NEW MECHANISMS FOR HETEROCLINIC CYCLES IN SYSTEMS WITH $\mathbb{Z}_2^3$ SYMMETRY

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We analyze the generating mechanisms for heteroclinic cycles in $\mathbb{Z}_2^3$–equivariant ODEs, not involving Hopf bifurcations. Such cycles have been observed in different areas of physics see [1, 8, 12], as well in modelling the geomagnetic field of Earth [15], and as far as we know, there is no available theoretical data explaining these phenomena. We use singularity theory to study the equivalence in the group-symmetric context, as well as the recognition problem for the simplest bifurcation problems with this symmetry group. Singularity results highlight different mechanisms for the appearance of heteroclinic cycles, based on the transition between the bifurcating branches.

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1. INTRODUCTION

The interest on the behavior offered by $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 := \mathbb{Z}_2^3$–equivariant ODEs increased considerably over the last decade, due to its applications in physics [8, 1, 12], and modeling the geomagnetic field of Earth [15]. The model proposed by Alexandre Rodrigues in 2013 [15], represents an important advance in reproducing the fluctuations of the geomagnetic field. Its main achievement is the capability of predicting the observed heteroclinic fluctuations of the geomagnetic field. By considering the chaotic switching between heteroclinic networks in ODE systems with $\mathbb{Z}_2^3$ symmetry, this model generalizes and improves the popular models considered by Melbourne *et al.* in [11]. Another application consists of the quantization of the supersymmetric action of the supermembrane, restricted by a topological condition, on a particular $G2$ manifold. The development of Belhaj *et al.* in [1] represents a new kind of supersymmetric quantum consistent models with potentially interesting properties from a phenomenological point of view.

Different mechanisms producing the heteroclinic cycles have been described such as the interaction between Hopf modes [9] or the symmetry breaking [2, 13, 6].

The present manuscript can be viewed as linking and extending the theoretical framework of Ian Melbourne in [9], with the oscillatory phenomena modeled in [15]. More specifically, in our paper we do not appeal to the Hopf bifurcation as in [9] or chaotic switching as in [15] or the symmetry breaking as in [2, 13, 6], to explain the heteroclinic cycles observed in systems with $\mathbb{Z}_2^3$ symmetry. The mechanisms proved here are new.

We will show that the bifurcation analysis from the singularity and group theoretical points of view as well as the weak–coupling case can offer valuable insight on the mechanisms leading to heteroclinic cycles in $\mathbb{Z}_2^3$–equivariant systems, with no involvement of the Hopf bifurcation.

So far, the main reference on the heteroclinic cycles appearing in $\mathbb{Z}_2^3$–equivariant systems is Melbourne's work [9]. In this article the author analyzes the interaction of three Hopf modes to show that locally a bifurcation gives rise to heteroclinic cycles between three periodic solutions. More specifically, he considers a vector f field with an equilibrium and assumes that the Jacobian matrix of f about this equilibrium has three distinct complex conjugate pairs of eigenvalues on the imaginary axis. He obtains three branches of periodic solutions arising at the Hopf point from the steady–state equilibrium, as the parameters are varied. He uses Birkhoff normal form, to approximate f close to the bifurcation point by a vector field commuting with the symmetry group of the three-torus.

In this paper we do not assume the existence of three Hopf modes to study the heteroclinic cycles in $\mathbb{Z}_2^3$–equivariant systems. Instead, we perform a detailed analysis of the bifurcation problem with $\mathbb{Z}_2^3$ symmetry, with results from singularity theory. More specifically, after analyze the action of the group $\mathbb{Z}_2^3$ on $\mathbf{R}^3$, we study the restrictions on bifurcation problems g commuting with $\mathbb{Z}_2^3$ symmetry; we analyze the equivalence and the recognition problem for the simplest bifurcation problems with $\mathbb{Z}_2^3$–symmetry. This allows us to identify two new possible mechanisms for obtaining heteroclinic cycles in these systems. They are based on the smooth transition and jumping between the bifurcating solution branches, respectively. Moreover, we carry out the the linearization of the normal form with $\mathbb{Z}_2^3$–symmetry. This allow us to obtain the explicit form of all possible eigenvalues for this problem. Up to this point, we owe our results to the application of the analysis methods developed in [4] and [5].

By providing two additional mechanisms capable of generating heteroclinic cycles in $\mathbb{Z}_2^3$–equivariant systems, we believe our paper affords significant additional insight to the original knowledge of these phenomena due to Melbourne [9].

In Section 2 we discuss the bifurcation problems with $\mathbb{Z}_2^3$ symmetry from three angles: the group action on $\mathbf{R}^3$, the restrictions on bifurcation problems

g commuting with this group, and define the solution types. In Section 3 we use singularity theory: to study the equivalence in the $\mathbb{Z}_2^3$–symmetric context; to analyze the recognition problem for the simplest bifurcation problems with this symmetry group and to analyze the linearized stability of the normal form. This section involves multiple but straightforward computations, especially in the proof of Theorem 3. We gave all the computation details, but, because of the dimensions of the involved matrices, we preferred to analyze the outcome rather than filling many pages with unnecessary rows/columns. We conclude this section by identifying two important mechanisms for generating heteroclinic cycles with no need of invoking Hopf bifurcation. Finally, we give the linearization calculations with the consequent stability results of the bifurcating branches.

2. BIFURCATION PROBLEMS WITH $\mathbb{Z}_2^3$ SYMMETRY

In this section we discuss the following points:

(a) The action of the group $\mathbb{Z}_2^3$ on $\mathbf{R}^3$;

(b) Restrictions on bifurcation problems g commuting with $\mathbb{Z}_2^3$;

(c) Solution types of the equation $g = 0$.

2.1. Preliminary notations

The application of Singularity Theory to our bifurcation analysis requires the use of many concepts developed in [4]. To facilitate the reading of this paper, we give here a brief description of them adapted to our case; for more details, the reader is invited to visit the mentioned reference which constitutes the main guidance for this section of the paper. Let $\mathbf{x} = (x, y, z) \in \mathbf{R}^3$. For the clarity of the explanations, we will use the explicit expression of the above equation only when it is required by the situation. By

$$\mathscr{E}_{\mathbf{x},\lambda}$$

we denote the space of all functions in three state parameters and one bifurcation parameter (λ), that are defined and C^∞ on some neighborhood of the origin. A germ is an equivalence class in $\mathscr{E}_{\mathbf{x},\lambda}$. We denote by

$$\mathscr{E}_{\mathbf{x},\lambda}(\Gamma)$$

the ring of Γ–equivariant germs. That is, if f is a germ, then

$$f(\gamma \cdot \mathbf{x}) = \gamma \cdot f(\mathbf{x}), \forall \mathbf{x} \in \mathbf{R}^3, \forall \gamma \in \Gamma.$$

The module $\overrightarrow{\mathscr{E}}(\Gamma)$ over the ring $\mathscr{E}(\Gamma)$ is defined as in the following Theorem.

THEOREM 1 (Poénaru, 1976, [14]). *Let Γ be a compact Lie group and let $g_1, \ldots, g_r$ generate the module $\overrightarrow{\mathscr{P}}(\Gamma)$ of Γ–equivariant polynomials over the ring $\mathscr{P}(\Gamma)$. Then $g_1, \ldots, g_r$ generate the module $\overrightarrow{\mathscr{E}}(\Gamma)$ over the ring $\mathscr{E}(\Gamma)$.*

Moreover, as in [4], we have the following definition for $\overleftrightarrow{\mathscr{E}}_{\mathbf{x},\lambda}(\Gamma)$:

$$\overleftrightarrow{\mathscr{E}}_{\mathbf{x},\lambda}(\Gamma) = \{3 \times 3 \text{ matrix germs } S(\mathbf{x}, \lambda) : S(\gamma \cdot \mathbf{x}, \lambda) = \gamma \cdot S(\mathbf{x}, \lambda)\}.$$

We define

$$\overrightarrow{\mathscr{M}}_{\mathbf{x},\lambda}(\Gamma) = \left\{ g \in \overrightarrow{\mathscr{E}}_{\mathbf{x},\lambda}(\Gamma) : g(\mathbf{0}, 0) = 0 \right\};$$

that is, $\overrightarrow{\mathscr{M}}_{\mathbf{x},\lambda}(\Gamma)$ consists of Γ–equivariant mappings that vanish at the origin. Finally, we need to define the Γ–equivariant restricted tangent space $RT(h, \Gamma)$ of a Γ–equivariant bifurcation problem $h \in \overrightarrow{\mathscr{E}}(\Gamma)$. In order to do this, we have to give first the following definition.

Definition 1. Let $g, h : \mathbf{R}^3 \times \mathbf{R} \to \mathbf{R}^3,\ g, h \in \overrightarrow{\mathscr{E}}_{\mathbf{x},\lambda}(\Gamma)$ be a bifurcation problem with three state variables. Then g and h are equivalent if there exists an invertible change of coordinates $(\mathbf{x}, \lambda) \mapsto (Z(\mathbf{x}, \lambda), \Lambda(\lambda))$ and S is a 3×3 invertible matrix depending smoothly on $\mathbf{x}$, such that

$$g(\mathbf{x}, \lambda) = S(\mathbf{x}, \lambda) h(Z(\mathbf{x}, \lambda), \Lambda(\lambda)) \tag{1}$$

where the mapping $\Phi(\mathbf{x}, \lambda) = (Z(\mathbf{x}, \lambda), \Lambda(\lambda))$ is preserving the orientation in λ; in particular,

$$\begin{aligned} &Z(\mathbf{0}) = 0, \Lambda(0) = 0, \det(dZ)_{(\mathbf{0})} \neq 0, \Lambda'(0) > 0, (dZ)_{\mathbf{0},0} \in \mathscr{L}_\Gamma(V)^0 \\ &Z(\gamma \cdot \mathbf{x}, \lambda) = \gamma \cdot Z(\mathbf{x}, \lambda), S(\gamma \cdot \mathbf{x}, \lambda) = \gamma \cdot S(\mathbf{x}, \lambda), S(\mathbf{0}, 0). \end{aligned} \tag{2}$$

We call g and h strongly Γ–equivalent if $\Lambda(\lambda) \equiv \lambda$.

We define the Γ–equivariant restricted tangent space of g to be

$$RT(g, \Gamma) = \{S \cdot g + (dg)Z,\ Z(\mathbf{0}, 0) = 0\},$$

while $S(\mathbf{x}, \lambda)$ and Z satisfy (2). In all of these cases, of course $\Gamma = \mathbb{Z}_2^3$.

LEMMA 1 (Nakayama's Lemma, [4]). *Let $\mathscr{I}$ and $\mathscr{J}$ be ideals in $\mathscr{E}_n$, and assume that $\mathscr{I} = \langle p_1, \ldots, p_l \rangle$ is finitely generated. Then $\mathscr{I} \subset \mathscr{J}$ if and only if $\mathscr{I} \subset \mathscr{J} + \mathscr{M} \cdot \subset \mathscr{I}$.*

2.2. The action of $\mathbb{Z}_2^3$ on $\mathbf{R}^3$

The group $\mathbb{Z}_2^3$ has eight elements (κ, ζ, ξ) where $\kappa = \pm 1$, $\zeta = \pm 1$, $\xi = \pm 1$. The group element (κ, ζ, ξ) acts on the point $(x, y, z) \in \mathbf{R}^3$ by

$$(\kappa, \zeta, \xi) \cdot (x, y, z) = (\kappa x, \zeta y, \xi z).$$

We may think of the action of (κ, ζ, ξ) on $\mathbf{R}^3$ as a linear mapping; the matrix associated to the action (κ, ζ, ξ) is the diagonal matrix

$$\begin{bmatrix} \kappa & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \xi \end{bmatrix}. \tag{3}$$

The behavior of the action of $\mathbb{Z}_2^3$ on $\mathbf{R}^3$ is different at different points in $\mathbf{R}^3$. We describe these differences in two ways: through orbits and through isotropy subgroups.

The orbit of a point (x, y, z) under the action of $\mathbb{Z}_2^3$ is the set of points

$$\left\{(\kappa, \zeta, \xi) \cdot (x, y, z) : (\kappa, \zeta, \xi) \in \mathbb{Z}_2^3\right\}.$$

There are eight orbit types:

$$\begin{array}{l}
\text{(a) The origin, } (0,0,0), \\
\text{(b) Points on the x} - \text{axis, } (\pm\text{x}, 0, 0) \text{ with x} \neq 0, \\
\text{(c) Points on the y} - \text{axis, } (0, \pm\text{y}, 0) \text{ with y} \neq 0, \\
\text{(d) Points on the z} - \text{axis, } (0, 0, \pm\text{z}) \text{ with z} \neq 0, \\
\text{(e) Points on the plane x} = 0, \ (0, \pm\text{y}, \pm\text{z}) \text{ with y} \neq 0, \ \text{z} \neq 0 \\
\text{(f) Points on the plane y} = 0, \ (\pm\text{x}, 0, \pm\text{z}) \text{ with x} \neq 0, \ \text{z} \neq 0 \\
\text{(g) Points on the plane z} = 0, \ (\pm\text{x}, \pm\text{y}, 0) \text{ with x} \neq 0, \ \text{y} \neq 0 \\
\text{(h) Points off the axes, } (\pm\text{x}, \pm\text{y}, \pm\text{z}) \text{ with x} \neq 0, \ \text{y} \neq 0, \ \text{z} \neq 0.
\end{array} \tag{4}$$

We see that orbits have either 1, 2, 4 or 8 points, the origin being the unique one-orbit point. The isotropy subgroup of a point (x, y, z) is the set of symmetries preserving that point. In symbols, the isotropy subgroup of the point (x, y, z) is

$$\left\{(\kappa, \zeta, \xi) \in \mathbb{Z}_2^3 : (\kappa, \zeta, \xi) \cdot (x, y, z) = (x, y, z)\right\}.$$

It is easy to see that there are eight isotropy subgroups:

(a) $\mathbb{Z}_2^3$ corresponding to the origin,

(b) $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \zeta, \xi\}$ corresponding to $(x, 0, 0)$ with $x \neq 0$,

(c) $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{\kappa, 1, \xi\}$ corresponding to $(0, y, 0)$ with $y \neq 0$,

(d) $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{\kappa, \zeta, 1\}$ corresponding to $(0, 0, z)$ with $x \neq 0$,

(e) $\mathbb{Z}_2 = \{\kappa, 1, 1\}$ corresponding to $(0, y, z)$ with $y \neq 0, \ z \neq 0$,

(f) $\mathbb{Z}_2 = \{1, \zeta, 1\}$ corresponding to $(x, 0, z)$ with $x \neq 0, \ z \neq 0$,

(g) $\mathbb{Z}_2 = \{1, 1, \xi\}$ corresponding to $(x, y, 0)$ with $x \neq 0, \ y \neq 0$,

(h) $\mathbb{1} = \{1, 1, 1\}$ corresponding to (x, y, z) with $x \neq 0, \ y \neq 0, \ z \neq 0$.

2.3. The form of $\mathbb{Z}_2^3$–symmetric bifurcation problems

Let $g : \mathbf{R}^3 \times \mathbf{R} \to \mathbf{R}^3$ be a bifurcation problem with three state variables; that is, let g be C^∞ and satisfy

$$g(0,0,0,0) = 0, \quad (dg)_{(0,0,0,0)} = 0.$$

We say that the bifurcation problem g commutes with $\mathbb{Z}_2^3$ if

$$g((\kappa,\zeta,\xi)\cdot(x,y,z),\lambda) = (\kappa,\zeta,\xi)\cdot g(x,y,z,\lambda). \tag{5}$$

We will need the following result for the next lemma.

Lemma 2. *If $f \in \mathscr{E}_{\mathbf{x},\lambda}$ is even in $\mathbf{x}$, then f may be expressed as a smooth function of $\mathbf{x}^2$ and λ; in symbols,*

$$f(\mathbf{x},\lambda) = a(\mathbf{x}^2,\lambda).$$

Proof. See Lema VI, 2.1 page 248 in [4]. □

We can now state

Lemma 3. *Let us consider the $g : \mathbf{R}^3 \times \mathbf{R} \to \mathbf{R}$ bifurcation problem in three state variables commuting with the action of $\mathbb{Z}_2^3$. Then there exist smooth functions $p(u,v,w,\lambda)$, $q(u,v,w,\lambda)$, $r(u,v,w,\lambda)$ such that*

$$\begin{aligned} g(x,y,z,\lambda) &= (p(x^2,y^2,z^2,\lambda)x,\ q(x^2,y^2,z^2,\lambda)y,\ r(x^2,y^2,z^2,\lambda)z), \\ & p(0,0,0,0)=0, \quad q(0,0,0,0)=0, \quad r(0,0,0,0)=0. \end{aligned} \tag{6}$$

Proof. We write g in coordinates

$$g(x,y,z,\lambda) = (a(x,y,z,\lambda),\ b(x,y,z,\lambda),\ c(x,y,z,\lambda)). \tag{7}$$

Commutativity with equation (5) implies

$$\begin{aligned} & a(\kappa x,\zeta y,\xi z,\lambda) = \kappa a(x,y,z,\lambda), \quad b(\kappa x,\zeta y,\xi z,\lambda) = \\ & \qquad = \zeta b(x,y,z,\lambda), \quad c(\kappa x,\zeta y,\xi z,\lambda) = \xi c(x,y,z,\lambda). \end{aligned} \tag{8}$$

The action of ξ is defined by $(x,y,z) \to (x,y,-z)$. Now κ transforms z into $\bar{z}$, i.e. $(x,y,z) \to (x,-y,z)$ and the action of ζ is $+1$. When $\kappa = -1$, $\zeta = +1$, $\xi = +1$, equation (8) shows that a is odd in x while b and c are even in x, respectively and c is even in z. When $\kappa = +1$, $\zeta = -1$, $\xi = +1$, equation (8) shows that a is even in y, b is odd in y, while c is even in y and z.

Conversely, if $\kappa = -1$, $\zeta = +1$, $\xi = -1$, we get that a is odd in x while b is even in x and c is odd in x and z while when $\kappa = 1$, $\zeta = +1$, $\xi = -1$ we get that a is even in y, b is odd in y, while c is odd in y and z. It follows from the Taylor's theorem that we may factor these functions

$$
\begin{aligned}
&a(x,y,z,\lambda) = \bar{a}(x,y,z,\lambda)x, \quad b(x,y,z,\lambda) = \\
&\qquad = \bar{b}(x,y,z,\lambda)y, \quad c(x,y,z,\lambda) = \bar{c}(x,y,z,\lambda)z
\end{aligned}
\tag{9}
$$

where $\bar{a}$, $\bar{b}$ and $\bar{c}$ are even in x, y and z. Applying Lemma 2 first to x, then to y and finally to z we conclude that g has the desired form (6). The linear terms in g vanish. The only linear terms compatible with the symmetry are

$$(p(0,0,0,0)x,\ q(0,0,0,0)y,\ r(0,0,0,0)z);$$

thus, $p(0,0,0,0) = q(0,0,0,0) = r(0,0,0,0) = 0$. $\square$

2.4. **Solution types for g**

Consider solving the equation $g = 0$ when g has the form (6). There are eight solution types which occur according as the first, the second or the third factor in $p(x^2,y^2,z^2,\lambda)x$ vanishes, the first, the second or the third factor in $q(x^2,y^2,z^2,\lambda)y$ vanishes or the first, the second or the third factor in $r(x^2,y^2,z^2,\lambda)r$ vanishes. Specifically, we have the solution types

(a) x=y=z=0,

(b) $p(x^2,0,0,\lambda) = 0$, $y = z = 0$, $x \neq 0$,

(c) $q(0,y^2,0,\lambda) = 0$, $x = z = 0$, $y \neq 0$,

(d) $r(0,0,z^2,\lambda) = 0$, $x = y = 0$, $z \neq 0$,

(e) $p(x^2,y^2,0,\lambda) = 0$, $q(x^2,y^2,0,\lambda) = 0$, $z = 0$, $x \neq 0$ $y \neq 0$,

(f) $p(x^2,0,z^2,\lambda) = 0$, $r(x^2,0,z^2,\lambda) = 0$, $y = 0$, $x \neq 0$ $z \neq 0$,

(g) $q(0,y^2,z^2,\lambda) = 0$, $r(0,y^2,z^2,\lambda) = 0$, $x = 0$, $y \neq 0$ $z \neq 0$,

(h) $p(x^2,y^2,z^2,\lambda) = 0$, $q(x^2,y^2,z^2,\lambda) = 0$, $r(x^2,y^2,z^2,\lambda) = 0$, $x \neq 0$, $y \neq 0$, $z \neq 0$.

These solution types correspond exactly to the orbit types listed in (4) of the action of $\mathbb{Z}_2^3$ on $\mathbf{R}^3$. As in [4] we use the following terminology for these five types of solutions:

(a) trivial solutions,

(b) x–mode solutions,

(c) y–mode solutions,

(d) z–mode solutions,

(e) xy–mixed mode solutions,

(f) xz–mixed mode solutions,

(g) yz–mixed mode solutions,

(h) xyz–mixed mode solutions.

Each solution type has its own characteristic multiplicity. The x–mode, y–mode and z–mode solutions always come in pairs $(\pm x, 0, 0)$, $(0, \pm y, 0)$, $(0, 0, \pm z)$ and mixed mode solutions on the one hand four at the time, and they are $(\pm x, \pm y, 0)$, $(\pm x, 0, \pm z)$ and $(0, \pm y, \pm z)$, while $(\pm x, \pm y, \pm z)$ come eight at a time.

3. SINGULARITY RESULTS

We divide this section into three subsections:

(1) Equivalence in the $\mathbb{Z}_2^3$– symmetric context;

(2) The recognition problem for the simplest bifurcation problems with $\mathbb{Z}_2^3$– symmetry;

(3) Linearized stability and $\mathbb{Z}_2^3$ symmetry.

3.1. $\mathbb{Z}_2^3$–equivalence

The singularities we describe here have codimension eight and modality six. We have the following remarks regarding Definition 1.

Remark 1. Since S in (1) is invertible, we see that

$$\Phi(\{(z, \lambda) : g(\mathbf{x}, \lambda) = 0\}) = \{(\mathbf{x}, \lambda) : h(\mathbf{x}, \lambda) = 0\}\,. \tag{10}$$

Thus equivalences preserve bifurcation diagrams. They also preserve the orientation of the parameter λ.

Let $g, h : \mathbf{R}^3 \times \mathbf{R} \to \mathbf{R}^3$ be a bifurcation problem with three state variables commuting with the action of $\mathbb{Z}_2^3$. We say that g and h are $\mathbb{Z}_2^3$–equivalent if g and h are equivalent in the sense of the Definition 1, and in addition the equivalence preserves the symmetry. Recall that g and h are equivalent if there exists a 3×3 invertible matrix $S(x, y, z, \lambda)$ depending smoothly on x, y, z and λ and a diffeomorphism $\Phi(x, y, z, \lambda) = (Z(x, y, z, \lambda), \Lambda(\lambda))$ satisfying

$$g(x, y, z, \lambda) = S(x, y, z, \lambda) h(Z(x, y, z, \lambda), \Lambda(\lambda)) \tag{11}$$

such that

$$\Phi(0, 0, 0, 0) = (0, 0, 0, 0) \text{ and } \Lambda'(0) > 0. \tag{12}$$

We say that the equivalence S, Φ preserves the symmetry if

$$\begin{aligned} &(a) \quad Z(\kappa x, \zeta y, \xi z, \lambda) = (\kappa, \zeta, \xi) \cdot Z(x, y, z, \lambda), \\ &(b) \quad S(\kappa x, \zeta y, \xi z, \lambda) \begin{bmatrix} \kappa & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \xi \end{bmatrix} = \begin{bmatrix} \kappa & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \xi \end{bmatrix} S(x, y, z, \lambda). \end{aligned} \tag{13}$$

Condition (13) restricts the form of Z and S in the following ways. By applying Lemma 3 one shows that

$$Z(x, y, z, \lambda) = (a(x^2, y^2, z^2, \lambda)x, b(x^2, y^2, z^2, \lambda)y, c(x^2, y^2, z^2, \lambda)z). \tag{14}$$

Therefore

$$(dZ)_{(0,0,0,0)} = \begin{bmatrix} a(0,0,0,0) & 0 & 0 \\ 0 & b(0,0,0,0) & 0 \\ 0 & 0 & c(0,0,0,0) \end{bmatrix}; \tag{15}$$

i.e $(dZ)_{(0,0,0,0)}$ is diagonal. Dealing now with S, we write out entries of S as

$$\begin{bmatrix} S_1(x,y,z,\lambda) & S_2(x,y,z,\lambda) & S_3(x,y,z,\lambda) \\ S_4(x,y,z,\lambda) & S_5(x,y,z,\lambda) & S_6(x,y,z,\lambda) \\ S_7(x,y,z,\lambda) & S_8(x,y,z,\lambda) & S_9(x,y,z,\lambda) \end{bmatrix}. \tag{16}$$

A calculation using (13) (b) shows that S_1, S_5 and S_9 are even in x, y and z, while S_2, S_3, S_4, S_6, S_7 and S_8 are odd in x, y and z. Therefore, Lemma 2 together with Taylor's theorem implies that

(17)

$$S(x,y,z,\lambda) = \begin{bmatrix} d_1(x^2,y^2,z^2,\lambda) & d_2(x^2,y^2,z^2,\lambda)xyz & d_3(x^2,y^2,z^2,\lambda)xyz \\ d_4(x^2,y^2,z^2,\lambda)xyz & d_5(x^2,y^2,z^2,\lambda) & d_6(x^2,y^2,z^2,\lambda)xyz \\ d_7(x^2,y^2,z^2,\lambda)xyz & d_8(x^2,y^2,z^2,\lambda)xyz & d_9(x^2,y^2,z^2,\lambda) \end{bmatrix}.$$

In particular $S(0, 0, 0, 0)$ is diagonal and has the form

$$S(x,y,z,\lambda) = \begin{bmatrix} d_1(0,0,0,0) & 0 & 0 \\ 0 & d_5(0,0,0,0) & 0 \\ 0 & 0 & d_9(0,0,0,0) \end{bmatrix}. \tag{18}$$

In order to have $\mathbb{Z}_2^3$–equivalences preserved linear stability (which will be discussed in detail in the next section), we shall require that $\mathbb{Z}_2^3$–equivalences satisfy

$$
\begin{aligned}
&a(0,0,0,0) > 0, \quad b(0,0,0,0) > 0, \quad c(0,0,0,0) > 0,\\
&d_1(0,0,0,0) > 0, \quad d_5(0,0,0,0) > 0, \quad d_9(0,0,0,0) > 0.
\end{aligned} \tag{19}
$$

So far we have proved

Proposition 1. *Two bifurcation problems g and h, both commuting with the group $\mathbb{Z}_2^3$, are $\mathbb{Z}_2^3$–equivalent if there exists S and $\Phi = (Z, \Lambda)$ as above satisfying* (11), (12), (13) (16), (17) *and* (18).

3.2. **The recognition problem for the simplest examples**

Let g be a bifurcation problem with three state variables commuting with the group $\mathbb{Z}_2^3$. Thus g has the form (6). We split off the lowest terms in (6), i.e.

$$
g(x,y,z,\lambda) = k(x,y,z,\lambda) + \text{hot} \tag{20}
$$

where

$$
\begin{aligned}
k(x,y,z,\lambda) = \big(&Ax^3 + Bxy^2 + Cxz^2 + \alpha\lambda x, Dyx^2 +\\
&+ Ey^3 + Fyz^2 + \beta\lambda y, Gzx^2 + Hzy^2 + Iz^3 + \gamma\lambda z\big)\,.
\end{aligned} \tag{21}
$$

The higher-order terms in (20) include monomials $x^r y^s \lambda^t$, $x^r z^s \lambda^t$ and $y^r z^s \lambda^t$ satisfying at least one of the following conditions:

(a) $r + s \geqslant 5$,

(b) $t = 1$, $r + s \geqslant 3$,

(c) $t \geqslant 2$.

Before proceeding with our analysis, we shall introduce the notion of nondegenerate bifurcation problem in three state variables. The bifurcation is nondegenerate if it satisfies several inequalities which are invariants of equivalence. Our first nondegeneracy condition is

$$
g_\lambda(0,0,0) \neq 0. \tag{22}
$$

Let

$$
\bar{k}(x,y,z) = (Ax^2 + By^2 + Cz^2, Dx^2 + Ey^2 + Fz^2, Gx^2 + Hy^2 + Iz^2).
$$

Then our second nondegeneracy condition is

$$\text{minors}(\det(J(\bar{k}))) \neq 0, \tag{23}$$

where $J(\bar{k})$ is the Jacobian matrix of $\bar{k}$. This condition is to constrain the three roots of the determinant of the Jacobian matrix to be different. Taking into account the generic nondegeneracy conditions (23) and (as it will be seen in Theorem 4), the specific conditions dictated by the choice of the parameters in (38), we have the following definition.

Definition 2. The bifurcation problem g in (20)–(21) is nondegenerate if all the following conditions are satisfied:

$$A \neq 0,\ E \neq 0,\ I \neq 0,\ \alpha \neq 0,\ \beta \neq 0,\ \gamma \neq 0,\ B|\beta| \neq |E\alpha|,$$

$$D|\alpha| \neq |A\beta|,\ G|\alpha| \neq |A\gamma, C|\gamma| \neq |I\alpha|,\ F|\gamma| \neq |I\beta|, \tag{24}$$

$$H|\beta| \neq |E\gamma|,\ AE \neq BD,\ BF \neq CE,\ AF \neq CD.$$

Our main goal in this subsection is to state and prove the Theorem 4, in which we solve the recognition problem for nondegenerate bifurcation problems commuting with the $\mathbb{Z}_2^3$. However, before even stating it, we need a sequence of three preliminary results. The first couple of results recall the Theorem XIV 1.3 and the Proposition XIV 1.4, both from [4], whose proofs can be found in the same reference. The third result is one of our particular developments; while it stands as a result on its own, it also constitutes the second part of the proof of Theorem 4. We have:

THEOREM 2 (Theorem XIV 1.3, [4]). *Let Γ be a compact Lie group acting on V. Let $h \in \overrightarrow{\mathscr{E}}_{\mathbf{x},\lambda}(\Gamma)$ be a Γ–equivariant bifurcation problem and let p be any germ in $\overrightarrow{\mathscr{E}}_{\mathbf{x},\lambda}(\Gamma)$. Suppose that*

$$RT(h + tp, \Gamma) = RT(h, \Gamma)$$

for all $t \in [0, 1]$. Then $h + tp$ is strongly Γ–equivalent to h for all $t \in [0, 1]$.

PROPOSITION 2 (Proposition XIV 1.4, [4]). *Let Γ be a compact Lie group acting on V and let $h \in \overrightarrow{\mathscr{E}}_{\mathbf{x},\lambda}(\Gamma)$. Then $RT(h, \Gamma)$ is a finitely generated sub-module of $\overrightarrow{\mathscr{E}}_{\mathbf{x},\lambda}(\Gamma)$ over the ring $\mathscr{E}_{\mathbf{x},\lambda}(\Gamma)$. Moreover, $RT(h, \Gamma)$ is generated by*

$$S_1 h, \ldots, S_t h; (dh)(X_1), \ldots, (dh)(X_s)$$

where $S_1, \ldots, S_t$ generate $\overleftrightarrow{\mathscr{E}}_{x,\lambda}(\Gamma)$ and $X_1, \ldots, X_s$ generate $\overrightarrow{\mathscr{M}}_{\mathbf{x},\lambda}(\Gamma)$.

THEOREM 3. *Suppose that g is a nondegenerate $\mathbb{Z}_2^3$–equivariant bifurcation problem. Then g is strongly $\mathbb{Z}_2^3$–equivalent to h.*

Proof. To perform the proof, we choose to work with $\mathbb{Z}_2^3$–invariant coordinates. For this purpose, we need to find the generators for $RT(h, \mathbb{Z}_2^3)$. In Lemma 3 we have taken care of the generators for $\mathscr{E}(\mathbb{Z}_2^3)$ and $\overrightarrow{\mathscr{E}}(\mathbb{Z}_2^3)$. Once the generators for $RT(h, \mathbb{Z}_2^3)$ are computed, then by working in invariant coordinates, the action of $\mathbb{Z}_2^3$ is effectively annihilated. Our purpose is to show that under the assumption of nondegeneracy,

$$RT(h + t\varphi, \mathbb{Z}_2^3) = RT(h, \mathbb{Z}_2^3), \ \forall t \in \mathbf{R}. \tag{25}$$

Then we will apply Theorem 2 to complete the proof. We start by identifying (working with invariant coordinates), an isomorphism between $\overrightarrow{\mathscr{E}}_{x,y,z,\lambda}(\mathbb{Z}_2^3)$ and $\overrightarrow{\mathscr{E}}_{u,v,w,\lambda}(\mathbb{Z}_2^3)$, where $u = x^2, \ v = y^2$ and $w = z^2$. That is,

$$g(x, y, z, \lambda) = (p(x^2, y^2, z^2, \lambda)x, q(x^2, y^2, z^2, \lambda)y, r(x^2, y^2, z^2, \lambda)z).$$

We write g in the form $[p(u, v, w, \lambda), q(u, v, w, \lambda), r(u, v, w, \lambda)]$ and work in $\overrightarrow{\mathscr{E}}_{u,v,w,\lambda}(\mathbb{Z}_2^3)$ which is a module over $\mathscr{E}_{u,v,w,\lambda}(\mathbb{Z}_2^3)$. A short calculation shows that the nine generators of $\overleftrightarrow{\mathscr{E}}(\mathbb{Z}_2^3)$ are the 3×3 matrices $S_k, k = 1, \ldots, 9$, each with eight zero entries while the ninth entry is $s_{ij} = 1$ if $i = j$, $s_{ij} = s_{ji}$ if $i \neq j$ and $s_{12} = xy$, $s_{13} = xz$, $s_{23} = yz$. Moreover, one observes that $RT(g, \mathbb{Z}_2^3)$ can be viewed as a submodule of $\overrightarrow{\mathscr{E}}_{u,v,w,\lambda}(\mathbb{Z}_2^3)$, which has the following twelve generators:

$$\begin{gathered}[p, 0, 0], \ [0, q, 0], \ [0, 0, r], \ [qv, 0, 0], \ [rw, 0, 0], \ [0, pu, 0], \ [0, rw, 0], \\ [0, 0, pu], \ [0, 0, qv], \ [up_u, uq_u, ur_u], \ [vp_v, vq_v, vr_v], \ [wp_w, wq_w, wr_w].\end{gathered} \tag{26}$$

We need to show that

$$\mathscr{M}^2_{u,v,w,\lambda} \overrightarrow{\mathscr{E}}_{u,v,w,\lambda} \subset RT(h + t\varphi, \mathbb{Z}_2^3). \tag{27}$$

In order to do this, let $\mathscr{I} \subset RT(g, \mathbb{Z}_2^3)$ be the submodule with the twenty–seven generators

$$\begin{gathered}\nu[p, 0, 0], \ \nu[0, q, 0], \ \nu[0, 0, r], \ \nu[qv, 0, 0], \ \nu[rw, 0, 0], \\ \nu[0, pu, 0], \ \nu[0, rw, 0], \ \nu[0, 0, pu], \ \nu[0, 0, qv], \ \nu[up_u, uq_u, ur_u], \\ \nu[vp_v, vq_v, vr_v], \ \nu[wp_w, wq_w, wr_w].\end{gathered} \tag{28}$$

where $\nu = u, v, w$ or λ and $g = h + t\varphi$. We claim that

$$\mathscr{M}^2_{u,v,w,\lambda} \overrightarrow{\mathscr{E}}_{u,v,w,\lambda} = \mathscr{I}. \tag{29}$$

If (29) is true then (27) is also true. In addition, if (29) is true then

$$\text{(a)} \quad \mathrm{RT}(\mathrm{h} + \mathrm{t}\varphi, \mathbb{Z}_2^3) = \mathscr{M}^2_{\mathrm{u,v,w},\lambda} \overrightarrow{\mathscr{E}}_{\mathrm{u,v,w},\lambda} + \mathrm{W}, \tag{30}$$

where

$$(31)\qquad \begin{aligned}(b)\quad \mathrm{W} = \mathbf{R}\,\{&[\mathrm{p},0,0],[0,\mathrm{q},0],[0,0,\mathrm{r}],\\ &u[p_u,q_u,r_u],v[p_v,q_v,r_v],w[p_w,q_w,r_w]\}\,.\end{aligned}$$

We compute now the elements composing the basis of W modulo terms in $\mathscr{I} = \mathscr{M}^2_{u,v,w,\lambda}\overrightarrow{\mathscr{E}}_{u,v,w,\lambda}$, that is, the terms that are quadratic in u, v, w, λ.

$$(32)\qquad \begin{aligned}
&(a)\quad [\mathrm{p},0,0] \equiv [\mathrm{Au}+\mathrm{Bv}+\mathrm{Cw}+\alpha\lambda,0,0] \quad (\mathrm{mod}\mathscr{I})\\
&(b)\quad [0,\mathrm{q},0] \equiv [0,\mathrm{Cu}+\mathrm{Dv}+\mathrm{Fw}+\beta\lambda,0] \quad (\mathrm{mod}\mathscr{I})\\
&(c)\quad [0,0,r] \equiv [0,0,Gu+Hv+Iw+\gamma\lambda] \quad (\mathrm{mod}\mathscr{I})\\
&(d)\quad \mathrm{u}[\mathrm{p_u},\mathrm{q_u},\mathrm{r_u}] \equiv [\mathrm{Au},\mathrm{Cu},\mathrm{Gu}] \quad (\mathrm{mod}\mathscr{I})\\
&(e)\quad \mathrm{v}[\mathrm{p_v},\mathrm{q_v},\mathrm{r_v}] \equiv [\mathrm{Bv},\mathrm{Dv},\mathrm{Hv}] \quad (\mathrm{mod}\mathscr{I})\\
&(f)\quad \mathrm{w}[\mathrm{p_w},\mathrm{q_w},\mathrm{r_w}] \equiv [\mathrm{Cw},\mathrm{Fw},\mathrm{Iw}] \quad (\mathrm{mod}\mathscr{I}).
\end{aligned}$$

From (31) and (32) it follows that

$$(33)\qquad \begin{aligned}&RT(h+t\varphi,\mathbb{Z}_2^3) =\\
&\mathscr{M}^2_{u,v,w,\lambda}\overrightarrow{\mathscr{E}}_{u,v,w,\lambda} \oplus \mathbf{R}\,\{[Au+Bv+Cw+\alpha\lambda,0,0],\\
&[0,Cu+Dv+Fw+\beta\lambda,0],[0,0,Gu+Hv+Iw+\gamma\lambda],\\
&[Au,Cu,Gu],[Bv,Dv,Hv],[Cw,Fw,Iw]\}\,.\end{aligned}$$

From (33) we conclude that $R(h+t\varphi,\mathbb{Z}_2^3)$ is independent of $t\varphi$, determining (25). But the proof of (25) is not complete yet. To achieve it, we have to determine (29). For this purpose, we will make use of the nondegeneracy of h. We know that all the generators of $\mathscr{I}$ in (28) are composed of quadratic or higher order terms in $u,\ v,\ w$ or λ. Therefore, the following inclusion happens:

$$\mathscr{I} \subset \mathscr{M}^2_{u,v,w,\lambda}\overrightarrow{\mathscr{E}}_{u,v,w,\lambda}.$$

To prove (29) we must show that the inverse inclusion is also true, i.e.

$$\mathscr{M}^2_{u,v,w,\lambda}\overrightarrow{\mathscr{E}}_{u,v,w,\lambda} \subset \mathscr{I}\,.$$

From Nakayama's Lemma 1, the above inclusion is true provided

$$(34)\qquad \mathscr{M}^2_{u,v,w,\lambda}\overrightarrow{\mathscr{E}}_{u,v,w,\lambda} \subset \mathscr{I} + \mathscr{M}^3_{u,v,w,\lambda}\overrightarrow{\mathscr{E}}_{u,v,w,\lambda}.$$

By inspection we see that $t\varphi$ consists of terms of quadratic or higher order in $u,\ v,\ w$ or λ. Hence, $t\varphi$ enters the generators of $\mathscr{I}$ in (28) only through cubic or higher order terms in $u,\ v,\ w$ or λ. Therefore, when checking (34) we can assume $t\varphi \equiv 0$. For the rest of the proof we consider the thirty generators of the module $\mathscr{M}^2_{u,v,w,\lambda}\overrightarrow{\mathscr{E}}_{u,v,w,\lambda}$; they are of the form

$$\begin{aligned}&[i,0,0],\ [0,i,0],\ [0,0,i],\\ &\qquad\qquad \text{where } i=\{u^2,\ v^2,\ w^2,\ \lambda^2,\ uv,\ uw,\ u\lambda,\ vw,\ v\lambda,\ w\lambda\}.\end{aligned} \tag{35}$$

Moreover, we want to express these thirty generators of $\mathscr{M}^2_{u,v,w,\lambda}\overrightarrow{\mathscr{E}}_{u,v,w,\lambda}$ in terms of the twenty–seven generators of $\mathscr{I}$ in (28). Since $t\varphi \equiv 0$, we can write

$$\begin{aligned}&\text{(a)}\qquad \mathrm{p} = \mathrm{Au}+\mathrm{Bv}+\mathrm{Cw}+\alpha\lambda,\\ &\text{(b)}\qquad \mathrm{q} = \mathrm{Du}+\mathrm{Ev}+\mathrm{Fw}+\beta\lambda,\\ &\text{(c)}\qquad \mathrm{r} = \mathrm{Gu}+\mathrm{Hv}+\mathrm{Iw}+\gamma\lambda.\end{aligned} \tag{36}$$

This yields a 30×27 matrix, which we call M. The idea is that if we show that the rank of this matrix is 27, then using basic algebra we can affirm that each generator of $\mathscr{I}$ in (28) can be written in terms of the generators in (35), hence (34) will follow. Now the size of the matrix M makes its explicit form impossible to be written in this paper. However, based on the nondegeneracy conditions (24), we will show that certain number of columns/rows can be removed, so in the end we will have showed that the rank of this matrix is 27, proving (34). We begin by taking into account the nondegeneracy conditions $\alpha\neq 0,\ \beta\neq 0,\ \gamma\neq 0,\ AE\neq BD,\ BF\neq CE,\ AF\neq CD$. This way we can remove from the matrix M the 12 columns $\lambda[p,0,0],\ \lambda[0,q,0],\ \lambda[0,0,r],\ i[p_u,q_u,r_u],\ i[p_v,q_v,r_v],\ i[p_w,q_w,r_w]$ where $i = uv,\ uw,\ vw$, and 12 rows $[\lambda^2,0,0],\ [0,\lambda^2,0],\ [0,0,\lambda^2],\ [i,0,0],\ [0,i,0],\ [0,0,i]$ where $i = uv,\ uw,\ vw$. This way we obtain a 18×15 matrix whose rank is precisely 15.

Next we use nondegeneracy assumptions $B|\beta|\neq|E\alpha|,\ D|\alpha|\neq|A\beta|,\ G|\alpha|\neq|A\gamma|, C|\gamma|\neq|I\alpha|,\ F|\gamma|\neq|I\beta|,\ H|\beta|\neq|E\gamma|$, to remove the 6 rows $[w^2,0,0],\ [0,v^2,0],\ [0,0,u^2],\ [\lambda w,0,0],\ [0,\lambda v,0],\ [0,0,\lambda u]$ and the 6 columns $w[p,0,0],\ v[0,q,0],\ u[0,0,r],\ [wr,0,0],\ [0,vq,0],\ [0,0,up]$. This yields a 9×6 matrix; we finally use the remaining nondegeneracy conditions $A\neq 0,\ E\neq 0,\ I\neq 0$ (see (24)) to show that this matrix does have rank 6. Therefore, the original matrix M has rank 27, and we have proved (34). $\square$

THEOREM 4. *Let $g:\mathbf{R}^3\times\mathbf{R}\to\mathbf{R}^3$ be a bifurcation problem in three state variables commuting with the group $\mathbb{Z}_2^3$ and satisfying the nondegeneracy* *conditions* (24). *Then* g *is* $\mathbb{Z}_2^3$*–equivalent to*

$$\begin{aligned} h(x,y,z,\lambda) = (\varepsilon_1 x^3 + m_1 xy^2 + n_1 xz^2 + \varepsilon_2 \lambda x, \varepsilon_3 y^3 + m_2 yx^2 + \\ + n_2 yz^2 + \varepsilon_4 \lambda y, \varepsilon_5 z^3 + m_3 zx^2 + n_3 zy^2 + \varepsilon_6 \lambda z) \end{aligned} \tag{37}$$

where

$$\begin{gathered} \varepsilon_1 = \operatorname{sgn}(A),\ \varepsilon_3 = \operatorname{sgn}(E),\ \varepsilon_5 = \operatorname{sgn}(I),\ \varepsilon_2 = \operatorname{sgn}(\alpha), \\ \varepsilon_4 = \operatorname{sgn}(\beta),\ \varepsilon_6 = \operatorname{sgn}(\gamma), \frac{B|\beta|}{|E\alpha|},\ m_2 = \frac{D|\alpha|}{|A\beta|}, \\ m_1 = \ m_3 = \frac{G|\alpha|}{|A\gamma|},\ n_1 = \frac{C|\gamma|}{|I\alpha|},\ n_2 = \frac{F|\gamma|}{|I\beta|},\ n_3 = \frac{H|\beta|}{|E\gamma|}. \end{gathered} \tag{38}$$

Moreover,

$$\begin{gathered} m_1 \neq \varepsilon_2\varepsilon_3\varepsilon_4,\ m_2 \neq \varepsilon_1\varepsilon_2\varepsilon_4,\ m_3 \neq \varepsilon_1\varepsilon_2\varepsilon_6,\ n_1 \neq \varepsilon_2\varepsilon_5\varepsilon_6, \\ n_2 \neq \varepsilon_4\varepsilon_5\varepsilon_6,\ n_3 \neq \varepsilon_3\varepsilon_4\varepsilon_5,\ m_1 m_2 \neq \varepsilon_1\varepsilon_3,\ m_1 n_2 \neq \varepsilon_3 n_1. \end{gathered} \tag{39}$$

Remark 2.

1. The normal form h in (37) depends on the six parameters $m_i, n_i, i = 1,2,3$ satisfying the nondegeneracy conditions (39). These are the six modal parameters promised at the very beginning of the Subsection 3.1.

2. The proof of Theorem 4 divides into two parts. In the first part, we use the linear $\mathbb{Z}_2^3$–equivalences to transform k to the normal form h. In the second part, we show that the higher–order terms can be annihilated by a nonlinear $\mathbb{Z}_2^3$–equivalence. This second part actually consists entirely on the proof of Theorem 3.

Proof. The most general linear $\mathbb{Z}_2^3$–equivalence is given by

$$Z(x,y,z,\lambda) = (ax, by, cz), \quad \Lambda(\lambda) = \sigma\lambda, \quad S(x,y,z,\lambda) = \begin{bmatrix} d & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & f \end{bmatrix},$$

where a, b, c, d, e and f are positive constants. Letting this equivalence act on $k(x,y,z,\lambda)$, which is given by (21), we find

$$\begin{aligned} &\begin{bmatrix} d & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & f \end{bmatrix} k(ax, by, cz, \sigma\lambda) = \\ &\qquad = \begin{bmatrix} Ada^3x^3 + Bdab^2xy^2 + Cdac^2xz^2 + d\alpha\lambda\sigma ax \\ Dea^2byx^2 + Eeb^3y^3 + Febc^2yz^2 + e\beta\lambda\sigma by \\ Gfa^2czx^2 + Hfb^2czy^2 + Ifc^3z^3 + f\gamma\lambda\sigma cz \end{bmatrix}. \end{aligned} \tag{40}$$

To obtain the normal form (37) we need

$$|A|da^3 = 1,\ da\sigma|\alpha| = 1,\ |E|eb^3 = 1, \tag{41}$$
$$eb\sigma|\beta| = 1,\ |I|fc^3 = 1,\ fc\sigma|\gamma| = 1.$$

We solve equations (41) to obtain

$$d = \frac{1}{a^3|A|},\ e = \frac{1}{b^3|E|},\ f = \frac{1}{c^3|I|},\ \sigma = \frac{a^2|A|}{|\alpha|}, \tag{42}$$
$$\frac{a}{b} = \sqrt{\left|\frac{E\alpha}{A\beta}\right|},\ \frac{a}{c} = \sqrt{\left|\frac{I\alpha}{C\gamma}\right|},\ \frac{b}{c} = \sqrt{\left|\frac{I\beta}{E\gamma}\right|}.$$

Substitution of (42) into the right–hand side of (40) yields the normal form (37) with $m_1,\ m_2,\ m_3,\ n_1,\ n_2$ and n_3 given in (38). Then we use Theorem 3 to complete the proof. $\square$

The preceding analysis of the $\mathbb{Z}_2^3$–equivariant bifurcation problem yields an example, namely, following form of equation (38)

$$G(x, y, z, \lambda) = \begin{pmatrix} \varepsilon_1 x^3 + m_1 xy^2 + n_1 xz^2 - \lambda x \\ \varepsilon_3 y^3 + m_2 yx^2 + n_2 yz^2 - \lambda y \\ \varepsilon_5 z^3 + m_3 zx^2 + n_3 zy^2 - \lambda z \end{pmatrix}. \tag{43}$$

We will state the next theorem whose proof is identical to the proof of Theorem 6.8 of [3].

Theorem 5. *Let $H(x, y, z, \lambda)$ be a bifurcation problem with symmetry group $\Gamma = \mathbb{Z}_2^3$. Suppose that H is a small perturbation of a non-degenerate problem (43) with modal parameters $m_{1_0}, m_{2_0}, m_{3_0}, n_{1_0}, n_{2_0}, n_{3_0}$. Then H is Γ–equivalent to*

$$F(x, y, z, \lambda) = \begin{pmatrix} \varepsilon_1 x^3 + m_1 xy^2 + n_1 xz^2 - \lambda x \\ \varepsilon_3 y^3 + m_2 yx^2 + n_2 yz^2 - (\varepsilon_2 + \lambda) y \\ \varepsilon_5 z^3 + m_3 zx^2 + n_3 zy^2 - (\varepsilon_2 + \varepsilon_6 + \lambda) z \end{pmatrix}, \tag{44}$$

where $(m_1, m_2, m_3, n_1, n_2, n_3, \lambda)$ is near $(m_{1_0}, m_{2_0}, m_{3_0}, n_{1_0}, n_{2_0}, n_{3_0}, 0)$.

The qualitative bifurcation diagrams illustrating four mode jumping possibilities are shown in Figure 1. To explain the derivation of the Figure 1, remark that setting (44) equal to zero yields the equations

$$
(45)\quad
\begin{aligned}
&(a)\ \ \mathrm{x}=0;\ \mathrm{y}=0;\ \mathrm{z}=0,\\
&(b)\ \ \mathrm{x}^2=\frac{\lambda}{\varepsilon_1};\ \mathrm{y}=0;\ \mathrm{z}=0,\\
&(c)\ \ \mathrm{x}=0;\ \mathrm{y}^2=\frac{\varepsilon_2+\lambda}{\varepsilon_3};\ \mathrm{z}=0,\\
&(d)\ \ \mathrm{x}=0;\ \mathrm{y}=0;\ \mathrm{z}^2=\frac{\varepsilon_2+\varepsilon_6+\lambda}{\varepsilon_5},\\
&(e)\ \ \varepsilon_1\mathrm{x}^2+\mathrm{m}_1\mathrm{y}^2=\lambda;\ \varepsilon_3\mathrm{y}^2+\mathrm{m}_2\mathrm{x}^2=\varepsilon_2+\lambda;\ \mathrm{z}=0,\\
&(f)\ \ \varepsilon_1\mathrm{x}^2+\mathrm{n}_1\mathrm{z}^2=\lambda;\ \mathrm{y}=0;\ \varepsilon_5\mathrm{z}^2+\mathrm{m}_3\mathrm{x}^2=\varepsilon_2+\varepsilon_6+\lambda,\\
&(g)\ \ \mathrm{x}=0;\ \varepsilon_3\mathrm{y}^2+\mathrm{n}_2\mathrm{z}^2=\varepsilon_2+\lambda;\ \varepsilon_5\mathrm{z}^2+\mathrm{n}_3\mathrm{y}^2=\varepsilon_2+\varepsilon_6+\lambda,\\
&(h)\ \ \begin{cases}\varepsilon_1x^2+m_1y^2+n_1z^2=\lambda;\ \varepsilon_3y^2+m_2x^2+n_2z^2=\varepsilon_2+\lambda;\\ \varepsilon_5z^2+m_3x^2+n_3y^2=\varepsilon_2+\varepsilon_6+\lambda.\end{cases}
\end{aligned}
$$

The first seven equations in (45) have real solutions; the last one can have periodic solutions. To avoid too complicated bifurcation diagrams we resume the possibility of jumping to the cases $(a)-(f)$. Computation of the conditions imposed in the parameter space in the caption of Figure 1 are easily obtained from the seven equation (45) and their explicit derivation is left as an exercise to the reader. It is important to remark that when

$$n_2,\varepsilon_2,\varepsilon_6<0,\ n_3,\varepsilon_1,\varepsilon_3,\varepsilon_4,\varepsilon_5>0,\ n_3<\varepsilon_3,\ \varepsilon_3+\varepsilon_6>n_3\ m_2,m_3<1,$$

a quasi-static variation of λ produces a smooth transition between the bifurcating branches in Figure 1, right, and a necessity for jumping between the branches in Figure 1, left.

3.3. Linearized stability and $\mathbb{Z}_2^3$ symmetry

Let $g:\mathbf{R}^3\times\mathbf{R}\to\mathbf{R}^3$ be a bifurcation problem commuting with the group $\mathbb{Z}_2^3$. We call a solution (x,y,z,λ) of the equation $g(x,y,z,\lambda)=0$ linearly stable if all three eigenvalues of dg at (x,y,z,λ) have positive linear part; unstable

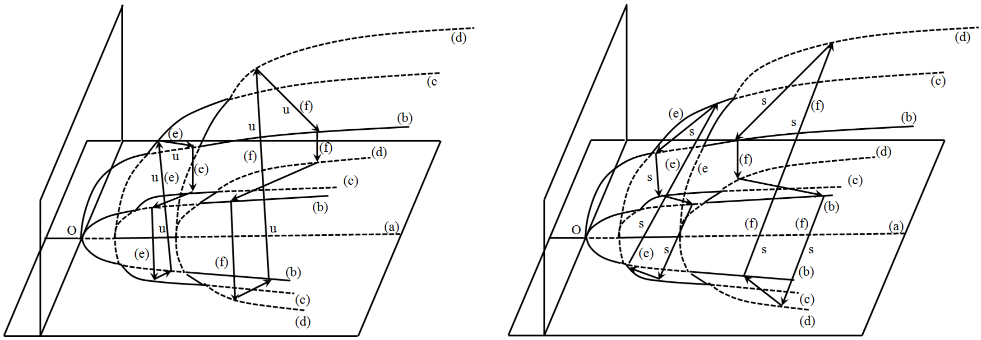

Figure 1 – Schematic representation of a structurally stable/unstable heteroclinic cycles derived from singularity results of the bifurcation problem with $\mathbb{Z}_2^3$ symmetry. Arrows are used to better show the jumps; they have no direction meanings. Solid lines represent stable the branches while dotted lines, the unstable ones. Parameters: $n_2, \varepsilon_2, \varepsilon_6 < 0$, $n_3, \varepsilon_1, \varepsilon_3, \varepsilon_4, \varepsilon_5 > 0$, $n_3 < \varepsilon_3$, $\varepsilon_3 + \varepsilon_6 > n_3$ and $m_2, m_3 < 1$.
Left figure: $\varepsilon_3\varepsilon_5 + n_2 n_3 < 0, m_1 m_2 < \varepsilon_1\varepsilon_3, n_1 m_3 < \varepsilon_1\varepsilon_5$.
Right figure: $\varepsilon_3\varepsilon_5 + n_2 n_3 > 0, m_1 m_2 > \varepsilon_1\varepsilon_3, n_1 m_3 > \varepsilon_1\varepsilon_5$.

if at least one of them has negative real part. We begin by calculating the eigenvalues of dg.

Since g has the form of equation (6) which we recall here

$$g(x, y, z, \lambda) = (p(u, v, w, \lambda)x,\ q(u, v, w, \lambda)y,\quad r(u, u, w, \lambda)z),$$

$$u = x^2,\quad v = y^2,\quad w = z^2,\quad p(0, 0, 0, 0) = 0, \tag{46}$$

$$q(0, 0, 0, 0) = 0, \quad r(0, 0, 0, 0) = 0.$$

The Jacobian matrix is then

$$dg = \begin{bmatrix} p + 2up_u & 2p_v xy & 2p_w xz \\ 2q_u xy & q + 2vq_v & 2q_w yz \\ 2r_u xz & 2r_v yz & r + 2wr_w \end{bmatrix}. \tag{47}$$

Let (x, y, z, λ) be a solution to $g = 0$. We find the following mode solutions:

(a) Trivial solution: when $x = y = z = 0$;

(b) x–mode solution: $p(x^2, 0, 0, \lambda) = 0,\ y = z = 0, x \neq 0$;

(c) y–mode solution: $q(0, y^2, 0, \lambda) = 0,\ x = z = 0, y \neq 0$;

(d) z–mode solution: $r(0, 0, z^2, \lambda) = 0,\ x = y = 0, z \neq 0$;

(e) xy–mode solution: $p(x^2, y^2, 0, \lambda) = q(x^2, y^2, 0, \lambda) = 0,\ z = 0,\ x \neq 0,\ y \neq 0$;

(f) xz–mode solution: $p(x^2, 0, z^2, \lambda) = r(x^2, 0, z^2, \lambda) = 0,\ y = 0,\ x \neq 0,\ z \neq 0$;

(g) yz–mode solution: $q(0, y^2, z^2, \lambda) = r(0, y^2, z^2, \lambda) = 0,\ x = 0,\ y \neq 0,\ z \neq 0$;

(h) xyz–mode solution: $p(x^2, y^2, z^2, \lambda) = q(x^2, y^2, z^2, \lambda) = r(x^2, y^2, z^2, \lambda) = 0,\ x \neq 0,\ y \neq 0,\ z \neq 0$;

To analyze the stability of these solutions we need the explicit form of the eigenvalues of the Jacobian matrix (47). We have

(a) Trivial solution: when $x = y = z = 0$ with eigenvalues: p, q, r.

(b) x–mode solution: $p(x^2, 0, 0, \lambda) = 0,\ y = z = 0, x \neq 0$ with eigenvalues: $2up_u, q, r$.

(c) y–mode solution: $q(0, y^2, 0, \lambda) = 0,\ x = z = 0, y \neq 0$ with eigenvalues: $p, 2vq_v, r$.

(d) z–mode solution: $r(0, 0, z^2, \lambda) = 0,\ x = y = 0, z \neq 0$ with eigenvalues: $p,\ q,\ 2wr_w$.

(e) xy–mode solution: $p(x^2, y^2, 0, \lambda) = q(x^2, y^2, 0, \lambda) = 0,\ z = 0,\ x \neq 0,\ y \neq 0$ with eigenvalues: r and

$$vq_v + up_u \pm \sqrt{u^2p_u^2 - 2up_uvq_v + v^2q_v^2 + 4p_vx^2y^2q_u}.$$

(f) xz–mode solution: $p(x^2, 0, z^2, \lambda) = r(x^2, 0, z^2, \lambda) = 0,\ y = 0,\ x \neq 0,\ z \neq 0$ with eigenvalues: q and

$$vq_v + up_u \pm \sqrt{w^2r_w^2 - 2up_uwr_w + u^2p_u^2 + 4r_ux^2z^2p_w}.$$

(g) yz–mode solution: $q(0, y^2, z^2, \lambda) = r(0, y^2, z^2, \lambda) = 0,\ x = 0,\ y \neq 0,\ z \neq 0$ with eigenvalues: p and

$$vq_v + wr_w \pm \sqrt{v^2q_v^2 - 2vq_vwr_w + w^2r_w^2 + 4q_wy^2z^2r_v}.$$

(h) xyz–mode solution: $p(x^2, y^2, z^2, \lambda) = q(x^2, y^2, z^2, \lambda) = r(x^2, y^2, z^2, \lambda) = 0,\ x \neq 0,\ y \neq 0,\ z \neq 0$. A short calculation with Matlab, for example, allows finding the explicit form of the eigenvalues μ_1, μ_2 and μ_3, which are too large to be exposed here.

In all these cases $(a) - (h)$ the stability of the solutions is given by the sign of the linear part, as indicated above.

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