}. \tag{3.53}$$

- *For all $T \in [|k|^{2/3}, \exp(|k|^{2/3})]$,*

$$\frac{f_{0,k}(T)}{\sqrt{H_k(T)}} \geqslant 1 - C\frac{\log(T)}{T}. \tag{3.54}$$

- *For all $T \geqslant |k|^{2/3}$,*

$$\frac{f_{0,k}(T)}{\sqrt{H_k(T)}} \geqslant 1 - C \exp(-T/|k|^{2/3}). \tag{3.55}$$

In particular, for $T \geqslant |k|^{4/3}$, we have

$$\frac{f_{0,k}(T)}{\sqrt{H_k(T)}} \geqslant 1 - C \exp(-\sqrt{T}). \tag{3.56}$$

Proof. Let $T \geqslant 2$. For $A \in [0,T]$, we introduce f_A the solution of $f'_{A,k} = H_k(A) - f_{A,k}^2$ on (A,∞) starting from $f_{A,k}(A) = 0$. Easy comparison arguments show that $f_{0,k}(T) \geqslant f_{A,k}(T)$. We then compute $f_{A,k}(T)$:

$$f_{A,k}(T) = \sqrt{H_k(A)} \frac{1 - e^{-2\sqrt{H_k(A)}(T-A)}}{1 + e^{-2\sqrt{H_k(A)}(T-A)}}.$$

Accordingly,

$$\frac{f_{0,k}(T)}{\sqrt{H_k(T)}} \geqslant \sup_{A \in [0,T]} \left\{ \frac{\sqrt{H_k(A)}}{\sqrt{H_k(T)}} \frac{1 - e^{-2\sqrt{H_k(A)}(T-A)}}{1 + e^{-2\sqrt{H_k(A)}(T-A)}} \right\}$$

$$\geqslant \sup_{A\in[0,T]} \left\{ \sqrt{\frac{1-e^{-2A/|k|^{2/3}}}{1-e^{-2T/|k|^{2/3}}}} \frac{1-e^{-2\sqrt{H_k(A)}(T-A)}}{1+e^{-2\sqrt{H_k(A)}(T-A)}} \right\}.$$

Let us first focus on the case $2 \leqslant T \leqslant |k|^{2/3}$. On one hand, we have, for some constants C independent of k, A and T and which may change from line to line, that,

$$\sqrt{\frac{1-e^{-2A/|k|^{2/3}}}{1-e^{-2T/|k|^{2/3}}}} = \sqrt{1-e^{-2T/|k|^{2/3}}\frac{e^{2(T-A)/|k|^{2/3}}-1}{1-e^{-2T/|k|^{2/3}}}}$$
$$\geqslant 1-Ce^{-2T/|k|^{2/3}}\frac{e^{2(T-A)/|k|^{2/3}}-1}{1-e^{-2T/|k|^{2/3}}} \geqslant 1-C\left(\frac{T-A}{T}\right).$$

On the other hand, assuming $A \geqslant T/2$, we get, for some positive constants $c>0$ independent of k, A, and T, which may change from line to line,

$$H_k(A) = |k|^{2/3}\left(1-e^{-2A/|k|^{2/3}}\right) \geqslant cA \geqslant cT,$$

so that

$$\frac{1-e^{-2\sqrt{H_k(A)}(T-A)}}{1+e^{-2\sqrt{H_k(A)}(T-A)}} \geqslant 1-2e^{-2\sqrt{H_k(A)}(T-A)} \geqslant 1-2\exp\left(-c\sqrt{T}(T-A)\right).$$

Let us now fix $\beta = \frac{3}{2\,c}$ where c is the constant appearing in the exponential in the last estimate. Then the choice $A = T-\beta\ln(T)/\sqrt{T}$ satisfies $A \geqslant T/2$ provided that T is greater than some fix constant T_0 depending on β only. With that choice, we obtain on one hand,

$$\sqrt{\frac{1-e^{-2A/|k|^{2/3}}}{1-e^{-2T/|k|^{2/3}}}} \geqslant 1-C\frac{\log(T)}{T\sqrt{T}},$$

and on the other hand,

$$\frac{1-e^{-2\sqrt{H_k(A)}(T-A)}}{1+e^{-2\sqrt{H_k(A)}(T-A)}} \geqslant 1-\frac{2}{T\sqrt{T}}.$$

Combining these last two estimates, and using that $T \mapsto 1/(T\sqrt{T})$ is negligible in front of $\log(T)/(T\sqrt{T})$ as T goes to infinity, we easily obtain (3.53).

For T satisfying $T \geqslant |k|^{2/3}$, and $T-A \leqslant |k|^{2/3}$, on one hand, we have

$$\sqrt{\frac{1-e^{-2A/|k|^{2/3}}}{1-e^{-2T/|k|^{2/3}}}} = \sqrt{1-e^{-2T/|k|^{2/3}}\frac{e^{2(T-A)/|k|^{2/3}}-1}{1-e^{-2T/|k|^{2/3}}}}$$
$$\geqslant 1-Ce^{-2T/|k|^{2/3}}\frac{e^{2(T-A)/|k|^{2/3}}-1}{1-e^{-2T/|k|^{2/3}}}$$

$$\geqslant 1 - Ce^{-2T/|k|^{2/3}} \left(\frac{T-A}{|k|^{2/3}} \right)$$
$$\geqslant 1 - Ce^{-2T/|k|^{2/3}} \frac{T}{|k|^{2/3}} \left(\frac{T-A}{T} \right).$$

On the other hand, for $T \geqslant |k|^{2/3}$, and $A \geqslant |k|^{2/3}/2$, $H_k(A) \geqslant c|k|^{2/3}$ for some c independent of k, and thus

$$\frac{1 - e^{-2\sqrt{H_k(A)}(T-A)}}{1 + e^{-2\sqrt{H_k(A)}(T-A)}} \geqslant 1 - 2e^{-2\sqrt{H_k(A)}(T-A)} \geqslant 1 - 2\exp\left(-c|k|^{1/3}(T-A)\right).$$

Therefore, for T satisfying $T \in [|k|^{2/3}, \exp(|k|^{2/3})]$, setting $A = T - \log T$, which indeed satisfies $A \geqslant |k|^{2/3}/2$ for k large enough, and $T - A \leqslant |k|^{2/3}$, we obtain

$$\frac{f_{0,k}(T)}{\sqrt{H_k(T)}} \geqslant \left(1 - Ce^{-2T/|k|^{2/3}} \frac{T}{|k|^{2/3}} \frac{\log(T)}{T}\right) \left(1 - 2\exp\left(-c|k|^{1/3}\log(T)\right)\right)$$
$$\geqslant 1 - Ce^{-2T/|k|^{2/3}} \frac{T}{|k|^{2/3}} \frac{\log(T)}{T} - \frac{C}{T^{c|k|^{1/3}}}.$$

This last estimate easily implies (3.54), since $\tau \mapsto \tau \exp(-\tau)$ is bounded on $[1, \infty]$, and $1/T^{c|k|^{1/3}}$ is negligible in front of $\log(T)/T$ as $T \to \infty$ for k large enough.

For $T \geqslant |k|^{2/3}$, we can also choose $A = T/2$, so that

$$\sqrt{\frac{1 - e^{-2A/|k|^{2/3}}}{1 - e^{-2T/|k|^{2/3}}}} \geqslant \sqrt{1 - e^{-2A/|k|^{2/3}}} \geqslant 1 - Ce^{-T/|k|^{2/3}},$$

while

$$\frac{1 - e^{-2\sqrt{H_k(A)}(T-A)}}{1 + e^{-2\sqrt{H_k(A)}(T-A)}} \geqslant 1 - 2\exp\left(-c|k|^{1/3}T\right).$$

These two estimates easily yield (3.55) for k large enough. □

Remark 3.16. It is not difficult to check that the proof of (3.53) can be adapted to obtain the following result: For all $C_0 > 0$, there exist k_0 and C such that (3.53) holds for all k with $|k| \geqslant k_0$ and $T \in [2, C_0|k|^{2/3}]$. This result will be useful later on.

LEMMA 3.17. *Let k_0 as in Lemma 3.15. Let $\varepsilon > 0$. There exists $A_\varepsilon \geqslant 0$ such that, for all $t \geqslant A_\varepsilon$ and $|k| \geqslant k_0$, $(1-\varepsilon)\sqrt{H_k(t)} \leqslant f_{0,k}(t)$.*

Proof. Lemma 3.17 is a simple corollary of Lemma 3.15 by noting that

$$\lim_{T\to\infty} \left(1 - \frac{C\log(T)}{T\sqrt{T}}\right) = 1, \quad \lim_{T\to\infty} \left(1 - C\frac{\log(T)}{T}\right) = 1,$$
$$\lim_{T\to\infty} \left(1 - \exp(-\sqrt{T})\right) = 1,$$

and using the estimates (3.53) for $T \in [1, |k|^{2/3}]$, (3.54) for $T \in [|k|^{2/3}, |k|^{4/3}]$ and (3.56) for $T \geqslant |k|^{4/3}$. $\square$

We now study the function $f_{1,k}$, starting with the following lemma.

LEMMA 3.18. *There exists $k_0 \in \mathbb{N}$ such that, for all $|k| \geqslant k_0$, $[-1/2, 0] \subset I_{1,k}$ and*

$$0 < \tan\left(\frac{1}{2}\right) \leqslant f_{1,k}(0) \leqslant 2.$$

Proof. We denote $t_0 = -\frac{1}{2}$. First, as for all $t \in [t_0, 0]$, we have $H_k(t) \leqslant 0$, we have $f_{1,k}' \leqslant -f_{1,k}^2$ on $[t_0, 0] \cap I_{1,k}$, hence $f_{1,k}(t) \leqslant f_{1,k}(t_0) = 2$ for all $t \in [t_0, 0] \cap I_{1,k}$.

Now, since H_k is increasing, we have $f_{1,k}' \geqslant H_k(t_0) - f_{1,k}^2 = -(|H_k(t_0)| + f_{1,k}^2)$ in I, which implies

$$\arctan\left(\frac{f_{1,k}}{\sqrt{|H_k(t_0)|}}\right)' \geqslant -\sqrt{|H_k(t_0)|}. \tag{3.57}$$

For each $t \in I_{1,k}$ with $t > t_0$, we integrate (3.57) in $[t_0, t]$, and we get

$$\arctan\left(\frac{f_{1,k}(t)}{\sqrt{|H_k(t_0)|}}\right) \geqslant \arctan\left(\frac{2}{\sqrt{|H_k(t_0)|}}\right) - \sqrt{|H_k(t_0)|}\,(t - t_0)$$

Seeing that $1 \leqslant \sqrt{|H_k(t_0)|} \leqslant \sqrt{1 + \frac{1}{|k|^{2/3}}}$, we have, for all $t \in [t_0, 0] \cap I_{1,k}$ such that $f_{1,k}(t) \geqslant 0$,

$$\begin{aligned}\arctan(f_{1,k}(t)) &\geqslant \arctan\left(\frac{f_{1,k}(t)}{\sqrt{|H_k(t_0)|}}\right)\\ &\geqslant \arctan\left(\frac{2}{\sqrt{1 + \frac{1}{|k|^{2/3}}}}\right) - \sqrt{1 + \frac{1}{k^{2/3}}}\left(t + \frac{1}{2}\right)\\ &\geqslant \arctan\left(\frac{2}{\sqrt{1 + \frac{1}{|k|^{2/3}}}}\right) - \frac{\sqrt{1 + \frac{1}{k^{2/3}}}}{2}.\end{aligned}$$

Since

$$\lim_{|k|\to\infty}\left(\arctan\left(\frac{2}{\sqrt{1 + \frac{1}{|k|^{2/3}}}}\right) - \frac{\sqrt{1 + \frac{1}{|k|^{2/3}}}}{2}\right) = \arctan(2) - \frac{1}{2} > \frac{1}{2},$$

we deduce that for all $|k|$ sufficiently large, $f_{1,k} > \tan\left(\frac{1}{2}\right) > 0$ in $[t_0, 0] \cap I_{1,k}$.

We hence have obtained that $\tan\left(\frac{1}{2}\right) \leqslant f_{1,k} \leqslant 2$ on $[t_0, 0] \cap I_{1,k}$, immediately implying that $[t_0, 0] \cap I_{1,k} = [t_0, 0]$, which ends the proof. $\square$

Lemma 3.19. *Under the assumptions of Lemma 3.18, we have* $I_{1,k} \cap [0,\infty) = [0,\infty)$*, and for all* $t \geqslant 0$*,* $f_{0,k}(t) < f_{1,k}(t)$*.*

Proof. As both $f_{0,k}$ and $f_{1,k}$ satisfy (3.51), and $f_{0,k}(0) < f_{1,k}(0)$, by Cauchy-Lipschitz theorem, we clearly have $f_{0,k}(t) < f_{1,k}(t)$ for all $t \in I_{1,k} \cap [0,\infty)$. Then, for all $t \in I_{1,k}$, $t \geqslant 0$, we have

$$f'_{1,k}(t) = H_k(t) - f_{1,k}(t)^2 < H_k(t) - f_{0,k}(t)^2 = f'_{0,k}(t),$$

Since $f_{0,k}$ is bounded by $|k|^{2/3}$ for all $t \geqslant 0$ by Lemma 3.14, we deduce $I_{1,k} \cap [0,\infty) = [0,\infty)$. □

Lemma 3.20. *There exists* $k_0 \in \mathbb{N}$ *such that for each* $|k| \geqslant k_0$*, there exists* $\tau_k \in (0,3]$ *such that*

- $f_{1,k}$ *is strictly decreasing on* $[-1/2, \tau_k)$*,*
- $f_{1,k}$ *is strictly increasing on* (τ_k, ∞)*.*

As a consequence, $f_{1,k}(t) \leqslant \sqrt{H_k(t)}$ *for all* $t \geqslant 3$*.*

Proof. Directly from the equation we get that $f'_{1,k}(t) < 0$ for all $t \in [-1/2, 0]$. For $t \geqslant 0$, we define the set

$$\mathscr{E}_k = \left\{ T \geqslant 0 : \; f'_{1,k}(t) < 0 \; \forall t \in [0,T] \right\}.$$

Since $f'_{1,k}(0) = -f_{1,k}(0)^2 < 0$, one has that $\mathscr{E}_k$ is nonempty and $\tau_k := \sup \mathscr{E}_k > 0$. By definition we have $f'_{1,k}(t) < 0$ for all t in $[0,\tau)$. Therefore, for all $t \in [0,\tau)$, $f_{1,k}(t) \leqslant f_{1,k}(0) \leqslant 2$ and then

$$f'_{1,k}(t) \geqslant H_k(t) - 4 \tag{3.58}$$

for all $t < \tau$. Note that the right hand side of (3.58) vanishes exactly at

$$\tau_{\star,k} := -\frac{|k|^{2/3}}{2} \ln\left(1 - \frac{4}{|k|^{2/3}}\right) \xrightarrow[|k|\to\infty]{} 2.$$

In particular, $\tau_{\star,k} \in [0,3]$ and then $\tau_k \leqslant \tau_{\star,k} \leqslant 3$ for all $|k|$ large enough.

On the other hand, given any $\tilde{\tau} > 0$ such that $f'_{1,k}(\tilde{\tau}) = 0$, we have $f''_{1,k}(\tilde{\tau}) = (H_k - f_{1,k}^2)'(\tilde{\tau}) = H'_k(\tilde{\tau}) > 0$. Then, a very simple reductio ad absurdum shows that $f'_{1,k}(t) > 0$ on $(\tilde{\tau}, \infty)$, which ends the proof. □

Lemma 3.21. *Let* k_0 *be fixed as in previous Lemma, and* $A > 0$*. There exists* $\delta_A > 0$ *such that for all* $|k| \geqslant k_0$ *and all* $t \in [0, A]$*,* $f_{1,k}(t) - f_{0,k}(t) \geqslant \delta_A$*.*

Proof. We observe that, for all $t \geqslant 0$ and all $k \geqslant 1$, $H_k(t) \leqslant 2t$. So, combining the results from Lemmas 3.19 and 3.20, we obtain that for all $t \in [0, A]$ and all $|k|$ greater than k_0,

$$f_{0,k}(t) \leqslant f_{1,k}(t) \leqslant 2 + \sqrt{H_k(t)} \leqslant 2 + \sqrt{2t} \leqslant 2 + \sqrt{2A}.$$

Setting, for all $t \in [0, A]$, $g_k(t) = f_{1,k}(t) - f_{0,k}(t)$, we easily check that g_k satisfies the equation $g_k'(t) + (f_{1,k}(t) + f_{0,k}(t))g_k(t) = 0$ on $[0, A]$. According to the above estimate, we obtain

$$\begin{aligned} f_{1,k}(t) - f_{0,k}(t) = g_k(0) \exp\left(-\int_0^t (f_{1,k}(s) + f_{0,k}(s))\, ds\right) \\ \geqslant \tan\left(\frac{1}{2}\right) \exp\left(-\int_0^A (f_{1,k}(s) + f_{0,k}(s))\, ds\right). \end{aligned}$$

The results follows, with

$$\delta_A = \tan\left(\frac{1}{2}\right) \exp\left(-A(4 + 2\sqrt{2A})\right).$$

□

Combining the results of Lemmas 3.14, 3.17, 3.18, 3.19, 3.20 and 3.21, we obtain Lemma 3.13.

4. **A NECESSARY CONDITION FOR THE RESOLVENT ESTIMATE: PROOF OF THEOREM 1.4**

Let $(\omega_n, k_n)_{n\in\mathbb{N}}$ be a sequence of elements in $\mathbb{R} \times \mathbb{Z}$ satisfying (1.23) and (1.24). For all $n \in \mathbb{N}$, we set

$$A_n = |k_n|^{2/3} \log\left(\frac{|k_n|}{|\omega_n| R_0}\right), \quad x^*_{0,n} = \log\left(\frac{|k_n|}{|\omega_n|}\right), \quad \tilde{x}^*_{0,n} = \min\left\{x^*_{0,n}, a_1\right\},$$

$$x^*_{-1,n} = \frac{1}{3}a_0 + \frac{2}{3}\tilde{x}^*_{0,n}, \quad x^*_{-2,n} = \frac{2}{3}a_0 + \frac{1}{3}\tilde{x}^*_{0,n}, \quad \delta_n = \min\left\{\frac{A_n}{3|k_n|^{2/3}}, \frac{a_1 - a_0}{3}\right\},$$

and we note that $a_0 < x^*_{-2,n} < x^*_{-1,n} < \tilde{x}^*_{0,n} \leqslant a_1$ with

$$\tilde{x}^*_{0,n} - x^*_{-1,n} = x^*_{-1,n} - x^*_{-2,n} = x^*_{-2,n} - a_0 = \delta_n.$$

Also note that, due to (1.24), δ_n is always of the order of $A_n/|k_n|^{2/3}$.

We then set, for $x \in [a_0, \tilde{x}^*_0]$,

$$\tilde{z}_{k_n}(x) = \exp\left(-\int_{a_0}^x X_{0,\omega_n,k_n}(y)\, dy\right), \tag{4.1}$$

where X_{0,ω_n,k_n} is defined in (3.40), and

$$z_{k_n}(x) = \tilde{z}_{k_n}(x)\eta_n(x) \quad \text{where} \quad \eta_n(x) = \eta\left(\frac{x - a_0}{\tilde{x}^*_{0,n} - a_0}\right), \tag{4.2}$$

where $\eta = \eta(s)$ is a smooth cut-off function taking value 1 for $s \leqslant 1/3$, vanishing for $s \geqslant 2/3$, and taking values in $[0, 1]$.

One then easily checks that z_{k_n}, extended by 0 for $x \in [\tilde{x}^*_{0,n}, a_1]$, satisfies (3.1) with

$$\begin{aligned} F_{2,k_n} &= 2\partial_x\eta_n\partial_x\tilde{z}_{k_n} + \partial_{xx}\eta_{k_n}\tilde{z}_{k_n}, \\ g_{k_n} &= -(\rho(\omega_n, k_n) + X_{0,\omega_n,k_n}(a_0)), \\ z_{k_n} &= 0 \text{ for } x \geqslant x^*_{-1,n}, \\ z_{k_n} &= \tilde{z}_{k_n} \text{ for } x \leqslant x^*_{-2,n}, \end{aligned}$$

Our next goal is to check that the resolvent condition (3.3) applied to z_{k_n} necessarily implies some condition. In order to do so, it is convenient to recall from Lemma 3.10 that, for n large enough, since $A_n \to \infty$ due to (1.23), we have

$$\forall x \in [a_0, x^*_{-1,n}], \qquad \frac{1}{2}\sqrt{H_{\omega_n,k_n}(x)} \leqslant X_{0,\omega_n,k_n}(x) \leqslant \sqrt{H_{\omega_n,k_n}(x)}. \tag{4.3}$$

We then check that $\sqrt{H_{\omega_n,k_n}(a_0)}$, $\sqrt{H_{\omega_n,k_n}(x^*_{-2,n})}$ and $\sqrt{H_{\omega_n,k_n}(x^*_{-1,n})}$ are all of the same order. Indeed,

$$H_{\omega_n,k_n}(x) = k_n^2\left(1 - e^{-2(x^*_{0,n}-x)}\right).$$

In view of the choice of $x^*_{-1,n}$ and the fact that H_{ω_n,k_n} is decreasing, it is thus clear that there exists a constant $C > 0$ independent of $n \in \mathbb{N}$ such that

$$\begin{aligned} \frac{1}{C}\sqrt{H_{\omega_n,k_n}(a_0)} \leqslant \sqrt{H_{\omega_n,k_n}(x^*_{-1,n})} &\leqslant \inf_{[a_0,x^*_{-1,n}]}\sqrt{H_{\omega_n,k_n}(x)} \\ &\leqslant \sup_{[a_0,x^*_{-1,n}]}\sqrt{H_{\omega_n,k_n}(x)} \leqslant \sqrt{H_{\omega_n,k_n}(a_0)} \end{aligned} \tag{4.4}$$

and

$$\frac{1}{C} \leqslant \frac{H_{\omega_n,k_n}(a_0)}{|k_n|^2\delta_n} \leqslant C, \tag{4.5}$$

where we used (1.24) to guarantee that $x^*_{0,n} - a_0$ is bounded uniformly in $n \in \mathbb{N}$. Using the above estimate, we immediately have constants $C > 0$ independent of n such that

$$\|z_{k_n}\|^2_{L^2(a_0,a_1)} \geqslant \|\tilde{z}_{k_n}\|^2_{L^2(a_0,x^*_{-2,n})} \geqslant \frac{1}{C|k_n|\sqrt{\delta_n}},$$

$$\|\partial_x z_{k_n}\|^2_{L^2(a_0,a_1)} \geqslant \|\partial_x \tilde{z}_{k_n}\|^2_{L^2(a_0,x^*_{-2,n})} \geqslant \frac{1}{C}|k_n|\sqrt{\delta_n},$$

$$\|\partial_x \eta_n \partial_x z_{k_n}\|^2_{L^2(a_0,a_1)} \leqslant \|\partial_x \eta_n\|^2_{L^\infty(a_0,a_1)} \|\partial_x z_{k_n}\|^2_{L^2(x^*_{-2,n},x^*_{-1,n})}$$
$$\leqslant \frac{C}{|\delta_n|^2} e^{-|k_n|\delta_n^{3/2}/C} |k_n|\sqrt{\delta_n},$$

$$\|\partial_{xx}\eta_n z_{k_n}\|^2_{L^2(a_0,a_1)} \leqslant \|\partial_{xx}\eta_n\|^2_{L^\infty(a_0,a_1)} \|z_{k,n}\|^2_{L^2(x^*_{-2,n},x^*_{-1,n})}$$
$$\leqslant \frac{C}{|\delta_n|^4} e^{-|k_n|\delta_n^{3/2}/C} \frac{1}{|k_n|\sqrt{\delta_n}}.$$

In particular, the estimate (3.3) applied to z_{k_n} yields, for all $n \in \mathbb{N}$,

$$|k_n|\sqrt{\delta_n} + \frac{|k_n|}{\sqrt{\delta_n}} \leqslant Ce^{-|k_n|\delta_n^{3/2}/C}\left(\frac{|k_n|}{\delta_n^{3/2}} + \frac{1}{|k_n|\delta_n^{9/2}}\right) + C\left(\rho(\omega_n,k_n) + X_{0,\omega_n,k_n}(a_0)\right)^2.$$

Moreover, since $\delta_n \leqslant (a_1 - a_0)/3$,

$$\frac{|k_n|}{\sqrt{\delta_n}} \geqslant C|k_n|\sqrt{\delta_n},$$

while, since $\delta_n \geqslant A_n/|k_n|^{2/3}$,

$$\frac{|k_n|}{\delta_n^{3/2}} \geqslant C\frac{1}{|k_n|\delta_n^{9/2}}.$$

We then deduce that for all $n \in \mathbb{N}$,

$$\frac{|k_n|}{\sqrt{\delta_n}} \leqslant Ce^{-|k_n|\delta_n^{3/2}/C}\frac{|k_n|}{\delta_n^{3/2}} + C\left(\rho(\omega_n,k_n) + X_{0,\omega_n,k_n}(a_0)\right)^2.$$

Then, it is not difficult to check that, since $|k_n| \to \infty$ and $A_n \gg \log^{2/3}(|k_n|)$ by (1.23), and δ_n is of the order of $A_n/(|k_n|^{2/3})$ due to (1.24), we get

$$\liminf_{n\to\infty}\left(\frac{|k_n|}{\sqrt{\delta_n}}\frac{\delta_n^{3/2}}{|k_n|e^{-|k_n|\delta_n^{3/2}/C}}\right) = \liminf_{n\to\infty}\left(\delta_n e^{|k_n|\delta_n^{3/2}/C}\right)$$
$$\geqslant \lim_{n\to\infty}\left(\frac{A_n}{|k_n|^{2/3}}e^{A_n^{3/2}/C}\right) = \infty,$$

so that the last condition reduces to

$$\frac{|k_n|}{\sqrt{\delta_n}} \leqslant C\left(\rho(\omega_n,k_n) + X_{0,\omega_n,k_n}(a_0)\right)^2,$$

for n large enough, thus showing that there exists $c_n \in [0,1]$ such that

$$\frac{|k_n|^2}{\sqrt{H_{\omega_n,k_n}(a_0)}} \leqslant C\left(\rho(\omega_n,k_n) + c_n\sqrt{H_{\omega_n,k_n}(a_0)}\right)^2. \tag{4.6}$$

The convergence $c_n \to 1$ immediately comes from (3.43) and that $A_n \to \infty$ as $n \to \infty$.

To be more precise on the rate at which c_n goes to 1, we specify that (ω_n, k_n) satisfies (1.24). This means

$$|x_{0,n}^* - a_0| = \log\left(\frac{|k_n|}{|\omega_n| R_0}\right) \leqslant C_0.$$

We can thus use Lemma 3.15 and Remark 3.16 to get the existence of C independent of n such that

$$1 - \frac{C}{|k_n| \log^{3/2}\left(\frac{|k_n|}{|\omega_n| R_0}\right)} \log\left(|k_n|^{2/3} \log\left(\frac{|k_n|}{|\omega_n| R_0}\right)\right) \leqslant \frac{X_{0,\omega_n,k_n}}{\sqrt{H_{\omega_n,k_n}(a_0)}} \leqslant 1,$$

i.e. the bound

$$1 - \frac{C}{|k_n| \log^{3/2}\left(\frac{|k_n|}{|\omega_n| R_0}\right)} \log\left(|k_n|^{2/3} \log\left(\frac{|k_n|}{|\omega_n| R_0}\right)\right) \leqslant c_n \leqslant 1. \tag{4.7}$$

on $c_n = X_{0,\omega_n,k_n}(a_0)/\sqrt{H_{\omega_n,k_n}(a_0)}$.

Then, due to (1.24), there exists $C > 0$ such that for all n,

$$\frac{1}{C}|k_n| \log\left(\frac{|k_n|}{|\omega_n| R_0}\right) \leqslant \sqrt{H_{\omega_n,k_n}(a_0)} \leqslant C|k_n| \log\left(\frac{|k_n|}{|\omega_n| R_0}\right).$$

Accordingly, the right hand-side of (4.6) can be bounded by

$$\begin{aligned} &\left(\rho(\omega_n, k_n) + c_n\sqrt{H_{\omega_n,k_n}(a_0)}\right)^2 \\ &\leqslant 2\left(\rho(\omega_n, k_n) + \sqrt{H_{\omega_n,k_n}(a_0)}\right)^2 + \frac{C}{\log\left(\frac{|k_n|}{|\omega_n| R_0}\right)} \log^2\left(|k_n|^{2/3} \log\left(\frac{|k_n|}{|\omega_n| R_0}\right)\right). \end{aligned} \tag{4.8}$$

We now simply remark that the left hand-side of (4.6) satisfies:

$$\frac{|k_n|^2}{\sqrt{H_{\omega_n,k_n}(a_0)}} \geqslant \frac{1}{C} \frac{|k_n|}{\log\left(\frac{|k_n|}{|\omega_n| R_0}\right)}.$$

Therefore, in the range (1.24), the second term in (4.8) is negligible compared to the left hand-side of (4.6). Accordingly, if (4.6) holds for c_n satisfying (4.7), we necessarily have (1.25). This concludes the proof of Theorem 1.4.

5. COMMENTS

5.1. Summary of the main results

In this article, we develop a strategy allowing to study the observability of wave equations in an annulus, when the observation is performed on the external boundary, for various boundary conditions on the internal boundary given by specific instances of microlocal operators.

This is done by using the Hautus test for the observability of linear abstract equations of the form $Y' = AY$ when A is a skew-adjoint operator to reduce the problem to the analysis of resolvent estimates for a family of 1d second order equations depending on the time and spherical Fourier parameters ω and k, given by (1.13), in which the boundary operator at R_0 is given in terms of some Fourier multiplier $\rho = \rho(\omega, k)$.

Our results underline that nice resolvent estimates can be obtained when the symbol $\rho(\omega, k)$ does not get close from $-\sqrt{k^2 - \omega^2 R_0^2}$ when $(|\omega|, |k|) \to \infty$ (see Theorem 1.2 and Corollary 1.3), and that this is in some sense sharp (see Theorem 1.4). Our approach also underlines that different estimates can be obtained on the boundary depending on the sign of $k^2 - \omega^2 R_0^2$, see (1.20).

5.2. Estimates on the time of observability

Although the approach we proposed here allows to deal with several models at once, we should underline that the time of observability provided by this approach is not known or badly estimated. In fact, it is known that if the resolvent estimate (1.12) holds, then we can derive a time estimate guaranteeing the observability inequality (1.11) depending on the constants in (1.12), see [18] and Theorem 1.1, of the form $T > M\pi$, where M is the constant in (1.12). However, this estimate on the time required for observability is, in general, far from optimal.

This remark suggests the development of approaches able to track down the time required for observability, maybe based on suitable weighted estimates in the spirit of Carleman estimates for the waves, see e.g. [15], able to precisely track down the boundary conditions, similarly to what has been done recently in the context of elliptic equations in [4] for instance.

Semiclassical approaches with detailed analysis of the boundary data could also be adapted to deal with general boundary conditions given by a kernel ρ, similarly to what has been developed in [2, 3]. This would probably be the first step to deal with more general geometries as well, and we refer also to the forthcoming work [10] for results in this direction.

5.3. More general kernels ρ

Our approach does not truly require the kernel $\rho = \rho(\omega, k)$ in (1.13) to be real valued, and the above proofs can easily be adapted to deal with cases in which ρ is complex valued. This remark can be useful to deal with wave models containing dissipative terms.

For instance, to our knowledge, the null-controllability of the damped wave equation with absorbing boundary conditions on the sphere $S(R_0)$ and a control function $u \in L^2((0,T) \times S(R_1))$ is still an open problem:

$$\tag{5.1} \begin{cases} \partial_{tt} z(t,x) - \Delta z(t,x) = 0, & \text{in } (0,T) \times A(R_0,R_1), \\ z(t,x) = u(t,x), & \text{on } (0,T) \times S(R_1), \\ \partial_\nu z(t,x) + \partial_t z(t,x) = 0, & \text{on } (0,T) \times S(R_0), \\ (z(0,\cdot), \partial_t z(0,\cdot)) = (z^0, z^1), & \text{in } A(R_0,R_1). \end{cases}$$

By duality, see e.g. [17, 21], such property would be equivalent to the observability inequality

$$\tag{5.2} \|(y(T), \partial_t y(T))\|_{H^1(A(R_0,R_1)) \times L^2(A(R_0,R_1))} \leqslant C \|\partial_\nu y|_{S(R_1)}\|_{L^2((0,T)\times S(R_1))}$$

for all solutions y of

$$\tag{5.3} \begin{cases} \partial_{tt} y(t,x) - \Delta y(t,x) = 0, & \text{in } (0,T) \times A(R_0,R_1), \\ y(t,x) = 0, & \text{on } (0,T) \times S(R_1), \\ \partial_\nu y(t,x) + \partial_t y(t,x) = 0, & \text{on } (0,T) \times S(R_0), \\ (y(0,\cdot), \partial_t y(0,\cdot)) = (y^0, y^1), & \text{in } A(R_0,R_1), \end{cases}$$

with $(y^0, y^1) \in H^1(A(R_0,R_1)) \times L^2(A(R_0,R_1))$ with $y^0|_{S(R_1)} = 0$.

It is then clear that resolvent estimates could be derived from our approach, with in this case, $\rho(\omega, k) = iR_0\omega$. However, we do not know if the resolvent estimates we would get this way would yield (5.2) in some time T.

Note that for this model, the propagation of regularity and of semi-classical measures has been studied and analyzed precisely in the literature, see [3, 1]. Despite of this, the usual compactness-uniqueness arguments to establish observability inequality do not apply since the wave equation (5.3) cannot be solved backward in time, so that to our knowledge, the observability inequality (5.2) for solutions of (5.3) is open.

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A. PROOF OF COROLLARY 1.3

The condition (1.16) clearly holds when $\rho(\omega,k)\geqslant 0$ (with $\delta=(1-\varepsilon)^2$ for instance), and we can thus focus on the case $\rho(\omega,k)\leqslant 0$. Now, the right hand-side of (1.16) is negative if $|\rho(\omega,k)+1/2|\geqslant\sqrt{k^2-\omega^2R_0^2+1/4}$, and in particular if

$$\rho(\omega,k)\leqslant-\sqrt{k^2-\omega^2R_0^2+\frac{1}{4}}-\frac{1}{2}. \tag{A.1}$$

Now, since in the range (1.15), $\omega^2R_0^2\geqslant k^2$ and $k^2-\omega^2R_0^2$ is of the order of $2\mathbf{r}|k|^{4/3}$ at least, when $\omega^2R_0^2+k^2$ is large enough, the first condition in (1.22) implies (A.1) and thus (1.16) obviously holds.

Finally, if $\rho(\omega,k)\leqslant 0$ and the second condition in (1.22) holds, then, necessarily $\gamma\leqslant 1$, and taking $\varepsilon=\gamma/2$, $I_\varepsilon(\omega,k)^2\geqslant\varepsilon^2(k^2-\omega^2R_0^2)$, while $k^2-\omega^2R_0^2-\rho(\omega,k)^2-\rho(\omega,k)\leqslant(k^2-\omega^2R_0^2)+1/4$. Since in the range (1.15), $\omega^2R_0^2\leqslant k^2$ and $k^2-\omega^2R_0^2$ is of the order of $2\mathbf{r}|k|^{4/3}$ at least, it is clear that (1.16) holds when $\omega^2R_0^2+k^2$ is large enough with $\delta=\varepsilon^2/2$.

REFERENCES

[1] L. Aloui, *Stabilisation Neumann pour l'équation des ondes dans un domaine extérieur*. J. Math. Pures Appl. (9) **81** (2002), *11*, 1113–1134.

[2] C. Bardos, G. Lebeau, and J. Rauch, *Un exemple d'utilisation des notions de propagation pour le contrôle et la stabilisation de problèmes hyperboliques*. Rend. Sem. Mat. Univ. Politec. Torino 1988, Special Issue, Nonlinear hyperbolic equations in applied sciences (1989), 11–31.

[3] C. Bardos, G. Lebeau, and J. Rauch, *Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary*. SIAM J. Control and Optim. **30** (1992), *5*, 1024–1065.

[4] M. Bellassoued and J. Le Rousseau, *Carleman estimates for elliptic operators with complex coefficients*. Part I: *Boundary value problems*. J. Math. Pures Appl. (9) **104** (2015), *4*, 657–728.

[5] I. Benabbas and D. E. Teniou, *Observability of wave equation with Ventcel dynamic condition*. Evol. Equ. Control Theory **7** (2018), *4*, 545–570.

[6] N. Burq and P. Gérard, *Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes*. C. R. Acad. Sci. Paris Sér. I Math. **325** (1997), *7*, 749–752.

[7] N. Burq and M. Zworski, *Control theory and high energy eigenfunctions*. In: *Journées Équations aux Dérivées Partielles*, Exp. No. XIII. École Polytech., Palaiseau, 2004.

[8] L. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*. Comm. Partial Differential Equations **32** (2007), *7-9*, 1245–1260.

[9] C. Conca, J. Planchard, and M. Vanninathan, *Fluids and periodic structures*. RAM: Research in Applied Mathematics, Vo. 38. John Wiley & Sons Ltd., Chichester, 1995.

[10] B. Dehman and E. Zuazua, *Sidewise energy estimates for the wave equation.* In preparation.

[11] S. Ervedoza and M. Vanninathan, *Controllability of a simplified model of fluid-structure interaction.* ESAIM: Control Optim. Calc. Var. **20** (2014), 547–575.

[12] F. G. Friedlander, *The wave front set of the solution of a simple initial-boundary value problem with glancing rays.* Math. Proc. Cambridge Philos. Soc. **79** (1976), *1*, 145–159.

[13] C. G. Gal and L. Tebou, *Carleman inequalities for wave equations with oscillatory boundary conditions and application.* SIAM J. Control Optim. **55** (2017), *1*, 324–364.

[14] L. F. Ho. *Observabilité frontière de l'équation des ondes.* C. R. Acad. Sci. Paris Sér. I Math. 302 (1986), *12*, 443–446.

[15] V. Isakov, *Inverse problems for partial differential equations.* Applied Mathematical Sciences, Vol. 127. Springer, New York, Second Edition, 2006.

[16] J.-L. Lions, *Contrôlabilité exacte, Stabilisation et Perturbations de Systèmes Distribués. Tome 1. Contrôlabilité exacte*, RMA, Vol. 8. Masson, 1988.

[17] J.-L. Lions, *Exact controllability, stabilization and perturbations for distributed systems.* SIAM Rev. **30** (1988), *1*, 1–68.

[18] L. Miller. *Controllability cost of conservative systems: resolvent condition and transmutation.* J. Funct. Anal. **218** (2005), *2*, 425–444.

[19] J.-P. Raymond and M. Vanninathan, *Exact controllability in fluid-solid structure: the Helmholtz model.* ESAIM Control Optim. Calc. Var. **11** (2005), *2*, 180–203.

[20] M. Tucsnak and M. Vanninathan, *Locally distributed control for a model of fluid-structure interaction.* Systems Control Lett. **58** (2009), *8*, 547–552.

[21] M. Tucsnak and G. Weiss, *Observation and control for operator semigroups.* Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser Verlag, Basel, 2009.

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