# ON THE DUAL OF BURCH'S INEQUALITY AND THE WIDTH OF CERTAIN GRADED ARTINIAN MODULES

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Let $(A, \mathfrak{m})$ be a commutative quasi-local ring with non-zero identity with infinite residue field and let $I$ be an ideal of $A$. Let $M$ be an Artinian $A$-module and $G(I, M)$ be a dual of associated graded module and we denote by $s(I, M)$ the analytic spread of $I$ with respect to $M$. The dual of Burch's inequality says that $s(I, M) + \inf\{\text{width}\,(0 :_M I^n) : n \geq 1\} \leq \operatorname{Kdim} M$, and it is well known that equality holds if $G(I, M)$ is co-Cohen-Macaulay. Thus, in that case one can compute the width of dual of the associated graded module $I$ as $\text{width}\, G(I, M) = s(I, M) + \inf\{\text{width}\,(0 :_M I^n) : n \geq 1\}$. We study when such an equality is also valid when $G(I, M)$ is not necessarily co-Cohen-Macaulay.

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## 1. INTRODUCTION

Throughout this paper, we assume that $(A, \mathfrak{m})$ is a commutative quasi-local ring (*i.e.* $A$ has exactly one maximal ideal $\mathfrak{m}$) with non-zero identity and $A/\mathfrak{m}$ is infinite and let $M$ be a non-zero Artinian $A$-module. For an ideal $I$ of $A$, Kirby in [5] introduced the following two graded modules dual to the Rees ring and associated graded ring $R(I, M) = \bigoplus_{n=-\infty}^{\infty} R(I, M)_n$, where $R(I, M)_n = M/(0 :_M I^{-n})$ if $n \leq 0$ and $R(I, M)_n = 0$ if $n > 0$, and $G(I, M) = \bigoplus_{n=-\infty}^{\infty} G(I, M)_n$, where $G(I, M)_n = (0 :_M I^{-n+1})/(0 :_M I^{-n})$ if $n \leq 0$ and $G(I, M)_n = 0$ if $n > 0$. Kirby used the two graded modules in the proofs of theorems about the Artin-Rees property and Hilbert polynomials for Artinian modules. Roberts in [10] defined the dual dimension $\operatorname{Kdim} M$ and proved that $\operatorname{Kdim} M = 0$ if and only if $M$ has finite length and $\operatorname{Kdim} M (> 0)$ is equal to the least integer $r$ for which there exists elements $a_1, ..., a_r \in \mathfrak{m}$ such that $\ell(0 :_M (a_1, ..., a_r)) < \infty$ (here $\ell(-)$ denotes length)(see also [6]). It is well-known that if $I$ is an ideal of $A$ with $\ell(0 :_M I) < \infty$, then $\operatorname{Kdim} G(I, M) = \operatorname{Kdim} M$ and if $\operatorname{Kdim} M > 0$, then $\operatorname{Kdim} R(I, M) = \operatorname{Kdim} M + 1$. For the proof of these facts see [12].

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Sharp and Taherizadeh in [11] proved that the two graded modules are very useful in the discussion of reductions and integral closures of ideals relative to an Artinian module. They defined that an ideal $J$ is a reduction of $I$ relative to $M$ if $J \subseteq I$ and there exists nonnegative integer $n$ such that $(0 :_M JI^n) = (0 :_M I^{n+1})$. If $J$ is a reduction of $I$ relative to $M$ and there is no reduction of $I$ relative to $M$ which is strictly contained in $J$ then it is said that $J$ is a minimal reduction of $I$ relative to $M$. When $A/\mathfrak{m}$ is infinite and $I \subseteq \mathfrak{m}$ is an ideal of $A$ with $\ell(0 :_M I) < \infty$, every reduction of $I$ relative to $M$ contains a minimal reduction of $I$ relative to $M$ and every minimal reduction of $I$ relative to $M$ is generated by a system of parameters for $M$ (see [11, Theorem 6.2]). The least nonnegative integer $n$ such that $(0 :_M JI^n) = (0 :_M I^{n+1})$ for some minimal reduction $J$ of $I$ relative to $M$ is called the reduction number of $I$ relative to $M$ and denoted by $r_J(I, M)$ and the reduction number of $I$ relative to $M$ denoted by $r(I, M)$ and is defined as
$r(I, M) = \min\{r_J(I, M), J \text{ is a minimal reduction of I relative to M }\}$.

Nishitani in [7] denoted by $deg(p)$ the degree of the polynomial function $p(i) = \ell((0 :_M I^i\mathfrak{m})/(0 :_M I^i))$ for any positive integer $i$. Sharp and Taherizadeh in [11] defined the analytic spread $s(I, M)$ by $1 + \deg(p)$. Indeed, the analytic spread $s(I, M)$ is defined as the minimal number of generators of a (any) minimal reduction of $I$ relative to $M$. Matlis in [8] defined that a sequence $x_1, ..., x_n \in \mathfrak{m}$ is an $M$-cosequence if $0 :_M (x_1, ..., x_{i-1}) \xrightarrow{x_i} 0 :_M (x_1, ..., x_{i-1})$ is surjective for $i = 1, ..., n$. The codepth of $M$, denoted by $\operatorname{width} M$, is defined as the length of a maximal $M$-cosequence in $\mathfrak{m}$. Then it is always true that $\operatorname{width} M \leq \operatorname{Kdim} M$ (see [9]). When the equality holds, it is said that $M$ is co-Cohen-Macaulay. Tang and Zakeri in [13] proved that $M$ is co-Cohen-Macaulay if and only if every system of parameters for $M$ is an $M$-cosequence (see also [14]). Note that for an ideal $I$, we use codepth of $M$ by $\operatorname{cograde}_I(M)$ and when $I = \mathfrak{m}$ we use codepth of $M$ by $\operatorname{width} M$.

Nishitani in [7] proved the dual of Burch's inequality

$$s(I, M) \leq \operatorname{Kdim} M - \operatorname{width}(0 :_M I^n) \quad (n \gg 0).$$

The Burch's inequality was given originally in [2, Corollary (i)]. Cheraghi and Mafi in [3] proved that the equality holds if $G(I, M)$ is co-Cohen-Macaulay. On the other hand, Burch's inequality says that for any ideal $I$,

$$\inf\{\operatorname{width}(0 :_M I^n), n \geq 1\} + s(I, M) \leq \operatorname{Kdim} M.$$

Hence, if $G(I, M)$ is co-Cohen-Macaulay one can write

$$\operatorname{width} G(I, M) = \inf\{\operatorname{width}(0 :_M I^n), n \geq 1\} + s(I, M).$$

To simplify, we shall call the value $\inf\{\operatorname{width}(0 :_M I^n), n \geq 1\}$ the Burch number of $I$ relative to $M$ and we denote it by $B(I, M)$. Note that, in [3, Corollary 3.7] Cheraghi and Mafi proved that the width of $(0:_M I^n)$ have a stable value (see also [1] for stability of depth). In this case, this asymptotic value coincides with $B(I,M)$ but this does not occur in general. In this paper we shall compute for the ideals under some conditions their Burch number and we compute the width of graded module dual to their associated graded modules.

## 2. THE RESULT

We start this section by the following fundamental lemma which is proved by Tang in [12, Theorem 3.2].

LEMMA 2.1. *Let $x_1,\dots,x_h \in I\backslash I^2$. Then $x_1^*,\dots,x_h^*$ is a $G(I,M)$-cosequence if and only if $x_1,\dots,x_h$ is an $M$-cosequence and*

$$(0:_M I^{n+1})+(0:_M (x_1,\dots,x_h))=(0:_M (x_1,\dots,x_h)I^n)$$

*for all $n\geq 0$, where $x^*=x+I^2$.*

LEMMA 2.2. *If $x_1,\dots,x_h \in I$ is an $M$-cosequence, then*

$$(0:_M (x_1,\dots,x_i))+(0:_M (x_{i+1}))=(0:_M (x_1,\dots,x_i)(x_{i+1}))$$

*for $i=0,1,\dots,h-1$.*

*Proof.* It is clear that

$$(0:_M (x_1,\dots,x_i))+(0:_M (x_{i+1}))\subseteq(0:_M (x_1,\dots,x_i)(x_{i+1}))$$

for $i=0,1,\dots,h-1$.

To prove the other side we assume $m\in(0:_M (x_1,\dots,x_i)(x_{i+1}))$ then we have $m(x_1,\dots,x_i)(x_{i+1})=0$ and so $m(x_{i+1})\subseteq(0:_M (x_1,\dots,x_i))=x_{i+1}(0:_M (x_1,\dots,x_i))$ for $i=0,1,\dots,h-1$. Thus there exists $m'\in(0:_M (x_1,\dots,x_i))$ such that $mx_{i+1}=m'x_{i+1}$. Then $m-m'\in(0:_M (x_{i+1}))$ and so $m\in(0:_M (x_1,\dots,x_i))+(0:_M (x_{i+1}))$ for $i=0,1,\dots,h-1$, as desired. □

The following result is a dual of [4, Lemma 3.1]

LEMMA 2.3. *Let $x_1,\dots,x_h$ be elements in $I\setminus I^2$ such that $x_1^*,\dots,x_h^*$ is a $G(I,M)$-cosequence.*

(a) *If* width$(0:_M I^2)<$ width$(0:_M I)$, *then*

$$\text{width}\,(0:_M (I^2+(x_1,\dots,x_h)))=\text{width}\,(0:_M I^2).$$

(b) *If* width$(0:_M I)<$ width$(0:_M I^2)$, *then*

$$\text{width}\,(0:_M (I^2+(x_1,\dots,x_h)))=\text{width}\,(0:_M I)-1.$$

(c) *If* $width\,(0:_M I) = width\,(0:_M I^2)$, *then*

$$width\,(0:_M (I^2 + (x_1, ..., x_h))) \geq width\,(0:_M I) - 1.$$

*Proof.* Since the family $x_1^*, ..., x_h^*$ is a $G(I, M)$-cosequence, then by Lemma 2.1, $x_1, ..., x_h$ forms an $M$-cosequence and we have that $(0:_M (x_1, ..., x_i)) + (0:_M I^2) = 0:_M (x_1, ..., x_i)I$ for all $i = 1, ..., h$. For all $i = 1, ..., h-1$, we consider the following exact sequence

$$0 \to 0:_M (I^2 + (x_1, ..., x_{i+1})) \to 0:_M (I^2 + (x_1, ..., x_i)) \\ \to \frac{0:_M (I^2 + (x_1, ..., x_i))}{0:_M (I^2 + (x_1, ..., x_{i+1}))} \to 0.$$

Since

$$\begin{aligned}
&0:_M (I^2 + (x_1, ..., x_i))/0:_M (I^2 + (x_1, ..., x_{i+1})) \\
&= 0:_M (I^2 + (x_1, ..., x_i))/0:_M (I^2 + (x_1, ..., x_i)) \cap 0:_M (x_1, ..., x_{i+1}) \\
&\cong 0:_M (I^2 + (x_1, ..., x_i)) + (0:_M (x_1, ..., x_{i+1}))/0:_M (x_1, ..., x_{i+1}) \\
&= ((0:_M I^2) + (0:_M (x_1, ..., x_{i+1}))) \cap (0:_M (x_1, ..., x_i))/0:_M (x_1, ..., x_{i+1}) \\
&= 0:_M (x_1, ..., x_{i+1})I \cap 0:_M (x_1, ..., x_i)/0:_M (x_1, ..., x_{i+1}) \\
&= 0:_M (x_1, ..., x_{i+1})I \cap 0:_M (x_1, ..., x_i)/0:_M (x_1, ..., x_{i+1}) \\
&\quad \cap 0:_M (x_1, ..., x_{i+1})I \\
&= 0:_M (x_1, ..., x_{i+1})I \cap 0:_M (x_1, ..., x_i)/0:_M (x_1, ..., x_i) \\
&\quad \cap 0:_M (x_{i+1}) \cap 0:_M (x_1, ..., x_{i+1})I \\
&= (0:_M (x_1, ..., x_{i+1})I \cap 0:_M (x_1, ..., x_i)) + 0:_M (x_{i+1})/0:_M (x_{i+1}) \\
&= (0:_M (x_{i+1})I \cap 0:_M (x_1, ..., x_i)) + 0:_M (x_{i+1})/0:_M (x_{i+1}) \\
&= (0:_M (x_{i+1})I) \cap (0:_M (x_1, ..., x_i) + 0:_M (x_{i+1}))/0:_M (x_{i+1}) \\
&= 0:_M (x_{i+1})I/0:_M (x_{i+1}) \cong (0:_M I),
\end{aligned}$$

for $i = 0, ..., h-1$, we have the following exact sequence

$$0 \longrightarrow 0:_M I^2 + (x_1, ..., x_{i+1}) \longrightarrow 0:_M I^2 + (x_1, ..., x_i) \longrightarrow (0:_M I) \longrightarrow 0. \quad (\dagger)$$

Assume that $\mathrm{width}\,(0:_M I^2) < \mathrm{width}\,(0:_M I)$. Then, by applying the width-Lemma (see [9, Proposition 3.16]) to the exact sequence ($\dagger$), for $i = 0, ..., h-1$, we obtain that $\mathrm{width}\,(0:_M I^2 + (x_1, ..., x_i)) = \mathrm{width}\,(0:_M I^2)$ for all $i = 1, ..., h$, and (a) is proved. Let $\mathrm{width}\,(0:_M I) < \mathrm{width}\,(0:_M I^2)$. Then, by [9, Proposition 3.16] and ($\dagger$) for $i = 0$, we obtain that $\mathrm{width}\,(0:_M I) = \mathrm{width}\,(0:_M I^2 + (x_1)) + 1$. In particular, $\mathrm{width}\,(0:_M I^2 + (x_1)) < \mathrm{width}\,(0:_M I)$ and by (a) we have now that $\mathrm{width}\,(0:_M I^2 + (x_1, ..., x_i)) = \mathrm{width}\,(0:_M I^2 + (x_1)) = \mathrm{width}\,(0:_M I) - 1$ for all $i = 1, ..., h$. This proves (b).

Assume that width $(0 :_M I) =$ width $(0 :_M I^2)$. In this case, by taking $i = 0$ in (†) and using [9, Proposition 3.16] we have

$$\operatorname{width}(0 :_M I) = \operatorname{width}(0 :_M I^2) \leq \operatorname{width}(0 :_M I^2 + (x_1)) + 1,$$

then we have the following three cases:

(i) $\operatorname{width}(0 :_M I) = \operatorname{width}(0 :_M I^2) = \operatorname{width}(0 :_M I^2 + (x_1)) + 1$

(ii) $\operatorname{width}(0 :_M I) = \operatorname{width}(0 :_M I^2) < \operatorname{width}(0 :_M I^2 + (x_1))$

(iii) $\operatorname{width}(0 :_M I) = \operatorname{width}(0 :_M I^2) = \operatorname{width}(0 :_M I^2 + (x_1))$

If $\operatorname{width}(0 :_M I^2 + (x_1)) = \operatorname{width}(0 :_M I) - 1$ or $\operatorname{width}(0 :_M I) < \operatorname{width}(0 :_M I^2 + (x_1))$, from (b) it follows that $\operatorname{width}(0 :_M I^2 + (x_1, ..., x_i)) = \operatorname{width}(0 :_M I^2 + (x_1)) = \operatorname{width}(0 :_M I) - 1$ for all $i = 1, ..., h$ and in this case we have $(c)$. If $\operatorname{width}(0 :_M I) = \operatorname{width}(0 :_M I^2 + (x_1))$ then, by [9, Proposition 3.16] and (†) for $i = 1$, we have $\operatorname{width}(0 :_M I) \leq \operatorname{width}(0 :_M I^2 + (x_1, x_2)) + 1$. Repeating the above argument if necessary for $i = 2, ..., h - 1$, we may conclude that $\operatorname{width}(0 :_M I^2 + (x_1, ..., x_i)) \geq \operatorname{width}(0 :_M I) - 1$ for all $i = 2, ..., h$. In any case we have $\operatorname{width}(0 :_M I^2 + (x_1, ..., x_i)) \geq \operatorname{width}(0 :_M I) - 1$ for all $i = 1, ..., h$ and (c) is proved. □

*Remark* 2.4. Sharp and Taherizadeh in [11] proved that the function $f_M : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by

$$f_M(n) = \ell_A(0 :_M \mathfrak{m}I^n / 0 :_M I^n)$$

is a polynomial function. If $d$ denotes the degree of this polynomial function, then $s(I, M) = d + 1$. By the proof of [11, Theorem(4.1) and Corollary (4.2)] and [5], if $x$ is a part of minimal reduction of $I$ relative to $M$ and $\overline{M} = (0 :_M x)$ then $s(I, \overline{M}) = s(I, M) - 1$.

Let $M$ be a co-Cohen-Macaulay Artinian $A$-module, we say that an ideal $I$ is equimultiple ideal if $\operatorname{cograde}_I(M) = s(I, M)$. For equimultiple ideals with reduction number 1, we may compute the Burch number in the following way:

Proposition 2.5. *Let $M$ be a co-Cohen-Macaulay Artinian $A$-module and let $I \subseteq A$ an equimultiple ideal with $r(I, M) \leq 1$. Then, $B(I, M) =$ width $(0 :_M I)$.*

*Proof.* We use induction on $h := \operatorname{cograde}_I(M)$. Let $J = (x_1, ..., x_h)$ be a minimal reduction of $I$ relative to $M$ with $(0 :_M I^2) = (0 :_M JI)$. Since $r(I, M) \leq 1$, by Lemma 2.1, the sequence $x_1^*, ..., x_h^*$ is a cosequence with respect to $G(I, M)$. If $h = 0$, for all $n \geq 2$ one has $0 :_M I^n = M$ and so $\operatorname{width}(0 :_M I^n) = \operatorname{width} M = \operatorname{Kdim} M \geq \operatorname{width}(0 :_M I)$ then we have

$B(I,M) = \operatorname{width}(0 :_M I)$. Suppose that $h > 0$, and let $\overline{M} = 0 :_M (x_1)$. Then, $\overline{M}$ is co-Cohen-Macaulay, $\operatorname{cograde}_I(\overline{M}) = s(I,\overline{M}) = h-1$ and $r(I,\overline{M}) \leq 1$. On the other hand, for all $n \geq 2$ we consider the following exact sequence

$$0 \longrightarrow 0 :_M I^n + (x_1) \longrightarrow 0 :_M I^n \longrightarrow 0 :_M I^n/0 :_M I^n + (x_1) \longrightarrow 0$$

where $0 :_M I^n/0 :_M I^n + (x_1) \cong 0 :_M I^{n-1}$, and hence by [9, Proposition 3.16],

$$\operatorname{width}(0 :_M I^n) \geq \min\{\operatorname{width}(0 :_M I^{n-1}), \operatorname{width}(0 :_M I^n + (x_1))\}$$

$= \min\{\operatorname{width}(0 :_M I^{n-1}), \operatorname{width}(0 :_{\overline{M}} I^n)\}$. By induction on $n$ and $h$ we have $\operatorname{width}(0 :_M I^n) \geq \operatorname{width}(0 :_M I)$ and $B(I,M) = \operatorname{width}(0 :_M I)$. $\square$

In the sequel we compute the width of the associated graded modules of equmultiple ideals with reduction number 1.

Proposition 2.6. *Let $M$ be a co-Cohen-Macaulay Artinian $A$-module and let $I \subseteq A$ an equimultiple ideal with $r(I,M) \leq 1$. Then width $G(I,M) =$ width $(0 :_M I) +$ cograde $_I(M)$.*

*Proof.* The proof is by reduction to the case $\operatorname{cograde}_I(M) = 0$. Put $h := \operatorname{cograde}_I(M)$. Assume that $h = 0$. Then the minimal reduction of $I$ relative to $M$ is 0 and so $(0 :_M I^2) = 0 :_M 0 = M$ and so $G(I,M) = (0 :_M I) \oplus M/(0 :_M I)$. Thus, $\operatorname{width} G(I,M) = \min\{\operatorname{width}(0 :_M I), M/(0 :_M I)\} = \operatorname{width}(0 :_M I)$.

Assume now that $h > 0$, $\overline{M} = 0 :_M (x_1, ..., x_h)$ and let $J = (x_1, ..., x_h) \subseteq I$ be a minimal reduction of $I$ relative to $M$ such that $(0 :_M I^2) = (0 :_M JI)$. Then, $x_1^*, ..., x_h^*$ is a $G(I,M)$-cosequence and by [12, Lemma 3.1] we have $(0 :_{G(I,M)} (x_1^*, ..., x_h^*)) \cong G(I, 0 :_M (x_1, ..., x_h))$. Hence, $\operatorname{width} G(I,M) = \operatorname{width} G(I, 0 :_M (x_1, ..., x_h)) + h$. Since $\operatorname{cograde}_I(\overline{M}) = s(I,\overline{M}) = 0$ and $r(I,\overline{M}) \leq 1$, we have that $\operatorname{width} G(I, 0 :_M (x_1, ..., x_h)) = \operatorname{width}(0 :_{\overline{M}} I) = \operatorname{width}(0 :_M I)$ and consequently $\operatorname{width} G(I,M) = \operatorname{width}(0 :_M I) + \operatorname{cograde}_I M$.
$\square$

Corrolary 2.7. *Let $M$ be a co-Cohen-Macaulay Artinian $A$-module and let $I \subseteq A$ be an equimultiple ideal with $r(I,M) \leq 1$. Then, width $G(I,M) = B(I,M) + s(I,M)$.*

*Proof.* By Proposition 2.5 we have $B(I,M) = \operatorname{width}(0 :_M I)$. Now by using Proposition 2.6 we have $\operatorname{width} G(I,M) = B(I,M) + s(I,M)$, as required.
$\square$

Proposition 2.8. *Let $M$ be a co-Cohen-Macaulay Artinian $A$-module and let $I \subseteq A$ an equimultiple ideal with $r(I,M) \leq 2$ and $(0 :_M I^2) + (0 :_M J) = (0 :_M JI)$ for any minimal reduction $J$ of $I$. Then $B(I,M) \geq \min\{$width $(0 :_M I^2)$, width $(0 :_M I) - 1\}$.*

*Proof.* We prove the proposition by induction on cograde$_I(M) = h$. Let $J = (x_1, x_2, ..., x_h)$ be a minimal reduction of $I$ relative to $M$ with $(0 :_M I^n) = (0 :_M JI^{n-1})$ for $n \geq 3$. Since $r(I, M) \leq 2$ and $(0 :_M I^2)+(0 :_M J) = (0 :_M JI)$ by Lemma 2.1, the sequence $x_1^*, ..., x_h^*$ is a cosequence with respect to $G(I, M)$. If $h = 0$, for all $n \geq 3$ one has $0 :_M I^n = M$ and so $B(I, M) \geq \min\{\text{width}\,(0 :_M I^2), \text{width}\,(0 : I) - 1\}$. Suppose that $h > 0$, and let $\overline{M} = 0 :_M (x_1)$. Then, $\overline{M}$ is a co-Cohen-Macaulay ring, cograde$_I(\overline{M}) = s(I, \overline{M}) = h - 1$ and $r(I, \overline{M}) \leq 2$. On the other hand, for all $n \geq 3$ we have the following exact sequence

$$0 \longrightarrow 0 :_M I^n + (x_1) \longrightarrow 0 :_M I^n \longrightarrow 0 :_M I^n/0 :_M I^n + (x_1) \longrightarrow 0,$$

where $0 :_M I^n/0 :_M I^n + (x_1) \cong 0 :_M I^{n-1}$, and hence by [9, Proposition 3.16],

$$\begin{aligned}
\text{width}\,(0 :_M I^n) &\geq \min\{\text{width}\,(0 :_M I^{n-1}), \text{width}\,(0 :_M I^n + (x_1))\} \\
&= \min\{\text{width}\,(0 :_M I^{n-1}), \text{width}\,(0 :_{\overline{M}} I^n)\} \\
&\geq \min\{\text{width}\,(0 :_M I^2), \text{width}\,(0 :_M I) - 1, \text{width}\,(0 :_{\overline{M}} I^2), \\
&\qquad\qquad \text{width}\,(0 :_{\overline{M}} I) - 1\} \\
&\geq \min\{\text{width}\,(0 :_M I^2), \text{width}\,(0 :_M I) - 1\}
\end{aligned}$$

such that the second inequality follows by induction on $n$ and $h$ and the third inequality follows by Lemma 2.3.

□

THEOREM 2.9. *Let $M$ be a co-Cohen-Macaulay Artinian $A$-module and let $I \subseteq A$ be an equimultiple ideal with $r(I, M) \leq 2$ and $(0 :_M I^2) + (0 :_M J) = (0 :_M JI)$ for any minimal reduction $J$ of $I$. Then*

$$\begin{aligned}
&\min\{\textit{width}\,(0 :_M I^2), \textit{width}\,(0 :_M I) - 1\} + \textit{cograde}_I(M) \\
&\leq \textit{width}\,(G(I, M)) \\
&\leq \min\{\textit{width}\,(0 :_M I^2), \textit{width}\,(0 :_M I)\} + \textit{cograde}_I(M)
\end{aligned}$$

*Proof.* We proceed by induction on cograde$_I(M) = h$. If $h = 0$, one has $(0 :_M I^n) = M$ for all $n \geq 3$ and so

$$G(I, M) = (0 :_M I) \oplus (0 :_M I^2/0 :_M I) \oplus (M/0 :_M I^2).$$

Therefore we have

$$\begin{aligned}
\text{width}\,(G(I, M)) = \min\{&\text{width}\,(0 :_M I), \text{width}\,(0 :_M I^2/0 :_M I), \\
&\text{width}\,(M/0 :_M I^2)\}.
\end{aligned}$$

Consider the following exact sequences

$$0 \longrightarrow 0 :_M I \longrightarrow M \longrightarrow M/0 :_M I \longrightarrow 0 \qquad (2.1)$$

$$0 \longrightarrow 0 :_M I^2 \longrightarrow M \longrightarrow M/0 :_M I^2 \longrightarrow 0 \tag{2.2}$$

$$0 \longrightarrow 0 :_M I \longrightarrow 0 :_M I^2 \longrightarrow 0 :_M I^2/0 :_M I \longrightarrow 0 \tag{2.3}$$

$$0 \longrightarrow 0 :_M I^2/0 :_M I \longrightarrow M/0 :_M I \longrightarrow M/0 :_M I^2 \longrightarrow 0. \tag{2.4}$$

We show that

$$\begin{aligned}&\min\{\text{width}\,(0 :_M I), \text{width}\,(0 :_M I^2/0 :_M I), \text{width}\,(M/0 :_M I^2)\}\\ &= \min\{\text{width}\,(0 :_M I^2), \text{width}\,(0 :_M I)\}.\end{aligned}$$

For this, we have the following three cases:
(i) If

$$\begin{aligned}\min\{\text{width}\,(0 :_M I), \text{width}\,(0 :_M I^2/0 :_M I), \text{width}\,(M/0 :_M I^2)\}\\ = \text{width}\,(0 :_M I),\end{aligned}$$

then by the exact sequence 2.3 and width-Lemma we have

$$\begin{aligned}\text{width}\,(0 :_M I^2) \geq \min\{\text{width}\,(0 :_M I), \text{width}\,(0 :_M I^2/0 :_M I)\}\\ = \text{width}\,(0 :_M I)\end{aligned}$$

and the assertion holds.
(ii) If

$$\begin{aligned}\min\{\text{width}\,(0 :_M I), \text{width}\,(0 :_M I^2/0 :_M I), \text{width}\,(M/0 :_M I^2)\}\\ = \text{width}\,(0 :_M I^2/0 :_M I),\end{aligned}$$

then $\text{width}\,(0 :_M I^2/0 :_M I) \leq \text{width}\,(0 :_M I)$. In fact we can assume that $\text{width}\,(0 :_M I^2/0 :_M I) < \text{width}\,(0 :_M I)$. If this is not hold, then $\text{width}\,(0 :_M I^2/0 :_M I) = \text{width}\,(0 :_M I)$ and by the previous case the assertion holds. Therefore by the exact sequence 2.3 and width-Lemma we have

$$\begin{aligned}\text{width}\,(0 :_M I^2) &\geq \min\{\text{width}\,(0 :_M I), \text{width}\,(0 :_M I^2/0 :_M I)\}\\ &= \text{width}\,(0 :_M I^2/0 :_M I)\\ &= \text{width}\,(0 :_M I^2).\end{aligned}$$

(iii) If

$$\begin{aligned}\min\{\text{width}\,(0 :_M I), \text{width}\,(0 :_M I^2/0 :_M I), \text{width}\,(M/0 :_M I^2)\}\\ = \text{width}\,(M/0 :_M I^2),\end{aligned}$$

then we can assume that

$$\text{width}\,(M/0 :_M I^2) < \min\{\text{width}\,(0 :_M I), \text{width}\,(0 :_M I^2/0 :_M I)\}.$$

On the other hand, by the previous case the assertion holds.

Therefore $\operatorname{width}(M/0 :_M I^2) < \operatorname{width}(M)$ and by width-Lemma $\operatorname{width}(0 :_M I^2) = \operatorname{width}(M/0 :_M I^2) - 1$ and this is contradiction by applying width-Lemma on the exact sequence 2.3. Thus we have

$$\operatorname{width}(G(I,M)) = \min\{\operatorname{width}(0 :_M I^2), \operatorname{width}(0 :_M I)\}.$$

Suppose that $h > 0$, and let $J = (x_1, x_2, ..., x_h)$ be a minimal reduction of $I$ relative to $M$ with $(0 :_M I^n) = (0 :_M JI^{n-1})$ for $n \geq 3$. Since $r(I,M) \leq 2$ and $(0 :_M I^2) + (0 :_M J) = (0 :_M JI)$, by Lemma 2.1, the sequence $x_1^*, ..., x_h^*$ is a cosequence with respect to $G(I,M)$. Then by [12, lemma 3.1]

$$0 :_{G(I,M)} (x_1^*, x_2^*, ...x_h^*) \cong G(I, (0 :_M (x_1, x_2, ..., x_h))$$

and

$$\operatorname{width}(G(I,M)) = \operatorname{width}(G(I, (0 :_M (x_1, x_2, ..., x_h))) + h.$$

Let $\overline{M} = 0 :_M (x_1, x_2, ..., x_h)$, then $\overline{M}$ is co-Cohen-Macaulay, $\operatorname{cograde}_I(\overline{M}) = s(I, \overline{M}) = 0$ and $r(I, \overline{M}) \leq 2$. Then by induction hypothesis we have

$$\operatorname{width}(G(I,M)) = \min\{\operatorname{width}(0 :_{\overline{M}} I^2), \operatorname{width}(0 :_{\overline{M}} I)\} + h.$$

By Lemma 2.3, if $\operatorname{width}(0 :_M I) > \operatorname{width}(0 :_M I^2)$, then

$$\operatorname{width}(G(I,M)) = \min\{\operatorname{width}(0 :_M I^2), \operatorname{width}(0 :_M I)\} + h$$

and if $\operatorname{width}(0 :_M I) \leq \operatorname{width}(0 :_M I^2)$, then

$$\operatorname{width}(G(I,M)) \geq \min\{\operatorname{width}(0 :_M I^2), \operatorname{width}(0 :_M I) - 1\} + h.$$

□

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