Dedicated to the memory of Professors Nicolae Popescu and Constantin Vraciu

INFINITE ALGEBRAIC EXTENSIONS OF DEDEKIND STRUCTURES

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Communicated by Alexandru Zaharescu

A Dedekind structure on a (commutative) field K is a pair (D, K), where D is a Dedekind domain and K is its quotient field. Let L/K be an arbitrary algebraic extension of K and let D_L be the integral closure of D in L. We give necessarily and sufficient conditions such that (D_L, L) is a Dedekind structure on L. As an application we also give necessarily and sufficient conditions such that the intersection of an arbitrary set of valuation rings in a Galois infinite extension L/K, which contain D, is a Dedekind domain. This is in a close relation with the main theorem of [6].

AMS 2020 Subject Classification: Primary 13F05, 13F99, 13B99; Secondary 13F30, 11R04, 11R99.

Key words: Dedekind domains, infinite algebraic extensions, algebraic number fields.

1. INTRODUCTION

Let K be a commutative field and let D be a Dedekind domain ([1], [4], [5], [7]) such that $K = Q(D)$ is its quotient field. We say that the pair (D, K) is a *Dedekind (arithmetical) structure*, or a *Dedekind pair* on K. Let L/K be an arbitrary algebraic extension of K and let D_L be the integral closure of D in L. For a fixed prime ideal $\mathfrak{p} \in Spec(D)$ and a finite extension K'/K, $K' \subset L$, we denote by $D_{K'}$ the integral closure of D in K' (it is a Dedekind domain [7]) and let

$$\mathfrak{p}D_{K'} = \mathfrak{P}_1^{e_{\mathfrak{p},1}} \cdot ... \cdot \mathfrak{P}_{s_{\mathfrak{p},K'}}^{e_{\mathfrak{p},s_{\mathfrak{p},K'}}},$$

be the decomposition of the ideal $\mathfrak{p}D_{K'}$ into powers of distinct prime ideals $\mathfrak{P}_1, ..., \mathfrak{P}_{s_{\mathfrak{p},K'}} \in Spec(D_{K'})$. We also write

$$e_{\mathfrak{p},K'} = \max\left\{e_{\mathfrak{p},j} : j = 1, 2, ..., s_{\mathfrak{p},K'}\right\}$$

To any fixed prime ideal $\mathfrak{p} \in Spec(D)$ we associate two invariants which depend only on the Dedekind pair (D, K) and on the algebraic extension L/K :

$$e_{\mathfrak{p}}^{L} = \sup_{K'} \left\{ e_{\mathfrak{p},K'} : K \subset K' \subset L, [K' : K] < \infty \right\} \in \mathbb{N}^{*} \cup \{\infty\} \tag{1.1}$$

and

$$s_{\mathfrak{p}}^{L} = \sup_{K'} \left\{ s_{\mathfrak{p},K'} : K \subset K' \subset L, [K' : K] < \infty \right\} \in \mathbb{N}^{*} \cup \{\infty\} \tag{1.2}$$

The main result of this note is the following theorem.

THEOREM 1. *With the above notation, definitions and hypotheses, the pair (D_L, L) is a Dedekind pair on L if and only if for each prime ideal $\mathfrak{p} \in Spec(D)$, $e_{\mathfrak{p}}^{L}$ and $s_{\mathfrak{p}}^{L}$ are finite natural numbers.*

We shall prove this result in Section 2.

In the following we freely use some notation and results from the classical valuation theory (see for instance [1], [2], [5]).

Let (D, K) be a Dedekind pair, let $\mathcal{F}$ be a nonempty family of distinct valuation rings V in an infinite Galois extension L/K such that $D \subset V$, and let $A = \bigcap_{V \in \mathcal{F}} V$ be the intersection ring of all the valuation ring of $\mathcal{F}$. For a valuation ring V we denote $m(V)$ its maximal ideal. The following result is a slight generalization of the main result of [6].

THEOREM 2. *With the above notation, definitions and assumptions, the ring A is a Dedekind domain in L (i.e. the pair (A, L) is a Dedekind pair) if and only if each $V \in \mathcal{F}$ is a discrete valuation ring and, for each fixed prime ideal $\mathfrak{p} \in Spec(D)$, there are only a finite number of valuation rings $V \in \mathcal{F}$ such that $\mathfrak{p} \subset m(V)$.*

One can find a proof of this theorem in Section 3.

Our proof of Theorem 2 can be also considered as a simpler and alternative proof of the main result of [6], with the restriction that L/K is a Galois extension.

In order to construct infinite algebraic extensions L such that (D_L, L) is a Dedekind structure on L, one can use a fundamental paper of Hasse [3] for the particular case $K = \mathbb{Q}$, the field of rational numbers, $D = \mathbb{Z}$, the ring of integer numbers and L a special infinite algebraic number fields. In Section 4 we also give a simple example of such a construction, without using the more sophisticated results of Hasse [3].

2. SOME GENERAL AUXILIARY RESULTS

In this Section we put together some general results on commutative algebra and elementary number theory, rather belonging to mathematical folklore.

In the following all the rings are domains, *i.e.* commutative unitary rings without zero divisors.

Lemma 1 ([4], Ch. VII). *Let $A \subset B$ be an integral extension of domains and let $\mathfrak{b} \neq (0)$ be a nonzero ideal of B. Then the ideal $A \cap \mathfrak{b}$ is a nonzero ideal in A. Moreover, A is a field if and only if B is a field. In particular, if $\mathfrak{p}$ is a nonzero prime ideal of B and if $\mathfrak{q} = \mathfrak{p} \cap A$, then $\mathfrak{p}$ is a maximal ideal of B if and only if $\mathfrak{q}$ is a maximal ideal of A.*

Definition 1. Let $A \subset B$ be an arbitrary integral extension of domains. A nonzero ideal $\mathfrak{b}$ of B is said to be a relative prime (maximal) ideal over A if for any finitely generated extension C of A, $C \subset B$, the ideal $\mathfrak{b} \cap C$ is a prime (maximal) ideal of C.

Lemma 2. *Let $A \subset B$ be an arbitrary integral extension of commutative rings. Then a nonzero ideal $\mathfrak{b}$ of B is a prime (maximal) ideal if and only if $\mathfrak{b}$ is a relative prime (maximal) ideal of B (over A).*

Proof. a) We prove that $\mathfrak{b}$ is a nonzero prime ideal of $B \Leftrightarrow \mathfrak{b}$ is relative prime of B over A. The nontrivial implication is "$\Leftarrow$." Let $\mathfrak{b}$ be a relative prime nonzero ideal of B. Let $x, y \in B$ such that $xy \in \mathfrak{b}$ and let $C = A[x, y] \subset B$ be the A-subalgebra of B generated by x and y. Since $\mathfrak{b} \cap C$ is a prime nonzero (see Lemma 1) ideal of C, and since $xy \in \mathfrak{b} \cap C$, we see that $x \in \mathfrak{b}$ or $y \in \mathfrak{b}$.

b) We prove that $\mathfrak{b}$ is a maximal ideal of $B \Leftrightarrow \mathfrak{b}$ is relative maximal of B over A.

"$\Rightarrow$": Assume that $\mathfrak{b}$ is a (nonzero) maximal ideal in B and let $C \subset B$ be a finitely generated A-subalgebra of B. Since $C/(\mathfrak{b} \cap C) \subset B/\mathfrak{b}$ is an integral extension of commutative rings, and since $B/\mathfrak{b}$ is a field, one sees that $C/(\mathfrak{b} \cap C)$ is also a field (Lemma 1), *i.e.* $\mathfrak{b} \cap C$ is a maximal ideal of C.

"$\Leftarrow$": We suppose now that $\mathfrak{b}$ is a relative maximal nonzero ideal of B such that $\mathfrak{b}$ is not a maximal ideal of B. Since $\mathfrak{b} \neq B$ ($1 \notin \mathfrak{b} \cap A$), let us take $z \in B$, $z \notin \mathfrak{b}$ and $\mathfrak{b} + Bz \neq B$. Let us take $C = A[z] \subset B$. Since $\mathfrak{c} = \mathfrak{b} \cap C$ is a maximal ideal of C, and since $z \notin \mathfrak{b}$, one sees that $\mathfrak{c} \subsetneq \mathfrak{c} + Cz \subseteq C$, i.e. $\mathfrak{c} + Cz = C$. Hence $1 = \beta + \gamma z$, $\beta \in \mathfrak{c} \subset \mathfrak{b}$, $\gamma \in C \subset B$. Thus $\mathfrak{b} + Bz = B$, a contradiction. Therefore $\mathfrak{b}$ must be a maximal ideal of B. $\square$

Proposition 1. *Let (D, K) be a Dedekind structure on a field K and let L/K be an infinite algebraic extension of K. Then any nonzero prime ideal of D_L (the integral closure of D in L) is a maximal ideal.*

Remark 1. Proposition 1 can be generalized as follows. We say that a domain D has property $[\mathcal{M}]$ if any nonzero prime ideal of D is a maximal ideal. Let E be a nonempty union of a totally ordered by inclusion family $\{E_\lambda\}_{\lambda \in \Lambda}$

of domains E_λ which are integral extensions of a given domain D. Then E has property $[\mathcal{M}]$ if and only if any E_λ, $\lambda \in \Lambda$, has property $[\mathcal{M}]$. Moreover, if D has property $[\mathcal{M}]$, if any $E_\lambda, \lambda \in \Lambda$ is an integral extension of D and if $\{E_\lambda\}_{\lambda\in\Lambda}$ is totally ordered by inclusion, then $E = \bigcup_{\lambda\in\Lambda} E_\lambda$ also has property $[\mathcal{M}]$.

The following result is very known in the mathematical folklore (see for instance, Cohen I. S., *Commutative rings with restricted minimum conditions*, Duke Math. J. **17**, no.1 (1950), 27–42).

LEMMA 3. *Let A be a unitary commutative ring such that any prime ideal of A is finitely generated. Then A is a Noetherian ring.*

In Section 4 we need the following two elementary results.

LEMMA 4. *Let F be a finite field of characteristic $p \neq 2$. Then F contains at least one element γ which is not a square in F (the index of ${F^*}^2$ in F^* is 2).*

As a consequence of this result we obtain a useful lemma.

LEMMA 5. *Let A be a unitary commutative ring and let $\mathfrak{m}_1, \mathfrak{m}_2, ..., \mathfrak{m}_k$ be k distinct nonzero maximal ideals of A such that all the fields $A/\mathfrak{m}_j$, $j = 1, 2, ..., k$ are finite fields with characteristic $p \neq 2$. Then there exists at least one element $a \in A$, which is not a square modulo $\mathfrak{m}_j$ for each $j = 1, 2, ..., k$.*

Proof. We apply Lemma 4 and the Chinese Reminder Theorem [1]. □

3. THE PROOF OF THEOREM 1

"$\Longrightarrow$". Let us assume that (D_L, L) is an arithmetical Dedekind structure, *i.e.* that D_L, the integral closure of D in L, is a Dedekind domain. Let $\mathfrak{p} \in Spec(D)$ be a fixed prime ideal of D and let

$$\mathfrak{p}D_L = \mathfrak{Q}_1^{e_1} \cdot ... \cdot \mathfrak{Q}_h^{e_h}, \tag{3.1}$$

be the decomposition of the ideal $\mathfrak{p}D_L$ into a product of powers of distinct prime ideals $\mathfrak{Q}_1, \mathfrak{Q}_2, ..., \mathfrak{Q}_h$ of D_L. We prove that

$$e_{\mathfrak{p}}^L \leq e = \max\{e_1, e_2, ..., e_h\}, \text{ and } s_{\mathfrak{p}}^L \leq h,$$

where the invariants $e_{\mathfrak{p}}^L$ and $s_{\mathfrak{p}}^L$ were defined by formulas (1.1) and (1.2) in Introduction.

Let K'/K be an arbitrary finite extension of K, $K' \subset L$. Let $D_{K'}$ be the integral closure of D in K' and let

$$\mathfrak{p}D_{K'} = \mathfrak{S}_1^{t_1} \cdot ... \cdot \mathfrak{S}_l^{t_l}, \tag{3.2}$$

be the decomposition of the ideal $\mathfrak{p}D_{K'}$ into a product of powers of distinct prime ideals $\mathfrak{S}_1, \mathfrak{S}_2, ..., \mathfrak{S}_l$ of $D_{K'}$ (here $D_{K'}$ is a Dedekind domain since $[K' : K] < \infty$ [7]). Let $t = \max\{t_1, t_2, ..., t_l\}$ and let

$$\mathfrak{S}_j D_L = \mathfrak{P}_{j,1}^{m_{j,1}} \cdot \mathfrak{P}_{j,2}^{m_{j,2}} \cdot ... \cdot \mathfrak{P}_{j,u_j}^{m_{j,u_j}}, j = 1, 2, ..., l \tag{3.3}$$

be the decomposition of the ideal $\mathfrak{S}_j D_L$ into a product of powers of distinct prime ideals $\mathfrak{P}_{j,1}^{m_{j,1}}, \mathfrak{P}_{j,2}^{m_{j,2}}, ..., \mathfrak{P}_{j,u_j}^{m_{j,u_j}}$ of D_L. Hence (see (3.1), (3.2), (3.3))

$$\mathfrak{p}D_L = \mathfrak{Q}_1^{e_1} \cdot ... \cdot \mathfrak{Q}_h^{e_h} = \prod_{j=1}^{l} \prod_{i=1}^{u_j} \mathfrak{P}_{j,i}^{m_{j,i}t_j}. \tag{3.4}$$

Since D_L is a Dedekind domain, one sees that the prime ideals $\mathfrak{P}_{j,i}$ $j = 1, 2, ..., l$, $i = 1, 2, ..., u_j$, are exactly the prime ideals $\mathfrak{Q}_1, \mathfrak{Q}_2, ..., \mathfrak{Q}_h$. It is clear now that $m_{j,i}t_j \le e$, in particular $t_j \le e$ for any $j = 1, 2, ..., l$. Thus $e_{\mathfrak{p},K'} = t \le e$ for any $K' \subset L$, $K \subset K'$, $[K' : K] < \infty$. Hence

$$e_{\mathfrak{p}}^L = \sup_{K'} \left\{e_{\mathfrak{p},K'} : K \subset K' \subset L, [K' : K] < \infty\right\} \le e.$$

Moreover, $s_{\mathfrak{p},K'} = l \le h$ for any $K' \subset L$, $K \subset K'$, $[K' : K] < \infty$. Hence

$$s_{\mathfrak{p}}^L = \sup_{K'} \{s_{\mathfrak{p},K'} : K \subset K' \subset L, [K' : K] < \infty\} \le h.$$

Since e and h depend only on $\mathfrak{p}$ and L, we see that $e_{\mathfrak{p}}^L < \infty$ and $s_{\mathfrak{p}}^L < \infty$ for any $\mathfrak{p} \in Spec(D)$.

"$\Longleftarrow$". Assume now that $e_{\mathfrak{p}}^L < \infty$ and $s_{\mathfrak{p}}^L < \infty$ for any $\mathfrak{p} \in Spec(D)$. Proposition 1 says that any nonzero prime ideal of D_L is a maximal ideal.

It remains to prove that D_L is a Noetherian ring. For this it is sufficient to prove that any nonzero prime ideal $\mathfrak{P} \in Spec(D_L)$ is finitely generated (see Lemma 3). Then $\mathfrak{p} = \mathfrak{P} \cap D \neq (0)$ is a nonzero maximal ideal in D.

Since $e_{\mathfrak{p}}^L < \infty$ and $s_{\mathfrak{p}}^L < \infty$, there exists a finite extension $K'' \supset K$, $[K'' : K] < \infty$, $K'' \subset L$, such that $e_{\mathfrak{p}}^L = e_{\mathfrak{p},K''}$ and $s_{\mathfrak{p}}^L = s_{\mathfrak{p},K''}$. Let $\mathfrak{P}'' = \mathfrak{P} \cap D_{K''} \in Spec(D_{K''})$, where $D_{K''}$ is the integral closure of D in K''. Since $\mathfrak{P}''$ does not ramify and does nor split in any finite extension of K'' contained in L, we see that $\mathfrak{P}''D_L = \mathfrak{P}$. Since $D_{K''}$ is a Dedekind domain, $\mathfrak{P}''$ is a finitely generated ideal of $D_{K''}$. Hence $\mathfrak{P}$ is also a finitely generated ideal of D_L, *i.e.* the pair (D_L, L) is a Dedekind structure on L, and the proof of Theorem 1 is completed.

Corollary 1. *Let (D, K) be a Dedekind pair, L/K be an infinite algebraic extension of K and let D_L be the integral closure of D in L. Then the following statements are equivalent:*

a) *(D_L, L) is a Dedekind structure on L.*

b) $e^L_{\mathfrak{p}} < \infty$ *and* $s^L_{\mathfrak{p}} < \infty$ *for any* $\mathfrak{p} \in Spec(D_L)$.
c) D_L *is a Noetherian ring.*
d) *Any ideal* $\mathfrak{Q} \in Spec(D_L)$ *is finitely generated.*

4. THE PROOF OF THEOREM 2

Let us recall (see Introduction) that $A = \cap_{V\in\mathcal{F}} V$ is the intersection of all the distinct valuation rings V in an infinite Galois extension L/K, of an arbitrary nonempty family $\mathcal{F}$ such that $D \subset V$ for any $V \in \mathcal{F}$, for a fixed Dedekind domain D in $K = Q(D)$. For any $V \in \mathcal{F}$, we denote $m(V)$ the maximal ideal of V and let $\mathfrak{p}_V = m(V) \cap D$, which is a nonzero prime ideal of D (the proof follows the same ideas as in the proof of Lemma 1). Since $1 \notin \mathfrak{p}_V$, we see that $\mathfrak{p}_V \in Spec(D)$ and the localization $D_{\mathfrak{p}_V}$ of D relative to $\mathfrak{p}_V$ is contained in V. Since D is a Dedekind domain, $D_{\mathfrak{p}_V}$ is a discrete valuation ring and so $V \cap K = D_{\mathfrak{p}_V}$ (see [2] and [7]). But we do not know that any V of $\mathcal{F}$ is an extension of a discrete valuation ring $D_{\mathfrak{q}}$ with $\mathfrak{q} \in Spec(D)$. Let $\mathcal{M} = \{\mathfrak{q} \in Spec(D) : \exists V \in \mathcal{F}, \text{ with } V \cap K = D_{\mathfrak{q}}\}$. Now we substitute the Dedekind pair (D, K) with the Dedekind pair $(D_{\mathcal{M}}, K)$, where $D_{\mathcal{M}}$ is the quotient ring of D w.r.t. the multiplicative system $\mathcal{S} = D \setminus \cup_{\mathfrak{q}\in\mathcal{M}} \mathfrak{q}$. Since $D = \cap_{\mathfrak{p}\in Spec(D)} D_{\mathfrak{p}}$ (see [1] or [7]), we see that $D_{\mathcal{M}} = \cap_{\mathfrak{q}\in\mathcal{M}} D_{\mathfrak{q}}$ is also a Dedekind domain with fewer prime ideals. Now, not all the valuation rings W of L, which extend a fixed $D_{\mathfrak{q}}$, $\mathfrak{q} \in \mathcal{M}$, belong to $\mathcal{F}$. Let $\mathcal{F}^*$ be the set of all valuation rings W of L, which extend at least one $D_{\mathfrak{q}}, \mathfrak{q} \in \mathcal{M}$. It is clear that $\mathcal{F}^* \supseteq \mathcal{F}$ and $A = \cap_{V\in\mathcal{F}} V \supseteq A^* = \cap_{W\in\mathcal{F}^*} W$. But A^* is the integral closure of $D_{\mathcal{M}}$ in L (see [2]). We apply Theorem 1 with a valuation language and find that A^* is a Dedekind ring if and only if any $W \in \mathcal{F}^*$ is a discrete valuation ring and the set $S^*_{\mathfrak{q}} = \{W \in \mathcal{F}^* : W \cap K = D_{\mathfrak{q}}\}$ is finite for any $\mathfrak{q} \in \mathcal{M}$. It remains to prove:

a) A and $A^*$are simultaneously Dedekind domains,
b) $\mathcal{F}$ and $\mathcal{F}^*$ simultaneously contain only discrete valuation rings, and
c) the sets $S_{\mathfrak{q}} = \{V \in \mathcal{F} : V \cap K = D_{\mathfrak{q}}\}$ and $S^*_{\mathfrak{q}}$ (defined above) are simultaneously finite sets for each $\mathfrak{q} \in \mathcal{M}$, whenever $\mathcal{F}$ contains only discrete valuation rings.

Let us denote $G = Gal(L/K)$ the Galois group of L over K. Since G acts transitively on the set of valuation rings $S^*_{\mathfrak{q}}$ for each $\mathfrak{q} \in \mathcal{M}$ (see [2]) and since the image of a discrete valuation ring through a $\sigma \in G$ is also a discrete valuation ring, the statement b) is clear. Let us assume that $A^*$is a Dedekind domain. Since A is a quotient ring of A^*, we see that also A is a Dedekind domain. Suppose now that A is a Dedekind domain. Since any prime ideal $\mathfrak{p} \in Spec(A^*)$ is of the form $\mathfrak{p} = \sigma(\mathfrak{q} \cap A^*)$, where $\mathfrak{q} \in Spec(A)$ and $\sigma \in G$, we see that A^* is a Noetherian ring (see Lemma 3) and that any prime ideal $\mathfrak{p} \in Spec(A^*)$ is a maximal ideal of A^*. Since $A^*$is an intersection of valuation rings, it is integrally closed in L. So A^* is also a Dedekind domain and the statement a) is proved.

Let us prove c). Since $S_\mathfrak{q} \subseteq S_\mathfrak{q}^*$, the finiteness of $S_\mathfrak{q}^*$ obviously implies the finiteness of $S_\mathfrak{q}$. We assume that $S_\mathfrak{q}$ is a finite set for any $\mathfrak{q} \in \mathcal{M}$. Theorem 1 says that A is a Dedekind domain. Thus, the statement a) implies that A^* is a Dedekind domain and, applying again Theorem 1, we find that $S_\mathfrak{q}^*$ is a finite set for any $\mathfrak{q} \in \mathcal{M}$, *i.e.* we finished to prove c) and Theorem 2 itself.

5. AN "EXOTIC" EXAMPLE

Let $p_1 = 3, p_2 = 5, p_3 = 7, ..., p_n, ...$ be the increasing sequence of all prime numbers $p > 2$, let $K_0 = \mathbb{Q}$ be the field of rational numbers and let $D = \left\{\frac{a}{2^s} : a \in \mathbb{Z}, s \in \mathbb{N}\right\}$ be the Dedekind domain obtained as a ring of quotients of $\mathbb{Z}$ with respect to the multiplicative system $S = \left\{1, \frac{1}{2}, \frac{1}{2^2}, ...\right\}$. We choose a root α_1 of the irreducible polynomial $f_1(X) = X^2 + 1$ over the finite field $D/Dp_1 \simeq \mathbb{Z}/p_1\mathbb{Z}$ and let $K_1 = K_0[\alpha_1]$. Since $f_1(X)$ is irreducible over D/Dp_1, p_1D does not ramify and does not split in K_1 (see [5]) Suppose that we just constructed a tower of fields in $\mathbb{C}$, the field of complex numbers:

$$K_0 \subset K_1 \subset K_2 \subset ... \subset K_n,$$

such that $[K_i : K_{i-1}] = 2$, $i = 1, 2, ..., n$ and the ideals $p_1D_j, p_2D_j, ..., p_jD_j$ (D_j is the integral closure of D in K_j) do not ramify and do not split in D_j, $j = 1, 2, ..., n$. Let $\mathcal{M}_n$ be the set of all maximal ideals $\mathfrak{P}_1, \mathfrak{P}_2, ..., \mathfrak{P}_h$ which may appear in the decompositions of the ideals $p_1D_n, p_2D_n, ..., p_{n+1}D_n$ in D_n (in fact only $p_{n+1}D_n$ might split or ramify in D_n). Let us choose now b_n in D_n such that b_n is not a square modulo all $\mathfrak{P}_j$, $j = 1, 2, ..., h$ (see Lemma 5). Let $\alpha_n = \sqrt{b_n} \in \mathbb{C}$ and let $K_{n+1} = K_n[\alpha_n]$ be the corresponding quadratic extension of K_n. Since the polynomial $X^2 - b_n$ is irreducible modulo $\mathfrak{P}_j$, $j = 1, 2, ..., h$, the ideals $\mathfrak{P}_1, \mathfrak{P}_2, ..., \mathfrak{P}_h$ do not ramify and do not split in D_{n+1}, the integral closure of D in K_{n+1} (see [5], the fundamental equality and the general rule of decomposition of a prime ideal in algebraic number field extensions). Let $L = \cup_{n=0}^{\infty} K_n \subset \mathbb{C}$ and let D_L be the integral closure of D in L. It is clear that $[L : \mathbb{Q}] = 2^\infty$ and $e_{p_i}^L < \infty$, $s_{p_i}^L < \infty$ for any $i = 1, 2, ...$ because p_iD_i does not ramify or decompose in any ring D_j with $j \geq i$. We apply Theorem 1 and find that the pair (D_L, L) is an infinite Dedekind structure on L.

This means that the infinite extension $L/\mathbb{Q}$ gives rise to an infinite extension of Dedekind structures $(D, \mathbb{Q}) \subset (D_L, L)$. This fact is not true in general. For instance, if A is the ring of all the algebraic integers, *i.e.* the integral closure of $\mathbb{Z}$ in the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ in $\mathbb{C}$, then the pair $(A, \overline{\mathbb{Q}})$ is not a Dedekind structure because A is not Noetherian. The importance of Theorem 1 consists in the possibility to construct a large class of infinite extensions of Dedekind structure $(D, K) \subset (D_L, L)$ not only with algebraic numbers as in the classical use of the main result of Hasse [3].

Acknowledgments. We are grateful to the referee for his patience in carefully reading this paper and for his useful advises which permitted us to improve the initial version of it.

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Received July 21, 2018

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