SEQUENTIALLY ALMOST COHEN-MACAULAY MODULES

SAMANEH TABEJAMAAT

Communicated by Lucian Beznea

Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring. In this paper, the notion of sequentially almost Cohen-Macaulay R-modules is introduced and some properties of these modules is investigated. Especially, permanence property of these modules with respect to $\mathfrak{m}$-adic completion, and flat base extension of R is described and also permanence property of some classes of these modules with respect to localization, and passing to non-zero divisor quotient is found. At the end, some examples of sequentially almost Cohen-Macaulay R-modules are given.

AMS 2020 Subject Classification: 13C14, 13D07, 13H10.

Key words: sequentially almost Cohen-Macaulay module, canonical module, Ext-functor.

1. INTRODUCTION

Throughout this paper, R denotes a commutative Noetherian ring with unit element, and M denotes a non-zero finitely generated R-module of dimension n.

The concept of sequentially Cohen-Macaulay modules was introduced by Stanley for finitely generated graded modules in [9]. Next, Herzog and Sbarra studied these modules in [4]. Later in the same as Stanely did, Cuong and Nhan in [3] introduced this concept for the local case instead of graded case, as following:

Definition 1.1. Let $(R, \mathfrak{m})$ be a local ring. M is called a sequentially Cohen-Macaulay R-module if there exists a filtration $(0) = M_0 \subset M_1 \subset ... \subset M_t = M$ of submodules of M such that the following statements hold.

(i) M_i/M_{i-1} is Cohen-Macaulay R-module for all $1 \leqslant i \leqslant t$.

(ii) $\dim M_1/M_0 < \dim M_2/M_1 < ... < \dim M_t/M_{t-1}$.

In this case, the filtration $(0) = M_0 \subset M_1 \subset ... \subset M_t = M$ is called Cohen-Macaulay filtration.

Also in [3] a definition of a dimension filtration is given as following:

Definition 1.2. A filtration $(0) = M_0 \subset M_1 \subset ... \subset M_s = M$ of submodules of M is called a dimension filtration if M_{i-1} is the largest submodule of M_i which has dimension strictly less than dimension of M_i for all $1 \leqslant i \leqslant s$.

[3, Lemma 4.4(a)] shows that the dimension filtration always exists and it is unique.

One year before introducing the sequentially Cohen-Macaulay modules by Cuong and Nhan, Kang introduced the concept of almost Cohen-Macaulay Modules in [7] as following:

Definition 1.3. M is called an almost Cohen-Macaulay R-module if $\operatorname{grade}(\mathfrak{p}, M) = \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Supp}_R(M)$.

R is called an almost Cohen-Macaulay ring if it is an almost Cohen-Macaulay R-module.

Clearly, every Cohen-Macaulay R-module is an almost Cohen-Macaulay R-module while the contrary is not necessarily true.

After Kang, several authors studied almost Cohen-Macaulay rings and modules (see for example [2],[5],[6],[8] and [10]). The following characterization of almost Cohen-Macaulay modules over local rings, which is used several times in this paper, is an immediate consequence of [7, Lemma 1.5 and Lemma 2.4].

LEMMA 1.4. *Let* $(R, \mathfrak{m})$ *be a local ring. Then* M *is an almost Cohen-Macaulay* R*-module if and only if* $\dim M \leqslant 1 + \operatorname{depth} M$.

From now on, R denotes a local ring with maximal ideal $\mathfrak{m}$.

In this paper, the concept of a sequentially almost Cohen-Macaulay module over a local ring is introduced and is shown that such a module may have more than one almost Cohen-Macaulay filtration in contrast with a sequentially Cohen-Macaulay module which has a uniqe Cohen-Macaulay filtration. Lemma 2.4 gives a sufficient condition under which an almost Cohen-Macaulay filtration is unique if it exists. Proposition 2.8 shows that $\mathfrak{m}$-adic completion of a sequentially almost Cohen-Macaulay module is also a sequentially almost Cohen-Macaulay module. Proposition 2.9 gives some condition under which localization of a sequentially almost Cohen-Macaulay module is sequentially almost Cohen-Macaulay module. Also, Proposition 2.11 gives some condition under which passing to non-zero divisor of a sequentially almost Cohen-Macaulay module is sequentially almost Cohen-Macaulay module. Theorem 2.12 checks the behaviour the almost Cohen-Macaulay filtration property under flat base extensions of the ground ring. The paper is closed by some examples of sequentially almost Cohen-Macaulay modules.

2. SEQUENTIALLY ALMOST COHEN-MACAULAY MODULES

Definition 2.1. A filtration $(0) = N_0 \subset N_1 \subset ... \subset N_t = M$ of submodules of M is called an almost Cohen-Macaulay filtration if the following statements hold:

(i) N_i/N_{i-1} is an almost Cohen-Macaulay R-module for all $1 \leqslant i \leqslant t$.

(ii) $\dim N_1/N_0 < \dim N_2/N_1 < ... < \dim N_t/N_{t-1}$.

Definition 2.2. M is called a sequentially almost Cohen-Macaulay R-module if there exists an almost Cohen-Macaulay filtration of M.

R is called a sequentially almost Cohen-Macaulay ring if it is a sequentially almost Cohen-Macaulay R-module.

Remark 2.3. In contrast with Cohen-Macaulay filtration, an almost Cohen-Macaulay filtration is not unique necessarily. For example if n = 1 and depth $M = 0$ then M is an almost Cohen-Macaulay R-module by Lemma 1.4. So $(0) = N_0 \subset N_1 = M$ is an almost Cohen-Macaulay filtration of M. Also $(0) = N_0 \subset \Gamma_{\mathfrak{m}}(M) \subset N_2 = M$ is an almost Cohen-Macaulay filtration of M as $\Gamma_{\mathfrak{m}}(M)$ is a Cohen-Macaulay R-module of dimension 0 and $M/\Gamma_{\mathfrak{m}}(M)$ is a Cohen-Macaulay R-module of dimension 1.

Lemma 2.5 gives a sufficient condition for the uniqueness of an almost Cohen-Macaulay fil- tration, if there is.

The following Remark which is used in the proof of Lemma 2.5 is an immediate consequence of [8, Lemma 2.1].

Remark 2.4. If M be an almost Cohen-Macaulay R-module, and N a submodule of M then $N = (0)$ or $\dim N = \dim M$ or $\dim N = \dim M - 1$.

LEMMA 2.5. *Suppose that* $(0) = M_0 \subset M_1 \subset ... \subset M_s = M$ *is a dimension filtration of M where* $\dim M_i < \dim M_{i+1} - 1$ *for all* $1 \leqslant i \leqslant s-1$. *If there is an almost Cohen-Macaulay filtration of M then it is unique and is exactly the dimension filtration of M.*

Proof. Let $(0) = M_0 \subset M_1 \subset ... \subset M_s = M$ be a dimension filtration of M where $\dim M_i < \dim M_{i+1} - 1$ for all $1 \leqslant i \leqslant s - 1$, and let $(0) = N_0 \subset N_1 \subset ... \subset N_t = M$ be an almost Cohen-Macaulay filtration of M. By Definition 1.2 one has $N_{t-1} \subseteq M_{s-1}$. Since by Definition 2.1, M/N_{t-1} is an almost Cohen-Macaulay R-module, and since $\dim M/N_{t-1} = n$ then from Remark 2.3 it follows that $M_{s-1}/N_{t-1} = (0)$ or $\dim M_{s-1}/N_{t-1} = n - 1$ or $\dim M_{s-1}/N_{t-1} = n$. Now, the condition $\dim M_{s-1} < n - 1$ yields that

$M_{s-1}/N_{t-1} = (0)$. Proceeding, one can see that $M_{s-i}/N_{t-i} = (0)$ for all $0 \leqslant i \leqslant s$ and consequently $s = t$, as required. □

Example 2.6. In the previous lemma, the assumption $\dim M_i < \dim M_{i+1} - 1$ for all $1 \leqslant i \leqslant s-1$, is necessary. Indeed, consider $R := \dfrac{\mathbb{K}[[x,y]]}{(x) \cap\ (x,y)^2}$ where $\mathbb{K}$ is a field. By Remark 2.3, $(0) \subset \Gamma_{\mathfrak{m}}(R) \subset R$ is a sequentially (almost) Cohen-Macaulay filtration of R, and thus from [3, Lemma 4.4(ii)] it follows that $(0) \subset \Gamma_{\mathfrak{m}}(R) \subset R$ is the dimension filtration of R. Clearly $\dim(0) = \dim \Gamma_{\mathfrak{m}}(R) - 1$ and $\dim \Gamma_{\mathfrak{m}}(R) = \dim R - 1$. But by Remark 2.3 the almost Cohen- Macaulay filtration of R is not unique.

Lemma 2.7. *Let M be a sequentially almost Cohen-Macaulay R-module with an almost Cohen-Macaulay filtration $(0) = N_0 \subset N_1 \subset ... \subset N_t = M$. Then for all $0 \leqslant i \leqslant t$, M/N_i is a sequentially almost Cohen-Macaulay R-module.*

Proof. $(0) = N_i/N_i \subset N_{i+1}/N_i \subset ... \subset N_t/N_i = M/N_i$ is an almost Cohen-Macaulay filtration of M/N_i. □

Proposition 2.8. *Let $\widehat{M}$ denotes the $\mathfrak{m}$-adic completion of M. If M is a sequentially almost Cohen-Macaulay R-module then $\widehat{M}$ is a sequentially almost Cohen-Macaulay $\widehat{R}$-module.*

Proof. Let $(0) = N_0 \subset N_1 \subset ... \subset N_t = M$ be an almost Cohen-Macaulay filtration of M. From [1, Corollary 2.1.8(a)] it follows that $(0) = \widehat{N_0} \subset \widehat{N_1} \subset ... \subset \widehat{N_t} = \widehat{M}$ where $\widehat{N_i}$ denotes $\mathfrak{m}$-adic completion of N_i for all $0 \leqslant i \leqslant t$, is an almost Cohen-Macaulay filtration of $\widehat{M}$. □

Proposition 2.9. *Let M be a sequentially almost Cohen-Macaulay R-module, and $\mathfrak{p} \in \mathrm{Supp}_R(M)$. If there exists an almost Cohen-Macaulay filtration $(0) = N_0 \subset N_1 \subset ... \subset N_t = M$ such that $\dim N_i/N_{i-1} < \dim N_{i+1}/N_i - 1$ for all $1 \leqslant i \leqslant t-1$ then $M_{\mathfrak{p}}$ is a sequentially almost Cohen-Macaulay $R_{\mathfrak{p}}$-module.*

Proof. Let $(0) = N_0 \subset N_1 \subset ... \subset N_t = M$ be an almost Cohen-Macaulay filtration of M such that $\dim N_i/N_{i-1} < \dim N_{i+1}/N_i$ for all $1 \leqslant i \leqslant t-1$. If $\mathfrak{p} \in \mathrm{Supp}_R(M/N_{t-1})$ then by [7, Lemma 2.6(1)], $M_{\mathfrak{p}}/(N_{t-1})_{\mathfrak{p}}$ is almost Cohen-Macaulay $R_{\mathfrak{p}}$-module. In this case set $t_* := t$, $t_* - 1 := t-1$. If $\mathfrak{p} \notin \mathrm{Supp}_R(M/N_{t-1})$ then $(N_{t-1})_{\mathfrak{p}} = M_{\mathfrak{p}}$. In this case consider the greatest index $i < t$ such that $\mathfrak{p} \in \mathrm{Supp}_R(N_i/N_{i-1})$. Now set $t_* := i$, $t_* - 1 := i-1$. Proceeding, one can find R-modules $N_{i_0}, N_{i_1}, ..., N_{i_{t_*}}$ from the family

$\{N_i\}_{i=0}^{t}$ such that every $(N_{i_l}/N_{i_{l-1}})_\mathfrak{p}$, $1 \leqslant l \leqslant t_*$, is almost Cohen-Macaulay. Using the dimension assumption yields that $\dim(N_{i_l}/N_{i_{l-1}})_\mathfrak{p} \leqslant \dim N_{i_l}/N_{i_{l-1}} - \dim R/\mathfrak{p} < \dim N_{i_{l+1}}/N_{i_l} - \dim R/\mathfrak{p} - 1$ for all $1 \leqslant l \leqslant t_* - 1$. Now since by [8, Theorem 2.3] for all $0 \leqslant l \leqslant t_* - 1$ one has $\dim N_{i_{l+1}}/N_{i_l} - \dim R/\mathfrak{p} \leqslant \dim(N_{i_{l+1}}/N_{i_l})_\mathfrak{p} + 1$ then $\dim(N_{i_l}/N_{i_{l-1}})_\mathfrak{p} < \dim(N_{i_{l+1}}/N_{i_l})_\mathfrak{p}$ for all $1 \leqslant l \leqslant t_* - 1$. Hence, $(0) = (N_{i_0})_\mathfrak{p} \subset (N_{i_1})_\mathfrak{p} \subset \ldots \subset (N_{i_{t_*}})_\mathfrak{p} = M_\mathfrak{p}$ is a an almost Cohen-Macaulay filtration of $M_\mathfrak{p}$. □

The following Lemma, which gives information on associated prime ideals of the components of dimension filtration and of some quotients of them, is needed in the proof of Theorem 2.11.

LEMMA 2.10. *Let* $(0) = M_0 \subset M_1 \subset \ldots \subset M_s = M$ *be the dimension filtration of* M *with* $\dim M_i = d_i$ *for all* $0 \leqslant i \leqslant s$. *Then*

(i) $\operatorname{Ass}_R(M_i) = \{\mathfrak{p} \in \operatorname{Ass}_R(M) | \dim R/\mathfrak{p} \leqslant d_i\}$,

(ii) $\operatorname{Ass}_R(M_i/M_{i-1}) = \{\mathfrak{p} \in \operatorname{Ass}_R(M) | \dim R/\mathfrak{p} = d_i\}$,

(iii) $\operatorname{Ass}_R(M/M_{i-1}) = \{\mathfrak{p} \in \operatorname{Ass}_R(M) | \dim R/\mathfrak{p} > d_i\}$.

Proof. It follows from [3, Lemma 4.4(i)]. □

PROPOSITION 2.11. *Let* $x \in \mathfrak{m} \setminus Z_R(M)$, *and* M *is a sequentially almost Cohen-Macaulay* R*-module such that its dimension filtration is an almost Cohen-Macaulay filtration of* M. *Then* M/xM *is a sequentially almost Cohen-Macaulay* R/xR*-module.*

Proof. Let M be a sequentially almost Cohen-Macaulay R-module such that $(0) = N_0 \subset N_1 \subset \ldots \subset N_t = M$ is the dimension filtration and an almost Cohen-Macaulay filtration of M. Assume that $x \in \mathfrak{m} \setminus Z_R(M)$. For all $1 \leqslant i \leqslant t$ set $\overline{N}_i := N_i/N_{i-1}$. From Lemma 2.10 and [7, Lemma 2.6(2)] it follows that $\overline{N}_i/x\overline{N}_i$ is almost Cohen-Macaulay for all $1 \leqslant i \leqslant t$. Now since by considering Lemma 2.10, for all $1 \leqslant i \leqslant t$ one has the equality $N_i \cap xM = xN_i$ then $\overline{N}_i/x\overline{N}_i \cong ((N_i, xM)/xM)/((N_{i-1}, xM)/xM)$. Set $M_i^* = (N_i, xM)/xM$ for all $0 \leqslant i \leqslant t$. $\{M_i^*\}_{i=0}^{t}$ is an almost Cohen-Macaulay filtration of M/xM. □

THEOREM 2.12. *Let* $(R, \mathfrak{m}) \longrightarrow (S, \mathfrak{n})$ *be a local flat homomorphism of Noetherian local rings, such that* $S/\mathfrak{m}S$ *is a Cohen-Macaulay ring. Let* M *be a finitely generated* R*-module. Then the following statements are equivalent.*

(i) M *has an almost Cohen-Macaulay filtration* $(0) = N_0 \subset N_1 \subset \ldots \subset N_t = M$.

(ii) *$M \otimes_R S$ has an almost Cohen-Macaulay filtration* $(0) = N_0 \otimes_R S \subset N_1 \otimes_R S \subset ... \subset N_t \otimes_R S = M \otimes_R S$.

Proof. (i)⇒(ii): By [1, Proposition 1.2.16(a) and Theorem A.11(b)] for every finitely generated R-module L one has $\dim_S L \otimes_R S = \dim L + \dim S/\mathfrak{m}S$ and $\operatorname{depth}_S L \otimes_R S = \operatorname{depth} L + \operatorname{depth} S/\mathfrak{m}S$. Hence by virtue Lemma 1.4 it follows that $(N_i \otimes_R S)/(N_{i-1} \otimes_R S)$ is almost Cohen-Macaulay S-module for all $1 \leqslant i \leqslant t$. Also, by definition it follows that $\dim(N_i \otimes_R S)/(N_{i-1} \otimes_R S) < \dim(N_{i+1} \otimes_R S)/(N_i \otimes_R S)$ for $1 \leqslant i \leqslant t-1$. Therefore $(0) = N_0 \otimes_R S \subset N_1 \otimes_R S \subset ... \subset N_t \otimes_R S = M \otimes_R S$ is an almost Cohen-Macaulay and non-Cohen-Macaulay filtration of $M \otimes_R S$.

(ii)⇒(i): The argument is the same method in (i)⇒(ii). □

Examples 2.13.

(i) Every almost Cohen-Macaulay R-module is a sequentially almost Cohen-Macaulay R-module with the almost Cohen-Macaulay filtration $(0) = M_0 \subset M_1 = M$. Also every sequentially Cohen-Macaulay R-module is a sequentially almost Cohen-Macaulay R-module.

(ii) If $\dim M \leqslant 2$ then M is a sequentially almost Cohen-Macaulay R-module because in the case where $\dim M - \operatorname{depth} M \leqslant 1$ the assertion is true by (i), and in the case where $\dim M - \operatorname{depth} M = 2$ there exists the almost Cohen-Macaulay filtration $(0) = M_0 \subsetneqq \Gamma_{\mathfrak{m}}(M) \subsetneqq M_2 = M$. (This example shows that the converse of (i) doesn't hold necessarily). So if C is a canonical module of R of dimension d, and M is an almost Cohen-Macaulay R-module of dimension 2 then $\operatorname{Ext}_R^{d-2}(M, C)$ is sequentially almost Cohen-Macaulay R-module (see [10, Theorem 3.6]).

(iii) Let M be an almost Cohen-Macaulay R-module. Then for every M-regular sequence $\underline{x}$ in $\mathfrak{m}$ of length greater than or equal to $\operatorname{depth} M - 1$ and for all $t \in \mathbb{N}$, $M/(\underline{x})^t M$ is a sequentially almost Cohen-Macaulay R-module. Indeed, let $\underline{x}$, in $\mathfrak{m}$, be an M-regular sequence of length $r \geqslant \operatorname{depth} M - 1$. Since by Lemma 1.4 one has $\dim M/(\underline{x})^t M = \dim M - r \leqslant 1 + \operatorname{depth} M - r \leqslant 2$ then the assertion follows by (ii).

(iv) If M is a sequentially almost Cohen-Macaulay R-module and R is an almost Cohen-Macaulay ring with $\dim R > \dim M$ then the idealization $R \ltimes M$ is a sequentially almost Cohen-Macaulay ring. To see this, let $(0) = N_0 \subset N_1 \subset ... N_t = M$ be an almost Cohen-Macaulay filtration of M. Then $(0) = (0) \ltimes N_0 \subset (0) \ltimes N_1 \subset ... \subset (0) \ltimes N_t = (0) \ltimes M \subset R \ltimes M$ is an almost Cohen-Macaulay filtration of $R \ltimes M$. It results from this fact that $(R \ltimes M) = ((0) \ltimes M) \cong R$ and $((0) \ltimes N_i) = ((0) \ltimes N_{i-1}) \cong N_i/N_{i-1}$ for all $1 \leqslant i \leqslant t$.

(v) Let $R[[x]]$ be a formal power series ring in variable x. Then $M[[x]]$ is sequentially almost Cohen-Macaulay $R[[x]]$-module $\Longleftrightarrow M$ is sequentially almost Cohen-Macaulay R-module. Let $R[[x]]$ be a formal power series ring in variable x. This fact follows [7, Lemma 2.6(2), *and* Lemma 2.7] as $M[[x]]/xM[[x]] \cong M$.

(vi) Let $N_1, N_2, ..., N_k$ be almost Cohen-Macaulay R-modules such that $\dim N_1 < \dim N_2 < ... < \dim N_k$, then $\bigoplus_{i=1}^{k} N_i$ is sequentially almost Cohen-Macaulay R-module with an almost Cohen-Macaulay filtration $(0) = N'_0 \subset N'_1 \subset ... \subset N'_k$ where $N'_0 = N_0$ and $N'_j = \oplus_{i=1}^{j} N_i$ for all $1 \leqslant j \leqslant k$.

Acknowledgments. The author would like to thank the referee for careful reading of the manuscript and helpful comments.

REFERENCES

[1] W. Bruns and J. Herzog, *Cohen-Macaulay rings.* Cambridge University Press, Cambridge, UK, 1998.

[2] L. Chu, Z. Tang, and H. Tang, *A note on almost Cohen-Macaulay modules.* J. Algebra Appl., **14** (2015), 1550136.

[3] N.T. Cuong and L.T. Nhan, *Pseudo Cohen-Macaulay and Pseudo generalized Cohen-Macaulay.* J. Algebra **267** (2003), 156–177.

[4] W. E. J. Herzog and E. Sbarra, *Sequentially Cohen-Macaulay modules and local cohomology, algebra, arithmetic and geometry*, Part I, II (Mumbai. 2000). Tata Inst. Fund. Res. Stud. Math. **16** (2002), 327–340.

[5] C. Ionescu, *More properties of almost Cohen-Macaulay rings.* J. Comm. Algebra **7** (2015), 363–372.

[6] C. Ionescu and S. Tabejamaat, *Tensor products and direct limits of almost Cohen-Macaulay modules.* J. Algebra Appl. **17** (2018), 1850221.

[7] M. Kang, *Almost Cohen-Macaulay modules.* Comm. Algebra **29** (2001), 781–787.

[8] A. Mafi and S. Tabejamaat, *Results on almost Cohen-Macaulay modules.* J. Algebraic Systems **3** (2016), 147–150.

[9] R.P. Stanley, *Combinatorics and commutative algebra.* Progress in Mathematics **41**, Birkhäuser, Boston, Inc. Boston, MA., 1983.

[10] S. Tabejamaat, A. Mafi, and K. Ahmadi Amoli, *Property of almost Cohen-Macaulay over extension modules.* Algebra Colloq. **24** (2017), 509–518.

Received May 28, 2018

Ayatollah Boroujerdi University
Department of Mathematics
Boroujerd, Iran
samanetabejamaat_golestan@yahoo.com