Dedicated to Marius Tucsnak on the occasion of his 60^{th} anniversary

T-COERCIVITY FOR THE HOMOGENIZATION OF SIGN-CHANGING COEFFICIENTS SCALAR PROBLEMS WITH EXTREME CONTRASTS

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Communicated by Jérôme Lohéac

We study the homogenization of a diffusion-type problem, for sign-changing conductivities with extreme contrasts. The weak limit, which is proved to be the same as in the elliptic case of positive conductivities, has an explicit dependence on the conductivities.

AMS 2010 Subject Classification: 35B27, 78M40, 35Q60, 35J20.

Key words: sign-changing coefficients, T-coercivity, homogenization.

1. INTRODUCTION

Our aim in this note is to study a homogenization problem of diffusion-type, stated in a classical periodic domain occupied by two materials periodically distributed in all directions with period ε, where ε is a small real positive parameter. There are two important features of the diffusion coefficients (that we classically call conductivities) of the two materials. Firstly, the conductivities are described by real constants of different signs. Materials with negative properties are encountered in practice in the frame of metamaterials, which are composite structures displaying unusual properties (see, *e.g.*, [14, 13]). In particular, electromagnetic metamaterials can exhibit, over some frequency range, negative dielectric permittivity and magnetic permeability, and hence a negative index of refraction. Secondly, the orders of magnitude of the conductivities are 1 and ε^2. Such small conductivities appear in "double-porosity" models used to describe single phase flows in fractured porous media (see [3], [1]), as well as in optics in the framework of *epsilon-near-zero* (ENZ) photonics (see [15] and the references therein). By using T-coercivity arguments (see [4, 10, 9, 5]) adapted to periodic problems (see [7], [8]), we prove that, for ε small enough, the initial problem is well-posed. The main difference with respect to the results obtained in [7] and [8] is that here the initial problem is well-posed for all the values (but zero) of the coefficients a_1 and a_2. This is due to the different order of magnitude of the conductivities, the contrast $a_1/(\varepsilon^2 a_2)$ being large (in modulus) when ε is small enough. Then, by using the *a priori* estimates and the two-scale convergence method (see [12], [1]), we pass to the limit in the variational formulation of the initial problem and get the homogenization result, stated in Theorem 3.2. We point out that the weak L^2 limit of the solution of the initial problem writes as the sum of two terms, the first one involving the solution of a classical homogenization problem stated in the whole domain Ω, with homogenized coefficients depending on one material only, and the second term involving the solution of a local problem stated in the second material. This is a characteristic result of the "double-porosity" models. Since the conductivities are constants, we obtain moreover an explicit expression of the limit function depending on the two conductivities.

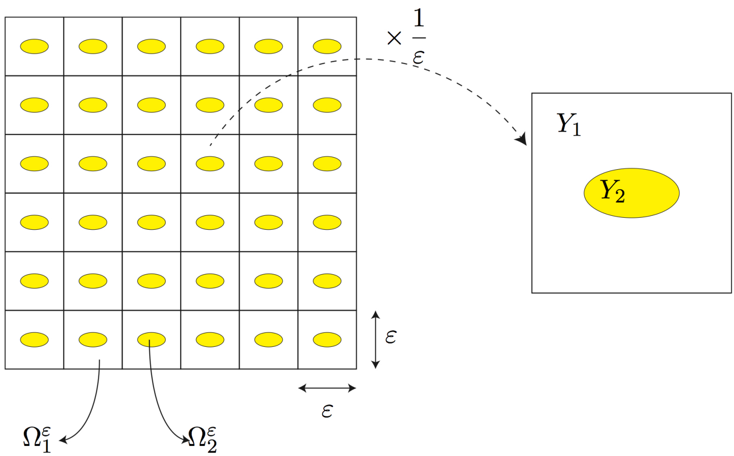

Fig. 1 – The composite periodic material and the corresponding reference cell Y $(d = 2)$.

Let Ω be an open, simply connected and bounded subset of $\mathbb{R}^d$, $d = 2, 3$, with Lipschitz-continuous boundary $\partial\Omega$. We consider two materials periodically distributed in Ω, with period εY, where ε is a small parameter and $Y = (0,1)^d$ is the unit cell. We assume that Y is composed of two materials occupying the two complementary open sets Y_1 and Y_2:

- a negative material, described by a negative constant $a_2\varepsilon^2 < 0$, and occupying a connected subset $Y_2 \subset Y$ with Lipschitz boundary ∂Y_2 and such that $\overline{Y_2} \subset Y$;
- a dielectric material, described by a positive constant $a_1 > 0$, and filling the region $Y_1 := Y \setminus \overline{Y_2}$.

Let us emphasize that the assumption $\overline{Y_2} \subset Y$, which implies that the metamaterial inclusions are disconnected from one cell to another, is crucial for our analysis. It would be interesting, but out of the scope of this paper, to investigate the (three dimensional) case of connected inclusions. Another interesting case to consider would be the one where Y_1 is not connected, like for instance the composite laminates.

For every $\varepsilon > 0$ and every integer vector $k \in \mathbb{Z}^d$, we define the shifted and scaled sets:

$$\tag{1.1} \begin{array}{rcl} Y_{1,k}^\varepsilon &:=& \{x \in \mathbb{R}^d \,|\, (x-k)/\varepsilon \in Y_1\}, \\ Y_{2,k}^\varepsilon &:=& \{x \in \mathbb{R}^d \,|\, (x-k)/\varepsilon \in Y_2\}, \\ Y_k^\varepsilon &:=& \{x \in \mathbb{R}^d \,|\, (x-k)/\varepsilon \in Y\}. \end{array}$$

In the macroscopic domain Ω, the metamaterial fills the region

$$\Omega_2^\varepsilon := \bigcup_{k \in \mathbb{Z}^d} \{Y_{2,k}^\varepsilon \mid Y_k^\varepsilon \subset \Omega\},$$

while the classical material fills the domain

$$\Omega_1^\varepsilon = \Omega \setminus \overline{\Omega_2^\varepsilon}.$$

We define the macroscopic function a^ε on Ω such that for all $x \in \Omega$:

$$\tag{1.2} a^\varepsilon(x) = a_1\, \mathbb{1}_{\Omega_1^\varepsilon}(x) + \varepsilon^2 a_2\, \mathbb{1}_{\Omega_2^\varepsilon}(x) = \begin{cases} a_1 > 0, & \text{for } x \in \Omega_1^\varepsilon \\ \varepsilon^2 a_2 < 0, & \text{for } x \in \Omega_2^\varepsilon. \end{cases}$$

Given $f \in L^2(\Omega)$, our goal is to analyze the asymptotic behavior, as ε tends to 0, of $u^\varepsilon \in H_0^1(\Omega)$, solution of the following problem:

$$\tag{1.3} \begin{cases} -\mathrm{div}(a^\varepsilon \nabla u^\varepsilon) = f, & \text{in } \Omega, \\ u^\varepsilon = 0, & \text{on } \partial\Omega. \end{cases}$$

Note that since the sign of a^ε is not constant, the well-posedness (*i.e.* existence, uniqueness and continuous dependence on the data) of the above problem is not obvious.

The weak formulation of the above problem reads as follows:

$$\tag{1.4} \text{Find } u^\varepsilon \in H_0^1(\Omega) \text{ such that for all } v \in H_0^1(\Omega): \quad \mathcal{A}^\varepsilon(u^\varepsilon, v) = \int_\Omega f(x) v(x) \,\mathrm{d}x,$$

where

$$\tag{1.5} \begin{array}{rcl} \mathcal{A}^\varepsilon(u,v) &=& \displaystyle\int_\Omega a^\varepsilon(x) \nabla u(x) \cdot \nabla v(x) \,\mathrm{d}x \\ &=& \displaystyle a_1 \int_{\Omega_1^\varepsilon} \nabla u(x) \cdot \nabla v(x) \,\mathrm{d}x + a_2 \varepsilon^2 \int_{\Omega_2^\varepsilon} \nabla u(x) \cdot \nabla v(x) \,\mathrm{d}x. \end{array}$$

2. WELL-POSEDNESS AND UNIFORM ENERGY ESTIMATES

The goal of this section is to obtain, using the T-coercivity method (see Definition 2.1), a well-posedness result and a uniform energy estimate for problem (1.3).

2.1. Uniform T-coercivity

We first recall the definition of T-coercivity.

Definition 2.1 (T-coercivity). Let V be a Hilbert space endowed with the norm $\|\cdot\|$ and let $\mathbf{T} \in \mathcal{L}(V)$ be a bounded linear operator on V. A bilinear form $a(\cdot,\cdot)$ defined on $V \times V$ is called T-coercive if there exists $\gamma > 0$ such that

$$a(u, \mathbf{T}u) \geqslant \gamma \|u\|^2, \qquad \forall u \in V.$$

The next useful result, proved in [8] (see Theorem 3.2 therein), shows that uniform T-coercivity yields well-posedness and uniform estimates for variational problems involving a parameter.

THEOREM 2.2. *Let V be a Hilbert space equipped with the norm $\|\cdot\|$ and let $\mathcal{A}^\varepsilon(\cdot,\cdot)$ be a bilinear form on V satisfying the following conditions:*

1. *$\mathcal{A}^\varepsilon(\cdot,\cdot)$ is symmetric: $\mathcal{A}^\varepsilon(u,v) = \mathcal{A}^\varepsilon(v,u)$, for all $u, v \in V$.*

2. *$\mathcal{A}^\varepsilon(\cdot,\cdot)$ is uniformly continuous: there exists $M > 0$ such that*

$$\mathcal{A}^\varepsilon(u,v) \leqslant M\|u\|\|v\|, \qquad \forall u, v \in V. \tag{2.1}$$

3. *There exists a family $(\mathbf{T}^\varepsilon)_{\varepsilon>0}$ of uniformly bounded linear operators on V and $\gamma > 0$ such that*

$$\mathcal{A}^\varepsilon(u, \mathbf{T}^\varepsilon u) \geqslant \gamma\|u\|^2, \qquad \forall u \in V. \tag{2.2}$$

Then, given a family $(\ell^\varepsilon)_{\varepsilon>0}$ in V', the space of linear forms on V, the variational problem:

$$\text{Find } u^\varepsilon \in V \text{ such that: } \mathcal{A}^\varepsilon(u^\varepsilon, v) = \ell^\varepsilon(v), \quad \forall v \in V \tag{2.3}$$

admits a unique solution $u^\varepsilon \in V$ for all $\varepsilon > 0$ and there exists $C > 0$ independent of ε such that

$$\|u^\varepsilon\| \leqslant C\|\ell^\varepsilon\|_{V'}. \tag{2.4}$$

2.2. **Well-posedness**

We denote by $\mathbf{P}$ the harmonic extension operator from Y_1 to Y_2. More precisely, given $u \in H^1(Y_1)$, $\mathbf{P}u \in H^1(Y)$ is such that $\mathbf{P}u = u$ in Y_1, while in Y_2, $\mathbf{P}u \in H^1(Y_2)$ is the unique solution of the Dirichlet boundary value problem

$$\begin{cases} -\Delta(\mathbf{P}u) &= 0 \qquad \text{in } Y_2, \\ \mathbf{P}u &= u \qquad \text{on } \partial Y_2. \end{cases}$$

Since we assumed that in the reference cell the boundary ∂Y_2 does not intersect ∂Y, according to [7, Lemma 2.2] (see also [11, Lemma 2.9]), there exists $\kappa_Y > 0$ such that

$$\|\nabla(\mathbf{P}u)\|^2_{L^2(Y)} \leqslant \kappa_Y \|\nabla u\|^2_{L^2(Y_1)}, \qquad \forall u \in H^1(Y_1). \tag{2.5}$$

Set

$$H^1_{0,\partial\Omega}(\Omega_1^\varepsilon) := \{u \in H^1(\Omega_1^\varepsilon); u_{|\partial\Omega} = 0\}.$$

We recall now the following classical result from homogenization theory (see, *e.g.*, [11, Theorem 2.10]) concerning extension operators from Ω_1^ε to Ω. As pointed out in [7, Proposition 2.4.], the (uniform) continuity constant for these extension operators is precisely κ_Y, the continuity constant in the reference cell.

PROPOSITION 2.3. *There exists a family $(\mathbf{P}^\varepsilon)_{\varepsilon>0}$ of linear bounded extension operators from Ω_1^ε to Ω such that*

1. $\mathbf{P}^\varepsilon \in \mathcal{L}\left(L^2(\Omega_1^\varepsilon), L^2(\Omega)\right) \cap \mathcal{L}\left(H^1_{0,\partial\Omega}(\Omega_1^\varepsilon), H^1_0(\Omega)\right)$ *and the family of operators $(\mathbf{P}^\varepsilon)$ is uniformly bounded in these spaces.*
2. *For all $u \in H^1_{0,\partial\Omega}(\Omega_1^\varepsilon)$: $(\mathbf{P}^\varepsilon u)(x) = u(x), \qquad \forall x \in \Omega_1^\varepsilon$.*
3. *If κ_Y is the constant defined in (2.5), one has for all $u \in H^1_{0,\partial\Omega}(\Omega_1^\varepsilon)$:*

$$\|\nabla(\mathbf{P}^\varepsilon u)\|^2_{L^2(\Omega^\varepsilon)} \leqslant \kappa_Y \|\nabla u\|^2_{L^2(\Omega_1^\varepsilon)}. \tag{2.6}$$

For $k = 1, 2$, let u_k be the restriction to Ω_k^ε of a function u defined on Ω. We have the following result.

PROPOSITION 2.4. *For $u \in H^1(\Omega)$, set:*

$$\mathbf{T}^\varepsilon u = \begin{cases} u_1 & \text{in } \Omega_1^\varepsilon, \\ -u_2 + 2\mathbf{P}^\varepsilon u_1 & \text{in } \Omega_2^\varepsilon. \end{cases} \tag{2.7}$$

Then, $(\mathbf{T}^\varepsilon)_{\varepsilon>0}$ is a family of uniformly boundedly invertible linear operators of $\mathcal{L}\left(H^1_0(\Omega)\right)$. Moreover, there exist ε_0 and $\gamma > 0$ such that the bilinear form $\mathcal{A}^\varepsilon(\cdot,\cdot)$ defined by (1.5) satisfies for all $\varepsilon \in (0, \varepsilon_0)$ and all $u \in H^1_0(\Omega)$:

$$\mathcal{A}^\varepsilon(u, \mathbf{T}^\varepsilon u) = \int_\Omega a^\varepsilon(x)\nabla u(x) \cdot \nabla(\mathbf{T}^\varepsilon u)(x) \,\mathrm{d}x \geqslant \gamma \left\{ \|\nabla u_1\|^2_{L^2(\Omega_1^\varepsilon)} + \|\varepsilon \nabla u_2\|^2_{L^2(\Omega_2^\varepsilon)} \right\}. \tag{2.8}$$

Proof. The first assertion follows by noting the fact that $(\mathbf{T}^\varepsilon)^2$ is the identity operator. Let us prove the second assertion. For all $u \in H_0^1(\Omega)$, we obtain by using successively Cauchy-Schwarz's inequality, Young's inequality and the estimate (2.6) that

$$\begin{aligned}
\mathcal{A}^\varepsilon(u, \mathbf{T}^\varepsilon u) &= \int_\Omega a^\varepsilon(x)\nabla u(x) \cdot \nabla(\mathbf{T}^\varepsilon u)(x) \,\mathrm{d}x \\
&= a_1 \int_{\Omega_1^\varepsilon} |\nabla u_1|^2 \,\mathrm{d}x + |a_2| \int_{\Omega_2^\varepsilon} |\nabla(\varepsilon u_2)|^2 \,\mathrm{d}x \\
&\qquad + 2\varepsilon a_2 \int_{\Omega_2^\varepsilon} \nabla(\varepsilon u_2) \cdot \nabla\left(\mathbf{P}^\varepsilon u_1\right) \mathrm{d}x \\
&\geqslant a_1 \|\nabla u_1\|^2_{L^2(\Omega_1^\varepsilon)} + |a_2| \|\nabla(\varepsilon u_2)\|^2_{L^2(\Omega_2^\varepsilon)} \\
&\qquad - 2\varepsilon |a_2| \|\nabla(\varepsilon u_2)\|_{L^2(\Omega_2^\varepsilon)} \|\nabla(\mathbf{P}^\varepsilon u_1)\|_{L^2(\Omega_2^\varepsilon)} \\
&\geqslant a_1 \|\nabla u_1\|^2_{L^2(\Omega_1^\varepsilon)} + |a_2| \|\nabla(\varepsilon u_2)\|^2_{L^2(\Omega_2^\varepsilon)} \\
&\qquad - \varepsilon |a_2| \left\{ \|\nabla(\varepsilon u_2)\|^2_{L^2(\Omega_2^\varepsilon)} + \|\nabla(\mathbf{P}^\varepsilon u_1)\|^2_{L^2(\Omega_2^\varepsilon)} \right\} \\
&\geqslant a_1 \|\nabla u_1\|^2_{L^2(\Omega_1^\varepsilon)} + |a_2| \|\nabla(\varepsilon u_2)\|^2_{L^2(\Omega_2^\varepsilon)} \\
&\qquad - \varepsilon |a_2| \left\{ \|\nabla(\varepsilon u_2)\|^2_{L^2(\Omega_2^\varepsilon)} + \kappa_Y \|\nabla u_1\|^2_{L^2(\Omega_1^\varepsilon)} \right\} \\
&= \left(a_1 - \varepsilon \kappa_Y |a_2|\right) \|\nabla u_1\|^2_{L^2(\Omega_1^\varepsilon)} + (1 - \varepsilon)\, |a_2| \|\nabla(\varepsilon u_2)\|^2_{L^2(\Omega_2^\varepsilon)}.
\end{aligned}$$

Let ε_0 be chosen small enough such that

$$\frac{2\kappa_Y |a_2| \varepsilon_0}{a_1} < 1 < \frac{1}{2\varepsilon_0}.$$

Then, for all $\varepsilon \in (0, \varepsilon_0)$, there holds

$$a_1 - \varepsilon \kappa_Y |a_2| > \frac{a_1}{2} \qquad \text{and} \qquad (1 - \varepsilon)|a_2| > \frac{|a_2|}{2}.$$

Consequently, we have proved that

$$\mathcal{A}^\varepsilon(u, \mathbf{T}^\varepsilon u) \geqslant \gamma \left\{ \|\nabla u_1\|^2_{L^2(\Omega_1^\varepsilon)} + \|\nabla(\varepsilon u_2)\|^2_{L^2(\Omega_2^\varepsilon)} \right\},$$

with $\gamma = \dfrac{1}{2} \min(a_1, |a_2|)$. $\square$

Corollary 2.5. *There exists ε_0 such that problem (1.3) admits a unique solution $u^\varepsilon \in H_0^1(\Omega)$ for all $\varepsilon \in (0, \varepsilon_0)$. Moreover, we have the a priori energy estimate:*

$$\|u^\varepsilon\|^2_{L^2(\Omega)} + \|\nabla u_1^\varepsilon\|^2_{L^2(\Omega_1^\varepsilon)} + \|\nabla(\varepsilon u_2^\varepsilon)\|^2_{L^2(\Omega_2^\varepsilon)} \leqslant C, \tag{2.9}$$

for some positive constant C independent of ε.

Proof. The well-posedness of problem (1.3) for fixed $\varepsilon \in (0, \varepsilon_0)$ follows immediately from Theorem 2.2 and Proposition 2.4. The uniform energy estimates (2.9) can be obtained from the T-coercivity property (2.8) by noting that:

$$\begin{aligned} \|\nabla u_1^\varepsilon\|^2_{L^2(\Omega_1^\varepsilon)} + \|\varepsilon \nabla u_2^\varepsilon\|^2_{L^2(\Omega_2^\varepsilon)} &\leqslant \gamma^{-1} \mathcal{A}^\varepsilon(u^\varepsilon, \mathbf{T}^\varepsilon u^\varepsilon) \\ &= \gamma^{-1} \int_\Omega f(x) \mathbf{T}^\varepsilon u^\varepsilon(x)\, \mathrm{d}x \\ &\leqslant \gamma^{-1} \|f\|_{L^2(\Omega)}\ \|\mathbf{T}^\varepsilon u^\varepsilon\|_{L^2(\Omega)} \\ &\leqslant C\|f\|_{L^2(\Omega)}\ \|u^\varepsilon\|_{L^2(\Omega)}, \end{aligned}$$

where we have used for the last inequality the fact that the family $(\mathbf{T}^\varepsilon)$, as bounded operators from $L^2(\Omega)$ onto itself, is uniformly bounded (just like $\mathbf{P}^\varepsilon$). To conclude, we need a Poincaré-type inequality to bound the L^2 norm of the solution appearing in the right hand side. In Ω_1^ε, the classical Poincaré inequality applies (since u^ε vanishes on $\partial\Omega$):

$$\|u_1^\varepsilon\|_{L^2(\Omega_1^\varepsilon)} \leqslant C\|\nabla u_1^\varepsilon\|_{L^2(\Omega_1^\varepsilon)}.$$

In Ω_2^ε, as in [2, Lemma 4.3], one can prove that

$$\|u_2^\varepsilon\|_{L^2(\Omega_2^\varepsilon)} \leqslant C\left\{\|u_1^\varepsilon\|_{L^2(\Omega_1^\varepsilon)} + \|\nabla(\varepsilon u^\varepsilon)\|_{L^2(\Omega)}\right\}.$$

Combining these two inequalities, we obtain the Poincaré-type inequality (given in [1, Remark 4.2]):

$$\|u^\varepsilon\|_{L^2(\Omega)} \leqslant C\left\{\|\nabla u_1^\varepsilon\|_{L^2(\Omega_1^\varepsilon)} + \|\nabla(\varepsilon u_2^\varepsilon)_{L^2(\Omega_2^\varepsilon)}\|\right\}.$$

□

3. TWO-SCALE CONVERGENCE

Our goal in this section is to pass to the limit $\varepsilon \to 0$ in the variational formulation (1.4) of the problem (1.3). To this end, we make use of the two-scale convergence method (see [1, 12]). Let us emphasize that this procedure (and the obtained two-scale limit) is the same as in the elliptic case ($a_2 > 0$).

The only difference comes from the derivation of the uniform *a priori* estimates.

Let us introduce the following functional spaces:

- $\mathcal{C}^\infty_\#(Y)$ the space of infinitely differentiable functions in $\mathbb{R}^d$ which are periodic of period Y,
- $H^1_\#(Y)$ the completion of $\mathcal{C}^\infty_\#(Y)$ for the norm of $H^1(Y)$,
- $H^1_{0\#}(Y_2)$ the subspace of $H^1_\#(Y)$ of functions vanishing on ∂Y_2.

We recall the definition of the weak two-scale convergence from [1, 12]:

Definition 3.1. A sequence of functions u^ε in $L^2(\Omega)$ two-scale converges to $u \in L^2(\Omega \times Y)$ (and we denote this by $u^\varepsilon \xrightarrow{2-s} u$) if, for every $\psi \in \mathcal{C}^\infty_0(\Omega; \mathcal{C}^\infty_\#(Y))$, one has

$$\lim_{\varepsilon \to 0} \int_\Omega u^\varepsilon(x)\psi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = \int_{\Omega \times Y} u(x,y)\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Our main result is the following theorem.

THEOREM 3.2. *The sequence of solutions u^ε of* (1.3) *two-scale converges to $u(x) + \mathbb{1}_{Y_2}(y)v(x,y)$, where*

- *$u \in H^1_0(\Omega)$ is given by*

$$u(x) = \frac{1}{a_1} u^H(x) \tag{3.1}$$

  and u^H is the unique solution of the homogenized problem

$$\begin{cases} -\mathrm{div}(a^H \nabla_x u^H) &= f, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

  in which the entries of the symmetric, positive definite, constant $d \times d$ matrix a^H are given by:

$$a^H_{ij} = \int_{Y_1} \nabla\left(w_i(y) + y_i\right) \cdot \nabla\left(w_j(y) + y_j\right) \mathrm{d}y, \qquad 1 \leqslant i,j \leqslant d,$$

  and w_i, $1 \leqslant i \leqslant d$, are the solutions of the cell problems ($\{\mathbf{e}_i,\ 1 \leqslant i \leqslant d\}$ denotes the canonical basis of $\mathbb{R}^d$):

$$\begin{cases} -\mathrm{div}_y\left(\nabla w_i(y) + \mathbf{e}_i\right) &= 0, \quad \text{in } Y_1, \\ \left(\nabla w_i(y) + \mathbf{e}_i\right) \cdot n &= 0, \quad \text{on } \partial Y_2, \\ w_i(y) \text{ is } Y-\text{periodic}. \end{cases} \tag{3.3}$$

- $v \in L^2(\Omega; H^1_{0\#}(Y_2))$ *is given by*

$$v(x, y) = \frac{1}{a_2} f(x)\zeta(y), \tag{3.4}$$

where ζ *is the unique solution of the following problem:*

$$\begin{cases} -\Delta_{yy}\zeta &= 1, \quad \text{in } Y_2, \\ \zeta &= 0, \quad \text{on } \partial Y_2. \end{cases} \tag{3.5}$$

Proof. From the *a priori* estimates (2.9), arguing as in [1, Lemma 4.7], it follows that there exist $u(x) \in H^1_0(\Omega)$, $v(x, y) \in L^2(\Omega; H^1_{0\#}(Y_2))$, and $u_1(x, y) \in L^2(\Omega; H^1_\#(Y_1)/\mathbb{R})$ such that, up to passing to a subsequence, we have

$$\begin{cases} u^\varepsilon \xrightarrow{\text{2-s}} u(x) + \mathbb{1}_{Y_2}(y)v(x, y), \\ \mathbb{1}_{Y_1}(x/\varepsilon)\nabla u^\varepsilon \xrightarrow{\text{2-s}} \mathbb{1}_{Y_1}(y)(\nabla u(x) + \nabla_y u_1(x, y)), \\ \varepsilon \mathbb{1}_{Y_2}(x/\varepsilon)\nabla u^\varepsilon \xrightarrow{\text{2-s}} \mathbb{1}_{Y_2}(y)\nabla_y v(x, y). \end{cases} \tag{3.6}$$

Here, $H^1_\#(Y_1)/\mathbb{R}$ denotes the quotient space of functions defined in $H^1_\#(Y_1)$ up to an additive real constant.

Now, in order to obtain the limit problem, following [1], let us take in the variational formulation of problem (1.3) an admissible test function of the form $v = \Phi(x) + \varepsilon\Phi_1(x, x/\varepsilon) + \Psi(x, x/\varepsilon)$, where $\Phi(x) \in \mathcal{D}(\Omega)$, $\Phi_1(x, y), \Psi(x, y) \in \mathcal{D}(\Omega; \mathcal{C}^\infty_\#(Y))$, with $\Psi(x, y) = 0$ for $y \in Y_1$. Then, passing to the two-scale limit with the aid of convergences (3.6), we are led to

$$\begin{aligned} &\int_{\Omega\times Y_1} a(y)\, [\nabla u(x) + \nabla_y u_1(x, y)] \cdot [\nabla\Phi(x) + \nabla_y \Phi_1(x, y)] \,\mathrm{d}x\,\mathrm{d}y \\ &\qquad + \int_{\Omega\times Y_2} a(y)\nabla_y v(x, y) \cdot \nabla_y \Psi(x, y)\,\mathrm{d}x\,\mathrm{d}y \\ &\qquad\qquad = \int_{\Omega\times Y} f(x)[\Phi(x) + \mathbb{1}_{Y_2}(y)\Psi(x, y)]\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{3.7}$$

By usual density arguments, (3.7) holds true for any $(\Phi, \Phi_1, \Psi) \in H^1_0(\Omega) \times L^2(\Omega; H^1_\#(Y_1)/\mathbb{R}) \times L^2(\Omega; H^1_{0\#}(Y_2))$. Let us first choose $\Psi = 0$ in (3.7). Then, by a standard procedure and by using the factorization

$$u_1(x, y) = \sum_{i=1}^{d} \frac{\partial u}{\partial x_i}(x) w_i(y), \tag{3.8}$$

where w_i, with $1 \leqslant i \leqslant d$, are the unique solutions of the cell problems (3.3), the two-scale limit problem can be decoupled and classical results from the theory of homogenization then imply that we arrive at (3.2). Choosing now $\Phi = 0$ and $\Phi_1 = 0$ in (3.7), we are led to (3.4) and (3.5). □

Remark 3.3. Let us notice that Theorem 3.2 implies that the sequence of solutions u^ε weakly converges in $L^2(\Omega)$ to the function

$$\frac{1}{a_1} u^H(x) + \frac{1}{a_2} \left(\int_{Y_2} \zeta(y)\, \mathrm{d}y \right) f(x). \tag{3.9}$$

The functions u^H and ζ, solutions of problems (3.2) and (3.5), respectively, are independent of the coefficients a_1 and a_2 and they only depend on the geometry of the domains Ω, Y_1, Y_2, and on the given function f. We thus obtain an explicit dependence of the weak L^2 limit of the sequence u^ε with respect to the conductivities a_1 and a_2. It is worth noticing that in [7, 6], where a similar problem was considered for a contrast supposed to be independent of ε, the dependence of the weak limit on the contrasts was not explicit.

Remark 3.4. Given $\alpha > 0$, the problem

$$\begin{cases} -\mathrm{div}(a^\varepsilon \nabla u^\varepsilon) + \alpha u^\varepsilon = f, & \text{in } \Omega, \\ u^\varepsilon = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.10}$$

seems more difficult to analyze than the case $\alpha = 0$ considered in this note. For fixed ε, one can easily prove that problem (3.10) fits in the framework of Fredholm theory (more precisely, the associated bilinear form is of the form $\mathtt{T}$-coercive+compact). Hence, it is well-posed for all values of α, except for a countable set of critical values. These forbidden values depend on ε and studying their asymptotic behavior as $\varepsilon \to 0$ seems to be not obvious.

Acknowledgments. The authors are very grateful to the anonymous referees for their valuable comments and suggestions, which improved the content of this paper. The last author wishes to thank Institut Elie Cartan de Lorraine and Université de Lorraine for the financial support and the warm hospitality.

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