EFFECTS OF EXPONENTIAL HEATING ON DOUBLE-DIFFUSE FREE CONVECTION FLOWS ON A MOVING VERTICAL PLATE

NEHAD ALI SHAH, AZHAR ALI ZAFAR, CONSTANTIN FETECAU, and ASIM NASEEM

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This investigation is carried out to find the momentary behavior of magnetohydrodynamics heat and mass transfer in double-diffusive free convection flow of an electrically conducting, incompressible viscous fluid over a vertical plate with time-exponential heating, constant concentration and first order chemical reaction. The influence of Lorentz force on the fluid motion is considered when the external magnetic field is fixed or it moves together with the plate. Governing partial differential equation of the mathematical model is solved by the Laplace transform method. An interesting property regarding the behavior of the fluid velocity is found when the magnetic field moves with the plate. In this case the fluid velocity is not zero far away of the plate. Particular cases of the plate motion (time-accelerating plate, oscillating plate) are studied. The influence of essential parameters on the fluid motion due to a slowly accelerating plate and the required time to reach the steady-state for oscillating motions are graphically underlined and discussed. Moreover, mechanical, thermal and concentration effects on the fluid motion are separately brought to light. The variation of thermal boundary layer thickness is also presented.

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1. INTRODUCTION

In the last time, the study of *magnetohydrodynamic* (MHD) natural convection flow of electrically conducting fluids with heat and mass transfer has received a special attention due to their multiple applications in meteorology, electrical power generation, solar physics, geophysics and chemical engineering. Exact solutions for such motions of incompressible viscous fluids over an infinite vertical plate have been developed for different sets of boundary conditions. Ghara *et al.* [9] have studied the radiation effects on the MHD free convection flow past an impulsively moving plate with ramp wall temperature. Nandkeolyar *et al.* [14] derived exact solutions for the same flow of a heat absorbing fluid with mass transfer. MHD natural convection flow with Newtonian heating and mass diffusion was analytically solved by Vieru *et al.* [26] and Fetecau *et al.* [8] when the plate applies an arbitrary time-dependent shear stress to the fluid or it moves in its plane with an arbitrary velocity. Fetecau *et al.* [5] investigated the slip effects on the radiative MHD free convection flow over a moving plate with mass diffusion and heat source. An exact solution for MHD natural convection flow through a vertical annular micro-channel has been obtained by Jha *et al.* [10] in the presence of velocity slip and temperature jump on the boundary. Recently, a general study of such a flow with radiative effects, heat source and shear stress on the boundary has been developed by Fetecau *et al.* [7]. However, in all these studies as well as in many other which have been previously published, the magnetic lines of force of the imposed magnetic field are fixed to the fluid. Recently, Narahari and Debnath [15] developed an interesting study of unsteady MHD free convection flow with constant heat flux and heat source when the magnetic lines of force are fixed to the fluid or to the plate. Their exact solutions are obtained for fluid motions due to an exponentially accelerated or constantly accelerating plate. Actually, the first exact solutions of this kind seem to be those obtained by Tokis [25]. His solutions correspond to motions induced by uniform, constantly accelerating or decaying oscillatory translations of the plate. Some numerical results have been recently obtained by Onyango *et al.* [18] for the hydro-magnetic Couette flow between two parallel plates with magnetic field lines fixed relative to the moving upper plate but none of these papers explore the fluid behaviour at infinity. On the other hand, the mass transfer that is essential in many biological and geophysical applications has not been taken into consideration in the above mentioned papers. However, free convection flows resulting from the coupled heat and mass transfer have been extensively studied due to their applications in chemical engineering and industrial activities. There are many papers [1-4], [11] [12], [17], [19], [22], [24], [27] containing exact solutions for hydro-magnetic free convection flows with heat and mass transfer, but they correspond to the case when the magnetic field lines of forces are fixed to the fluid and the fluid velocity at infinity tends to zero. Furthermore, the mass transfer due to the concentration differences affects the rate and heat transfer and corresponding buoyancy effect cannot be neglected. In this note we present a general study of hydro-magnetic natural convection flow over a moving infinite vertical plate with exponential heating, constant concentration and chemical reaction. However, our purpose is not only to extend Narahari and Debnath's results by including the mass transfer, but we also want to provide new results both for general and oscillating motions. The thickness of the thermal boundary layer, for instance, is also determined. It is worth pointing out the fact that "the fluid velocity does not remain zero at infinity if the magnetic field is fixed to the plate". Moreover, the fluid velocity is presented as a sum of mechanical, thermal and concentration components whose contribution to the fluid motion is graphically underlined and discussed for slowly accelerating motions of the plate. Solutions corresponding to oscillating motions of the plate are presented as a sum of steady-state (permanent) and transient solutions and the required time to reach the steady-state is graphically determined.

2. WORDING OF THE PROBLEM

Let us consider the unsteady free convection flow of an electrically conducting incompressible viscous fluid over a non-conducting infinite vertical plate in the presence of a uniform magnetic field of strength B. The magnetic field is applied perpendicular to the plate and its magnetic lines of force are fixed to the fluid or to the plate. Initially, the plate and the fluid are at rest at the constant temperature T_∞ and the species concentration C_∞. After the time $t = 0^+$, the plate begins to slide in its plane against the gravitational field with the velocity $Vf(t)$ and its temperature is maintained at the value $T_\infty + T_w\left(1 - ae^{-bt}\right)$. Here, V is a constant velocity, $f(\cdot)$ is a piecewise continuous function with $f(0) = 0$ and a, b and T_w are also constants. The plate is also maintained at a constant concentration C_w.
Following Narahari and Debnath [15], we also assume that all physical properties are constant except the density variation with temperature in the body force and the induced magnetic field is negligible in comparison with the applied magnetic field B. Furthermore, radiative effects and the chemical reaction between the fluid and the species concentration are taken into consideration while the viscous dissipation and Joule heating are neglected. In these conditions, choosing a suitable Cartesian coordinate system x, y, z and using the usual Boussinesqs approximation, our problem reduces to the next set of partial differential equations Narahari and Debnath [15], Shah *et al.* [23]

(2.1)
$$\frac{\partial \mathrm{v}}{\partial t} = \upsilon\frac{\partial^2 \mathrm{v}}{\partial y^2} + g\beta_T\left(T - T_\infty\right) + g\beta_C\left(C - C_\infty\right) - \frac{\sigma B_o^2}{\rho}\left(\mathrm{v} - \in Vf(t)\right);\ y,\ t > 0,$$

(2.2) $$\rho C_p\frac{\partial T}{\partial t} = k\frac{\partial^2 \mathrm{T}}{\partial y^2} - \frac{\partial q_r}{\partial y},\ \ \frac{\partial C}{\partial t} = D\frac{\partial^2 \mathrm{C}}{\partial y^2} - R\left(C - C_\infty\right);\ y,\ t > 0,$$

(2.3) $$\mathrm{v}(y, 0) = 0,\ T(y, 0) = T_\infty,\ C(y, 0) = C_\infty;\ y > 0,$$

(2.4) $$\mathrm{v}(0, t) = Vf(t),\ T(0, t) = T_\infty + T_w\left(1 - ae^{-bt}\right),\ C(0, t) = C_w;\ t > 0,$$

$$\text{(2.5)} \qquad \mathrm{v}(y,t) < \infty,\ T(y,t) \to T_\infty,\ C(y,t) \to C_\infty \text{ as } \mathrm{y} \to \infty.$$

In the above equations, the unknown functions $\mathrm{v}(y,t)$, $T(y,t)$ and $C(y,t)$ are the velocity, the temperature and the species concentration respectively, while ν, g, β_T, β_C, σ, ρ, C_p, k, D, R and q_r are kinematic viscosity, acceleration due to gravity, thermal expansion coefficient, concentration expansion coefficient, electrical conductivity, density of the fluid, specific heat at constant pressure, thermal conductivity, mass diffusivity, chemical reaction parameter and the radiative heat flux. The parameter $\in$ is 0 when the *magnetic field is fixed relative to the fluid* (MFFRF) and 1 (one) when the *magnetic field is fixed relative to the plate* (MFFRP). By adopting the Rosseland diffusion approximation for an optically thick fluid (see Seth *et al.* [21] or Narahari and Dutta [15])

$$\text{(2.6)} \qquad q_r = -\frac{4}{3}\frac{\rho}{k_R}\frac{\partial T^4}{\partial y}$$

where σ is the Stefan-Boltzman constant and k_R is the Rosseland mean attenuation coefficient and assuming the temperature difference between the fluid temperature T and the free stream temperature T_∞ is small enough, the energy equation $(2.2)_1$ can be written in the form Fetecau *et al.* [6]

$$\text{(2.7)} \qquad \Pr_{eff}\frac{\partial T(y,t)}{\partial t} = \frac{\partial^2 \mathrm{T}(y,t)}{\partial y^2};\ y,\ t > 0$$

where $\Pr_{eff} = \frac{\Pr}{1+Nr}$ is the effective Prandtl number Magyari and Pantokratoras [13] and $\Pr = \frac{\mu C_p}{k}$, $Nr = \frac{16}{3}\frac{\sigma}{kk_R}T_\infty^3$ are Prandtl number, respectively the radiation-conduction parameter. Introducing the next dimensionless variables, functions and parameters

$$\text{(2.8)} \qquad \begin{aligned} &y^* = \tfrac{V}{v}y,\ t^* = \tfrac{V^2}{v}t,\ \mathrm{v}^* = \tfrac{\mathrm{v}}{V},\ T^* = \tfrac{T-T_\infty}{T_w},\ C^* = \tfrac{C-C_\infty}{C_w-C_\infty},\\ &b^* = \tfrac{v}{V^2}b,\ R^* = \tfrac{v}{V^2}R,\ f^*(t^*) = f\left(\tfrac{v}{V^2}t^*\right) \end{aligned}$$

dropping out the star notation and choosing the characteristic velocity V to be equal with $\sqrt[3]{vg\beta_T T_w}$, our problem reduce to the following dimensionless partial differential equations

(2.9)
$$\frac{\partial \mathrm{v}(y,t)}{\partial t} = \frac{\partial^2 \mathrm{v}(y,t)}{\partial y^2} + T(y,t) + NC(y,t) - M\left(\mathrm{v}(y,t) - \in f(t)\right);\ y,\ t > 0,$$

(2.10)
$$\Pr_{eff}\frac{\partial T(y,t)}{\partial t} = \frac{\partial^2 \mathrm{T}(y,t)}{\partial y^2},\quad \frac{\partial C(y,t)}{\partial t} = \frac{1}{Sc}\frac{\partial^2 \mathrm{C}(y,t)}{\partial y^2} - RC(y,t);\ y,\ t > 0$$

with the initial and boundary conditions

$$\text{(2.11)} \qquad \mathrm{v}(y,0) = 0,\ T(y,0) = 0,\ C(y,0) = 0;\ y \geq 0,$$

$$\mathrm{v}\left(0,t\right)=f\left(t\right),\ T\left(0,t\right)=1-ae^{-bt},\ C\left(0,t\right)=1;\ t>0, \tag{2.12}$$

$$\mathrm{v}\left(y,t\right)<\infty,\ T\left(y,t\right),\ C\left(y,t\right)\rightarrow 0 \text{ as y}\rightarrow\infty. \tag{2.13}$$

In the above relations, R is the dimensionless chemical reaction parameter while N the ratio of the buoyancy forces, M the magnetic parameter and Schmidt number Sc are defined by

$$N=\frac{\beta_C\left(C_w-C_\infty\right)}{\beta_T T_w}, M=\frac{\sigma B^2}{\rho}\frac{v}{V^2}, Sc=\frac{v}{D}. \tag{2.14}$$

It is worth pointing out that Pr_{eff} and Sc are transport parameters regarding the thermal and mass diffusivity, while N represents the relative contribution of the mass transport rate on the free convection flow Narahari and Dutta [16]. As β_C can be positive (or negative) and $\beta_T>0$, N can be also positive or negative. If $N=0$ the buoyancy force effect from mass diffusion is absent.

3. SOLUTION OF THE PROBLEM

As the temperature and concentration fields corresponding to this problem can be easily obtained from previous works, therefore here we are only here interested in the fluid velocity only and the boundary layer thickness for temperature. The corresponding boundary layer thickness for concentration is given in [8, Eq. (56)]. However, in order to determine the fluid velocity using the Laplace transform technique, we need the Laplace transforms of $T\left(y,t\right)$ and $c\left(y,t\right)$, namely (see Rubbab *et al.* [20], Eq. (15), respectively Shah *et al.* [23], Eq. (20)).

$$\bar{T}\left(y,q\right)=\left(\frac{1}{q}-\frac{q}{q+b}\right)e^{-y\sqrt{\mathrm{Pr}_{eff}q}},\ \bar{C}\left(y,q\right)=\frac{1}{q}e^{-y\sqrt{Sc(q+R)}} \tag{3.1}$$

corresponding to our problem. Here, q is the transform parameter. In order to determine the differential equation of the thermal boundary layer thickness, we integrate Eq. $(2.10)_2$ with respect to y from 0 to ∞ and introduce a measure of the thermal layer

$$\delta_T\left(t\right)=\int_0^{\delta_{1T}}T\left(y,t\right)\mathrm{d}y \tag{3.2}$$

where δ_{1T} is the thermal boundary layer thickness. It results that

$$\Pr_{eff}\frac{\mathrm{d}\delta_T\left(t\right)}{\mathrm{d}t}=-\left.\frac{\partial T\left(y,t\right)}{\partial y}\right|_{y=0}. \tag{3.3}$$

To determine Eq. (17), we also used the fact that

$$\lim_{\mathrm{y}\to\infty} T(y,t) = \lim_{\mathrm{y}\to\infty} \frac{\partial T(y,t)}{\partial y} = 0 \tag{3.4}$$

Applying the Laplace transform to Eq. (3.3) and using Eq. $(3.1)_1$ and the fact that $\delta_T(0) = 0$, we find that

$$\Pr_{eff} q\bar{\delta}_T(q) = \sqrt{\Pr_{eff} q}\left(\frac{1}{q} - \frac{a}{q+b}\right) \tag{3.5}$$

or equivalently

$$\bar{\delta}_T(q) = \frac{1}{\sqrt{\Pr_{eff} q}}\left(\frac{1}{q} - \frac{a}{q+b}\right) = \frac{1}{\sqrt{\Pr_{eff}}}\left(\frac{1}{q^{3/2}} - \frac{a}{(q+b)\sqrt{q}}\right) \tag{3.6}$$

Now, applying the inverse Laplace transform to Eq. (3.6) and using the convolution theorem for the last term, it results that

$$\begin{aligned}\delta_T(t) &= \frac{1}{\sqrt{\Pr_{eff}\pi}}\left[2\sqrt{t} - a\int_0^t \frac{1}{\sqrt{s}} e^{-b(t-s)}\mathrm{d}s\right] \\ &= \frac{2}{\sqrt{\Pr_{eff}\pi}}\left[\sqrt{t} - \frac{ae^{-bt}}{\sqrt{b}}\int_0^{\sqrt{bt}} e^{s^2}\mathrm{d}s\right].\end{aligned} \tag{3.7}$$

For velocity, we apply the Laplace transform to Eq. (2.9) and bear in mind the corresponding initial and boundary conditions. We find the ordinary differential equation

$$q\bar{\mathrm{v}}(y,q) = \frac{\partial^2\bar{\mathrm{v}}(y,q)}{\partial y^2} + \bar{T}(y,q) + N\bar{C}(y,q) - M\left(\bar{\mathrm{v}}(y,q) - \in F(q)\right), \tag{3.8}$$

with the boundary conditions

$$\bar{\mathrm{v}}(0,q) = F(q),\ \bar{\mathrm{v}}(y,q) \to 0 \text{ as } \mathrm{y} \to \infty. \tag{3.9}$$

Of course, $\bar{\mathrm{v}}(y,q)$ and $F(q)$ denote the Laplace transforms of $\mathrm{v}(y,t)$, respectively $f(t)$. Introducing Eqs. (3.1) into (3.8), it results that

$$\begin{aligned}&\frac{\partial^2\bar{\mathrm{v}}(y,q)}{\partial y^2} - (q+M)\,\bar{\mathrm{v}}(y,q) \\ &= -\in MF(q) - \left(\frac{1}{q} - \frac{a}{q+b}\right)e^{-y\sqrt{\Pr_{eff} q}} - N\frac{1}{q}e^{-y\sqrt{Sc(q+R)}}.\end{aligned} \tag{3.10}$$

The solution of the ordinary differential equation (3.10) with the boundary conditions (3.9), is

(3.11)
$$\bar{\mathrm{v}}\left(y,q\right)=F\left(q\right)e^{-y\sqrt{q+M}}+\in M\tfrac{F(q)}{q+M}\left(1-e^{-y\sqrt{q+M}}\right)+$$
$$+\tfrac{(1-a)q+b}{q(q+b)\left(\left(1-\mathrm{Pr}_{eff}\right)q+M\right)}\left(e^{-y\sqrt{\mathrm{Pr}_{eff}q}}-e^{-y\sqrt{q+M}}\right)+\tfrac{N\left(e^{-y\sqrt{Sc(q+R)}}-e^{-y\sqrt{q+M}}\right)}{q((1-Sc)q-(ScR-M))}$$

Now, introducing the relations
$$\tfrac{(1-a)q+b}{q(q+b)\left(\left(1-\mathrm{Pr}_{eff}\right)q+M\right)}=\tfrac{1}{1-\mathrm{Pr}_{eff}}\left(\tfrac{1}{C}\tfrac{1}{q}-\tfrac{a}{C-b}\tfrac{1}{q+b}+\tfrac{(1-a)C-b}{C(b-C)}\tfrac{1}{q+C}\right),$$
$$\tfrac{1}{q((1-Sc)q-(ScR-M))}=\tfrac{1}{M-ScR}\left(\tfrac{1}{q}-\tfrac{1}{q+D}\right);\ C=\tfrac{M}{1-\mathrm{Pr}_{eff}},\ D=\tfrac{ScR-M}{Sc-1},$$
into Eq. (3.11), applying the inverse Laplace transform and using the convolution theorem and Eqs. (A1) and (A2) from Appendix, we can present the velocity field under the form

$$\mathrm{v}\left(y,t\right)=\mathrm{v}_m\left(y,t\right)+\mathrm{v}_T\left(y,t\right)+\mathrm{v}_C\left(y,t\right) \tag{3.12}$$

where

$$\begin{aligned}\mathrm{v}_m\left(y,t\right)=&\frac{y}{2\sqrt{\pi}}\int_0^t\frac{f\left(t-s\right)}{s\sqrt{s}}\exp\left(-\frac{y^2}{4s}-Ms\right)\mathrm{d}s\\&+\in M\int_0^t f\left(t-s\right)e^{-Ms}erf\left(\frac{y}{2\sqrt{s}}\right)\mathrm{d}s,\end{aligned} \tag{3.13}$$

(3.14)
$$\mathrm{v}_T\left(y,t\right)=\left(\tfrac{\left[\psi\left(y\sqrt{\mathrm{Pr}_{eff}},t;0,0\right)-\psi(y,t;M,0)\right]}{C}+a\tfrac{\left[\psi\left(y\sqrt{\mathrm{Pr}_{eff}},t;0,-b\right)-\psi(y,t;M,-b)\right]}{b-C}+\right.$$
$$\left.+\tfrac{(1-a)C-b}{C(b-C)}\left[\psi\left(y\sqrt{\mathrm{Pr}_{eff}},t;0,-C\right)-\psi\left(y,t;M,-C\right)\right]\right)\tfrac{1}{1-\mathrm{Pr}_{eff}},$$

$$\begin{aligned}\mathrm{v}_C\left(y,t\right)=&\frac{N}{M-ScR}\left(\psi\left(y\sqrt{Sc},t;R,0\right)-\psi\left(y,t;M,0\right)\right.\\&\left.-\psi\left(y\sqrt{Sc},t;R,-D\right)+\psi\left(y,t;M,-D\right)\right)\end{aligned} \tag{3.15}$$

are its mechanical, thermal and concentration components and the function $\psi\left(y,t;a,b\right)$ is defined in Appendix. It is not difficult to show that $\mathrm{v}\left(y,t\right)$, given by Eqs. (3.12)-(3.15), satisfies the imposed initial and boundary conditions. In order to verify the boundary condition $(2.12)_1$, for instance, we rewrite $\mathrm{v}_m\left(y,t\right)$

in the equivalent form

$$\begin{aligned} \mathrm{v}_m(y,t) &= \frac{y}{\sqrt{\pi}} \int\limits_{\frac{y}{2\sqrt{t}}}^{\infty} f\left(t-\frac{y^2}{4s}\right) \exp\left(-s^2-\frac{My^2}{4s^2}\right) \mathrm{d}s \\ &+ \in M \int\limits_0^t f(t-s)\, e^{-Ms} erf\left(\frac{y}{2\sqrt{s}}\right) \mathrm{d}s. \end{aligned} \tag{3.16}$$

As regards the limit of velocity at infinity, it results that

$$\lim_{\mathrm{y}\to\infty} \mathrm{v}(y,t) = \begin{cases} 0 & if \ \in = 0 \\ M \int\limits_0^t f(t-s)\, e^{-Ms} \mathrm{d}s & if \ \in = 1 \end{cases} \tag{3.17}$$

Consequently, in the case when the magnetic field is fixed relative to the plate, the fluid does not remain at rest far away of the plate.

From physical point of view, it is also important to determine the skin friction or shear on the plate. Introducing Eq. (3.11) into

$$\tau = -\left.\frac{\partial \mathrm{v}(y,t)}{\partial y}\right|_{y=0} = -L^{-1}\left\{\left.\frac{\partial \bar{\mathrm{v}}(y,q)}{\partial y}\right|_{y=0}\right\}, \tag{3.18}$$

we find that (see also Eqs. (A3)-(A5) from Appendix)

$$\tau = \tau_m + \tau_T + \tau_C, \tag{3.19}$$

where

$$\begin{aligned} \tau_m &= \int\limits_0^t f'(t-s)\left[\sqrt{M}\, erf\left(\sqrt{Ms}\right) + \frac{e^{-Ms}}{\sqrt{\pi s}}\right] \mathrm{d}s \\ &- \in \sqrt{M} \int\limits_0^t f'(t-s) erf\left(\sqrt{Ms}\right) \mathrm{d}s, \end{aligned} \tag{3.20}$$

$$\begin{aligned} \tau_T = \tfrac{1}{1-\mathrm{Pr}_{eff}} &\left\{\sqrt{\mathrm{Pr}_{eff}}\left[\tfrac{1}{C\sqrt{\pi t}} + \tfrac{a}{b-C}\phi(t;0,b) + \tfrac{(1-a)C-b}{C(b-C)}\phi(t;0,C)\right] - \right. \\ &\left. -\tfrac{a}{b-C}\phi(t;M,b) - \tfrac{(1-a)C-b}{C(b-C)}\phi(t;M,C)\right\}, \end{aligned} \tag{3.21}$$

$$\tau_C = \frac{N}{ScR-M}\left\{\phi(t;M,D) - \phi(t;M,0) + \sqrt{Sc}\left[\phi(t;R,0) - \phi(t;R,D)\right]\right\}, \tag{3.22}$$

are the mechanical, thermal and concentration components of the skin friction and the function $\phi(t;a,b)$ is defined in the Appendix.

Finally, for validation, let us take $f(t) = H(t)$ (the Heaviside unit step function) in our relations (3.13) and (3.20) and use Eqs. (A6) and (A7) from Appendix. As it was to be expected, the corresponding results are identical to those obtained by Narahari and Debnath [15], Eqs. (11a), (13) with a_0 and Tokis [26], Eqs. (12) and (13) in the absence of thermal and concentration effects. Consequently, the general solutions (3.12) and (3.19) are correct and the present problem is completely solved. Indeed, assigning to $f(\cdot)$ suitable forms, we can determine exact solutions for any motion with technical relevance of this type. In the following in order to get some physical insight of present results and to avoid repetition, we shall here consider the flows due to slowly accelerating or oscillating plate.

3.1. **Case I, $f(t) = H(t)t^{\alpha}$. Variably accelerating plate**

The thermal and concentration components of velocity do not depend on the plate motion. However, the heat and mass transfer can influence the fluid motion and we have to know if their influence is significant or it can be neglected in some motions with possible engineering applications. Taking $f(t) = H(t)t^{\alpha}$ with $\alpha > 0$, the equations (3.13) and (3.20) take the forms

$$\begin{aligned} \mathrm{v}_m(y,t) &= \frac{y}{2\sqrt{\pi}} \int_0^t \frac{(t-s)^{\alpha}}{s\sqrt{s}} \exp\left(-\frac{y^2}{4s} - Ms\right) \mathrm{d}s \\ &+ \in M \int_0^t (t-s)^{\alpha} e^{-Ms} erf\left(\frac{y}{2\sqrt{s}}\right) \mathrm{d}s, \end{aligned} \tag{3.23}$$

$$\begin{aligned} \tau_m &= \alpha \int_0^t (t-s)^{\alpha-1} \left[\sqrt{M}\, erf\left(\sqrt{Ms}\right) + \frac{e^{-Ms}}{\sqrt{\pi s}}\right] \mathrm{d}s \\ &- \in \alpha\sqrt{M} \int_0^t (t-s)^{\alpha-1} erf\left(\sqrt{Ms}\right) \mathrm{d}s; \quad \alpha > 0, \end{aligned} \tag{3.24}$$

which corresponds to motions induced by a slowly, constantly or highly accelerating plate. The solutions corresponding to $\alpha = 0$ namely

$$\mathrm{v}_{0m}(y,t) = \psi(y,t;M,0)\,H(t) + \in H(t)\left[1 - \psi(y,t;M,0) - e^{-Mt} erf\left(\frac{y}{2\sqrt{t}}\right)\right], \tag{3.25}$$

$$\tau_{0m} = \left[\sqrt{M}\, erf\left(\sqrt{Ms}\right) + \frac{e^{-Ms}}{\sqrt{\pi s}}\right] H(t) - \in \sqrt{M} H(t) erf\left(\sqrt{Mt}\right), \tag{3.26}$$

can have a very important role. More exactly, the solutions corresponding to $\alpha = n$ (a natural number) can be written as simple or multiple integrals of these solutions, namely

$$\begin{aligned} \mathrm{v}_{nm}(y,t) &= \int_0^t \int_0^{s_1} \int_0^{s_2} \ldots \int_0^{s_{n-1}} \mathrm{v}_{0m}(y, s_n)\, \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3 \ldots \mathrm{d}s_n, \\ \tau_{nm}(y,t) &= \int_0^t \int_0^{s_1} \int_0^{s_2} \ldots \int_0^{s_{n-1}} \tau_{0m}(y, s_n)\, \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3 \ldots \mathrm{d}s_n. \end{aligned} \tag{3.27}$$

However, in the following we shall be interested of the solutions corresponding to motions due to a slowly accelerating plate (when $\alpha < 1$).

3.2. CaseII, $f(t) = H(t)\cos(\omega t)$ or $H(t)\sin(\omega t)$

Introducing $f(t) = H(t)\cos(\omega t)$ or $H(t)\sin(\omega t)$ into Eqs. (3.13) and (3.20) and using the fact that $H'(t) = \delta(t)$ and

$$\int_0^t \delta(t-s) f(s)\mathrm{d}s = \int_0^t \delta(s) f(t-s)\mathrm{d}s = f(t)$$

where $\delta(\cdot)$ is the Dirac delta function, we find that

$$\begin{aligned} \mathrm{v}_{cm}(y,t) = &\frac{y}{2\sqrt{\pi}} \int_0^t \frac{\cos[\omega(t-s)]}{s\sqrt{s}} \exp\left(-\frac{y^2}{4s} - Ms\right) \mathrm{d}s \\ &+ \in M \int_0^t \cos[\omega(t-s)]\, e^{-Ms} erf\left(\frac{y}{2\sqrt{s}}\right) \mathrm{d}s, \end{aligned} \tag{3.28}$$

$$\begin{aligned} \mathrm{v}_{sm}(y,t) = &\frac{y}{2\sqrt{\pi}} \int_0^t \frac{\sin[\omega(t-s)]}{s\sqrt{s}} \exp\left(-\frac{y^2}{4s} - Ms\right) \mathrm{d}s \\ &+ \in M \int_0^t \sin[\omega(t-s)]\, e^{-Ms} erf\left(\frac{y}{2\sqrt{s}}\right) \mathrm{d}s, \end{aligned} \tag{3.29}$$

$$\tau_{cm} = H(t)\left\{\sqrt{M}\, erf\left(\sqrt{Mt}\right) + \frac{e^{-Mt}}{\sqrt{\pi t}} - \in \sqrt{M} erf\left(\sqrt{Mt}\right)\right\} \tag{3.30}$$

$$- \omega \int_0^t \sin[\omega(t-s)] \left[\sqrt{M}\, erf\left(\sqrt{Ms}\right) + \frac{e^{-Ms}}{\sqrt{\pi s}}\right] \mathrm{d}s$$

$$+\in\omega\sqrt{M}\int_0^t \sin\left[\omega\left(t-s\right)\right] erf\left(\sqrt{Ms}\right)\mathrm{d}s,$$

$$\tau_{sm}=\omega\int_0^t \cos\left[\omega\left(t-s\right)\right]\left[\sqrt{M}\, erf\left(\sqrt{Ms}\right)+\frac{e^{-Ms}}{\sqrt{\pi s}}\right]\mathrm{d}s$$

(3.31)

$$+\in\omega\sqrt{M}\int_0^t \cos\left[\omega\left(t-s\right)\right] erf\left(\sqrt{Ms}\right)\mathrm{d}s,$$

As expected, for $\omega=0$ the solutions (3.28) and (3.29) reduce to those given by Eqs. (3.25) and (3.26) corresponding to the motion with uniform velocity on the boundary.

The dimensionless velocities $\mathrm{v}_{cm}\left(y,t\right)$ and $\mathrm{v}_{sm}\left(y,t\right)$ describe the fluid motion some time after its initiation. After that time, when the transients disappear, they reduce to the steady-state (permanent) solutions

$$\mathrm{v}_{cmp}\left(y,t\right)=\frac{y}{2\sqrt{\pi}}\int_0^t \frac{\cos\left[\omega\left(t-s\right)\right]}{s\sqrt{s}}\exp\left(-\frac{y^2}{4s}-Ms\right)\mathrm{d}s$$

(3.32)

$$+\in M\int_0^t \cos\left[\omega\left(t-s\right)\right]e^{-Ms} erf\left(\frac{y}{2\sqrt{s}}\right)\mathrm{d}s,$$

$$\mathrm{v}_{smp}\left(y,t\right)=\frac{y}{2\sqrt{\pi}}\int_0^t \frac{\sin\left[\omega\left(t-s\right)\right]}{s\sqrt{s}}\exp\left(-\frac{y^2}{4s}-Ms\right)\mathrm{d}s$$

(3.33)

$$+\in M\int_0^t \sin\left[\omega\left(t-s\right)\right]e^{-Ms} erf\left(\frac{y}{2\sqrt{s}}\right)\mathrm{d}s$$

which are periodic in time and independent of the initial conditions. However, they satisfy the governing equations and the boundary conditions.

Lengthy but straightforward computations show that the steady-state solutions (3.32) and (3.33) can be written in the simple but elegant forms (see Eqs. (A8) and (A9))

(3.34)

$$\mathrm{v}_{cmp}\left(y,t\right)=e^{-my}\cos\left(\omega t-ny\right)+\frac{\in M}{\sqrt{M^2+\omega^2}}\Big\{\cos\left(\omega t-\varphi\right)$$
$$-e^{-my}\cos\left(\omega t-ny-\varphi\right)\Big\},$$

$$\mathrm{v}_{smp}(y,t) = e^{-my}\sin(\omega t - ny) + \frac{\in M}{\sqrt{M^2+\omega^2}}\Big\{\sin(\omega t-\varphi) - e^{-my}\sin(\omega t - ny - \varphi)\Big\}, \tag{3.35}$$

where $m = \sqrt{\frac{\sqrt{M^2+\omega^2}+M}{2}}$, $n = \sqrt{\frac{\sqrt{M^2+\omega^2}-M}{2}}$ and $\varphi = arctg\left(\frac{\omega}{M}\right)$. A simple analysis clearly shows that these solutions satisfy the boundary conditions and governing equations (2.9) in the absence of thermal effects and concentration. Consequently, in the absence of these effects, our fluid flows according to the steady-state solutions (3.34) and (3.35) after a characteristic time. This time will be graphically determined in the next section both for cosine and sine oscillations of the plate. It is also worth to mention that our steady-state solution (3.34) is a little bit different from of the similar result of Toki [25], Eq. (36). This is due to the last but one term of his equation (20) that was wrongly rewritten into Eq. (36). Taking the limit of Eqs. (3.34) and (3.35) , when y $\to \infty$, we find that

$$\mathrm{v}_{cmp}(\infty,t) = \begin{cases} 0 & if \ \in= 0 \\ \frac{M}{\sqrt{M^2+\omega^2}}\cos(\omega t-\varphi) & if \ \in= 1 \end{cases} \tag{3.36}$$

respectively

$$\mathrm{v}_{smp}(\infty,t) = \begin{cases} 0 & if \ \in= 0 \\ \frac{M}{\sqrt{M^2+\omega^2}}\sin(\omega t-\varphi) & if \ \in= 1 \end{cases} \tag{3.37}$$

Now, for comparison, let us put take $f(t) = H(t)cos(\omega t)$ or $H(t)sin(\omega t)$ into Eq. (3.17). The obtained results, namely

$$\mathrm{v}_{c}(\infty,t) = \begin{cases} 0 & if \ \in= 0 \\ -\frac{M}{M^2+\omega^2}e^{-Mt} + \frac{M}{\sqrt{M^2+\omega^2}}\cos(\omega t-\varphi) & if \ \in= 1 \end{cases} \tag{3.38}$$

and

$$\mathrm{v}_{s}(\infty,t) = \begin{cases} 0 & if \ \in= 0 \\ \frac{M\omega}{M^2+\omega^2}e^{-Mt} + \frac{M}{\sqrt{M^2+\omega^2}}\sin(\omega t-\varphi) & if \ \in= 1 \end{cases} \tag{3.39}$$

are in accordance with those from Eqs. (3.36) and (3.37). The second relations of Eqs. (3.38) and (3.39) are also contain the transient components of velocity at infinity when the transverse magnetic field is fixed to the plate. Finally for later use, as well as for a simple correction of Eq. (38) from Toki [25], we also provide here the shear stresses

$$\tau_{cmp}(y,t) = \sqrt{m^2+n^2}e^{-my}\sin(\omega t - ny - \gamma) - \in Me^{-my}\sin(\omega t - ny - \phi - \gamma) \tag{3.40}$$

and

$$\tau_{smp}\left(y,t\right)=\sqrt{m^2+n^2}e^{-my}\cos\left(\omega t-ny-\gamma\right) - \in Me^{-my}\cos\left(\omega t-ny-\phi-\gamma\right) \tag{3.41}$$

corresponding to the steady-state. Here $\gamma = arctg\left(\frac{m}{n}\right)$. By now letting into Eqs. (3.40) and (3.41), we find

$$\tau_{cmp}\left(0,t\right)=\sqrt{m^2+n^2}\sin\left(\omega t-\gamma\right)- \in M\sin\left(\omega t-\phi-\gamma\right) \tag{3.42}$$

and

$$\tau_{smp}\left(0,t\right)=\sqrt{m^2+n^2}\cos\left(\omega t-\gamma\right)- \in M\cos\left(\omega t-\phi-\gamma\right) \tag{3.43}$$

which represent the skin frictions corresponding to the steady-state in the absence of thermal and concentration effects.

4. NUMERICAL RESULTS AND DISCUSSION

In this section, exact general solutions are determined for dimensionless velocity and skin friction corresponding to the hydro-magnetic natural convection flow over a moving vertical plate with exponential heating, constant concentration and chemical reaction. Radiative effects are taken into consideration and the magnetic field is fixed to the fluid or to the plate. In order to get some physical insight of obtained results and to avoid repetition, two special cases are considered and the influence of essential parameters N, Sc, R and t on the fluid motion is graphically underlined and discussed. Magnetic effects have been previously discussed by Narahari and Debnath [15], while, according to Magyari and Pantokratoras [13], the investigation of heat transfer characteristics with or without thermal radiation is the same problem and we just wanted to remember it.

Figsures 1 and 2, for comparison, present the diagrams of the dimensionless velocity $v(y,t)$ respectively of its mechanical component $v_m(y,t)$ against y at different times for a slowly accelerating motion of the plate. As expected, both velocities are increasing functions of time and the combined contribution of thermal and concentration components is substantial and cannot be neglected. Furthermore, the velocities corresponding to MFFRP are appreciably increased as compared with MFFRF. In all cases, the corresponding velocities smoothly decrease from maximum values on the boundary to asymptotical values for increasing y . However, as it is clearly seen from these figures, the asymptotic values of both velocities are not zero at infinity if the magnetic field is fixed to the plate.

Effects of the ratio of buoyancy forces N on the dimensionless fluid velocity are shown in Fig. 3 at time $t = 1.5$ for aiding ($N > 0$) and opposing ($N < 0$) flows. In the first case, the buoyancy force due to the species diffusion assists the thermal buoyancy force and the fluid velocity increases for increasing values of N. More precisely, the two forces acts in the same direction and the fluid velocity increases due to the increase in density of the solution. In the second case, the negative buoyancy force causes the occurrence of a reverse flow just away of the plate.

The variation of velocity with respect to Schmidt number Sc and the chemical reaction parameter R is given in Figs. 4 and 5. These figures clearly show that the velocity is a decreasing function with respect to both parameters. Moreover, as expected, it is again observed that the velocity profiles are higher in the case when the magnetic field is fixed to the plate. These profiles, as before, are everywhere higher/lower for different values of Sc or R. The contributions of the three components of dimensionless velocity on the fluid motion are presented in Figs. 6 both for (MFFRF) and (MFFRP). It can be clearly seen from these figures that each component has a significant influence on the fluid velocity and cannot be neglected.

Figures 7 and 8 correspond to motions due to an oscillating plate. They present the diagrams of the starting solutions (3.28) and (3.29) and of their steady-state components (3.34) respectively (3.35). At small values of time, the differences between these solutions are appreciable but they disappear in time and the required time to reach the steady-state is lower for motions due to cosine oscillations of the plate as compared to that corresponding to sine oscillations of the plate. This is due to the fact that, at time $t = 0$ the velocity of the wall is zero in the case of sine oscillations of the plate. However, this time is almost the same for the two cases (MFFRF) and (MFFRP).

Finally, for completeness, the variations of the skin frictions $\tau_{\alpha m}$, $\tau_{\alpha m} + \tau_T$ and $\tau_{\alpha m} + \tau_T + \tau_C$ ($\alpha = 0, 5$) with dimensionless time t are depicted in Figs. 9a and 9b both for MFFRF and MFFRP and of the thickness of the boundary layer in Fig. 10. The skin frictions corresponding to MFFRP are slower as compared to those of MFFRF. This should be expected because the corresponding velocities are higher in the case of MFFRP and thermal and concentration effects on the skin friction are negligible for small values of time. Moreover, the skin friction corresponding to the combined mechanical and thermal effects has the lowest values. Consequently, as it results from Figures 6 and 9, the thermal effects imply an increase of the fluid velocity on the whole flow domain and a diminution of the skin friction in time. As regards the concentration effects, they also imply an increase of fluid velocity in all flow domain and a deminiuation of the skin friction for values of t less than a critical value t_c between 0.6 and 0.7. The thermal boundary layer thickness, as it results from Fig. 10, is a decreasing function with respect to Pr_{eff} and tends to an asymptotic value for large value of t.

5. CONCLUSIONS

Hydro-magnetic natural convection flow of an electrically conducting, incompressible viscous fluid over a moving infinite vertical plate with exponentially heating, constant concentration and chemical reaction is analytically and graphically studied. Viscous dissipation and Joule heating are neglected but the radiative effects are taken into consideration. The plate is moving with arbitrary time-dependent velocity in its plane while the transverse magnetic field is fixed to the fluid or to the moving plate and our interest is focused on the fluid motion. Consequently, exact general expressions for the dimensionless velocity and the corresponding skin friction are established in simple forms in terms of error and complementary error functions of Gauss and the problem under consideration is completely solved. Both the velocity and the skin friction are presented as sum of their mechanical, thermal and concentration components. An exact expression is also determined for the thermal boundary layer thickness.

However, in order to obtain some physical insight of results that have been obtained as well as to avoid repetition, two special cases are considered and some graphical representations are depicted for different values of time and of physical parameters The solutions corresponding to oscillating motions of the plate are written as sum of steady-state and transient solutions and the required time to reach the steady-state is graphically determined. Finally, the contributions of mechanical, thermal and concentration components of velocity and skin friction on the fluid motion are brought to light for a slowly accelerating motion of the plate. The main conclusions are:

- Contrary to our expectations, the fluid velocity does not remain zero at infinity if the magnetic lines of force of the magnetic field are fixed relative to the plate. More exactly, the fluid does not remains at rest far away from the plate (cf. Eq. (3.11)).
- The dimensionless velocity of the fluid significantly increases in the case MFFRP in comparison to the case MFFRF.
- For aiding flows (when $N > 0$), the fluid velocity increases for increasing values of N. A reverse flow appears for opposing flows (when $N < 0$).
- Contributions of mechanical, thermal and concentration components of velocity and the skin friction on the fluid motion are significant and they cannot be neglected.

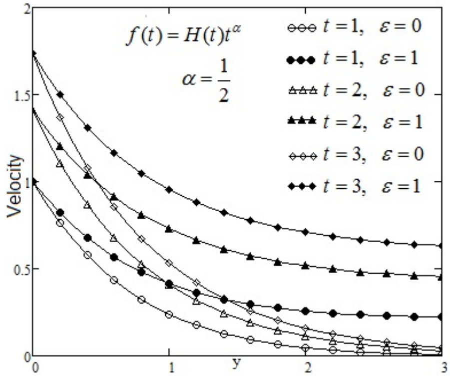

Fig. 1 – Profiles of the mechanical component $V_{\alpha m}(y,t)$ of the dimensionless velocity against y for $M = 0.5$ and different values of t

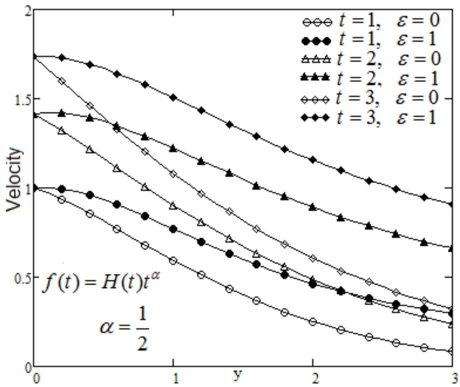

Fig. 2 – Profiles of the dimensionless velocity $v(y,t)$ against y for $a = 0.75$, $b = 0.15$, $M = 0.5$, $N = 2$, $Pr_{eff} = 5$, $Sc = 0.5$, $R = 0.7$ and different values of t.

- The required time to reach the steady-state is lower for motions due to cosine as compared with sine oscillations of the plate. This is obvious, since at time $t = 0$ the velocity of the wall is zero for sine oscillations of the plate.

- The thickness of thermal boundary layer smoothly increases from zero value up to the asymptotic value for large values of t. It is a decreasing function with respect to Pr_{eff}.

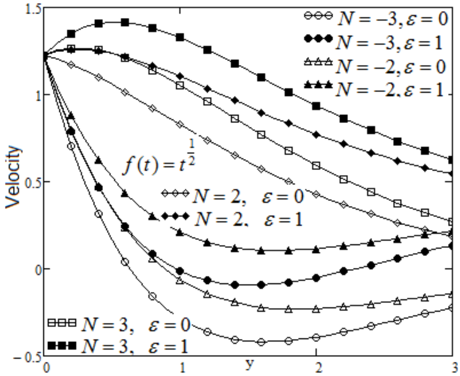

Fig. 3 – Profiles of the dimensionless velocity $v(y,t)$ against y at $t = 1.5$ for $a = 0.75, b = 0.15$, $M = 0.5$, $Pr_{eff} = 5$, $Sc = 0.5$, $R = 0.7$ and different values of N.

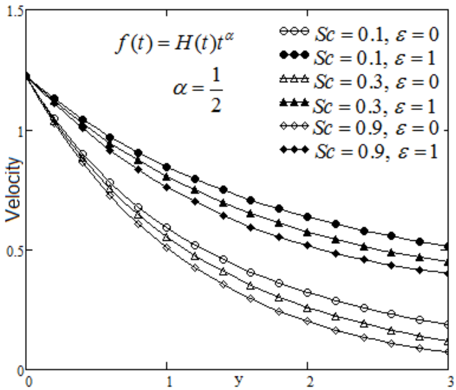

Fig. 4 – Profiles of the dimensionless velocity $v(y,t)$ against y at $t = 1.5$ for $a = 0.75$, $b = 0.15$, $M = 0.5$, $N = 0.5$, $Pr_{eff} = 0.5$, $R = 0.7$ and different values of Sc.

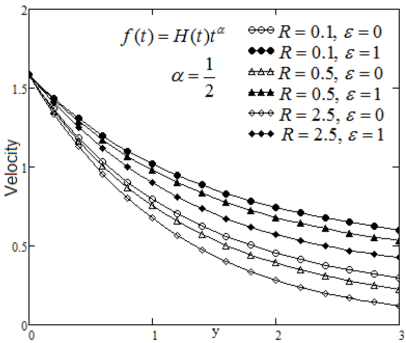

Fig. 5 – Profiles of the dimensionless velocity $v(y,t)$ against y at $t = 2.5$ for $a = 0.75, b = 0.15, M = 0.5, Pr_{eff} = 0.5, N = 0.5, Sc = 0.2$ and different values of R.

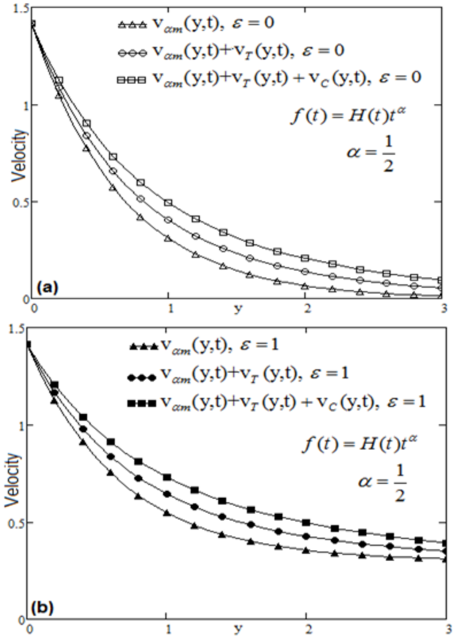

Fig. 6 – Profiles of the dimensionless velocities $v_{\alpha m}(y,t)v_{\alpha m}(y,t) + v_c(y,t)$ and $v_{\alpha m}(y,t) + v_T(y,t) + v_c(y,t)$ against y at $t = 1.5$ for $a = 0.75, b = 0.15$, $M = 0.5$, $Pr_{eff} = 0.5$, $N = 0.5$, $Sc = 0.5$ and $R = 0.7$.

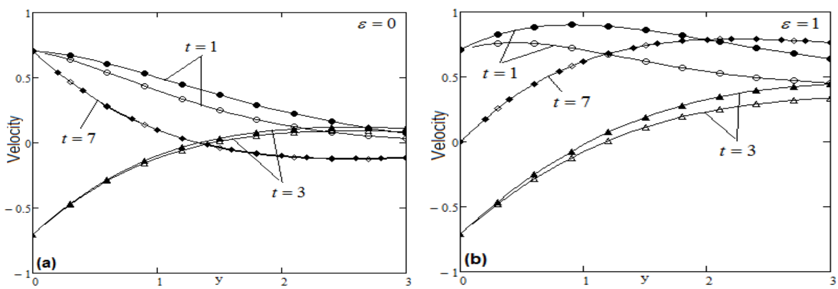

Fig. 7 – Required time to reach the steady-state for the motion due to cosine oscillation of the plate for $M = 0.1$ and $\omega = \pi/4$.

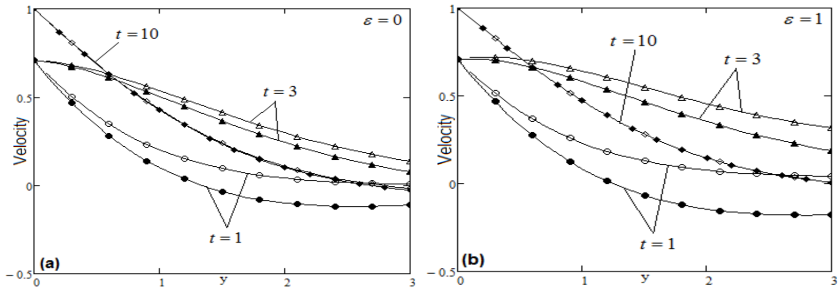

Fig. 8 – Required time to reach the steady-state for the motion due to sine oscillation of the plate for $M = 0.1$ and $\omega = \pi/4$.

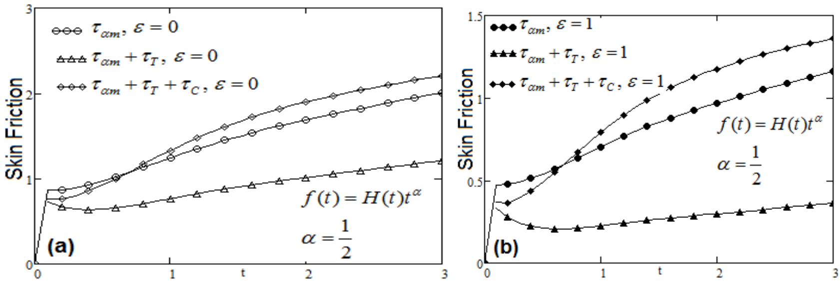

Fig. 9 – Profiles of the dimensionless skin frictions $\tau_{\alpha m}, \tau_{\alpha m} + \tau_T$ and $\tau_{\alpha m} + \tau_T + \tau_C$ against y for $a = 0.75, b = 0.15,\ M = 0.5,\ Pr_{eff} = 0.5,\ N = 0.5,$ $Sc = 0.5$ and $R = 0.7.$ and $R = 0.7.$

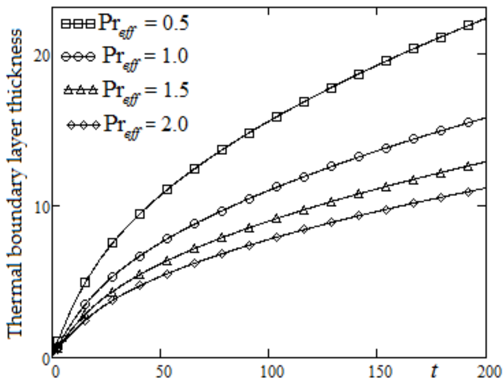

Fig. 10 – Time variation of thermal boundary layer thickness for different values of effective Prandtl number.

Appendix

(A1) $L^{-1}\left\{e^{-y\sqrt{q}}\right\}=\frac{y}{2t\sqrt{\pi}}\exp\left(-\frac{y^2}{4t}\right),\ L^{-1}\left\{\frac{e^{-y\sqrt{q}}}{q}\right\}=erfc\left(\frac{y}{2\sqrt{t}}\right),$
$L^{-1}\left\{\frac{e^{-y\sqrt{q+a}}}{q-b}\right\}=\psi\left(y,t;a,b\right).$

$$(A2)\ \psi\left(y,t;a,b\right)=\frac{e^{bt}}{2}\left[e^{-y\sqrt{a+b}}erfc\left(\frac{y}{2\sqrt{t}}-\sqrt{\left(a+b\right)t}\right)\right.$$
$$\left.+e^{y\sqrt{a+b}}erfc\left(\frac{y}{2\sqrt{t}}+\sqrt{\left(a+b\right)t}\right)\right].$$

(A3) $L^{-1}\left\{qF(q)\right\}=f'(t)+\delta(t)f(0)\ if\ L^{-1}\left\{F(q)\right\}=f(t)$ ($\delta(\cdot)$ is the Dirac delta function).

(A4) $L^{-1}\left\{\frac{1}{(q+b)\sqrt{q+a}}\right\}=\frac{e^{-bt}}{\sqrt{a-b}}erf\left(\sqrt{\left(a-b\right)t}\right),\ L^{-1}\left\{\frac{1}{\sqrt{q}}\right\}=\frac{1}{\sqrt{\pi t}}.$

(A5) $L^{-1}\left\{\frac{\sqrt{q+a}}{q+b}\right\}=\frac{e^{-at}}{\sqrt{\pi t}}+\frac{e^{-bt}}{\sqrt{a-b}}erf\left(\sqrt{\left(a-b\right)t}\right)=\phi\left(t;a,b\right).$

$$(A6)\ \int_0^t\frac{1}{\sqrt{s}}\exp\left(-\frac{y^2}{4s}-as\right)\mathrm{d}s=\frac{\sqrt{\pi}}{2\sqrt{a}}\left\{e^{-y\sqrt{a}}erfc\left(\frac{y}{2\sqrt{t}}-\sqrt{at}\right)\right.$$
$$\left.-e^{y\sqrt{a}}erfc\left(\frac{y}{2\sqrt{t}}+\sqrt{at}\right)\right\}.$$

$$(A7)\ \int_0^t\frac{1}{s\sqrt{s}}\exp\left(-\frac{y^2}{4s}-as\right)\mathrm{d}s=\frac{\sqrt{\pi}}{y}\left\{e^{-y\sqrt{a}}erfc\left(\frac{y}{2\sqrt{t}}-\sqrt{at}\right)\right.$$
$$\left.-e^{y\sqrt{a}}erfc\left(\frac{y}{2\sqrt{t}}+\sqrt{at}\right)\right\}.$$

$$(A8)\ \int_0^\infty e^{-p^2s^2-\frac{q^2}{s^2}}\cos\left(a^2s^2+\frac{b^2}{s^2}\right)\mathrm{d}s=\frac{\sqrt{\pi}}{2^4\sqrt{p^4+a^4}}e^{-2c\cos(\alpha+\beta)}$$
$$\times\cos\left[\alpha+2c\sin\left(\alpha+\beta\right)\right].$$

$$(A9)\ \int_0^\infty e^{-p^2s^2-\frac{q^2}{s^2}}\sin\left(a^2s^2+\frac{b^2}{s^2}\right)\mathrm{d}s=\frac{\sqrt{\pi}}{2^4\sqrt{p^4+a^4}}e^{-2c\cos(\alpha+\beta)}$$

$$\times \sin\left[\alpha + 2c \sin\left(\alpha + \beta\right)\right].$$

where $\alpha = \frac{1}{2}\mathrm{arctg}\left(\frac{\mathrm{a}^2}{\mathrm{p}^2}\right)$, $\beta = \frac{1}{2}\mathrm{arctg}\left(\frac{\mathrm{b}^2}{\mathrm{q}^2}\right)$, and $c = \sqrt[4]{(p^4 + a^4)\,(q^4 + b^4)}$.

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Nehad Ali Shah and Asim Naseem Abdus Salam School of Mathematical Sciences
GC University Lahore, Pakistan

Azhar Ali Zafar
GC University
Department of Mathematics
Lahore, Pakistan
azharalizafar@gmail.com

Constantin Fetecau
Academy of Romanian Scientists
Bucuresti 050094, Romania