QUASICONTINUITY AND XING PRINCIPLE FOR m-SUBHARMONIC FUNCTIONS WITH RESPECT TO m-POSITIVE CLOSED CURRENT

JAWHAR HBIL

Communicated by Mihai Putinar

The aim of this paper is to introduce and study some classes of m-subharmonic functions ($\mathcal{E}^T_{p,m}(\Omega)$ and $\mathcal{F}^T_m(\Omega)$) where the operator $(dd^c.)^q \wedge T$ is well defined for a given m-positive closed current T of bidimension (q,q) defined on an m-hyperconvex domain Ω of $\mathbb{C}^n$. We prove first the quasicontinuity, with respect to a new capacity defined by the Monge-Ampere measure, of all m-subharmonic function that belong either to $\mathcal{E}^T_{p,m}(\Omega)$ or $\mathcal{F}^T_m(\Omega)$. This will allow us to prove that the well-known Xing comparison principle is valid on to the class $\mathcal{E}^T_{p,m}(\Omega)$.

AMS 2020 Subject Classification: 32U05, 32U15, 32U20, 32W20.

Key words: plurisubharmonic function, currents, Capacity, Monge-Ampére operator.

1. INTRODUCTION

Let Ω be an m-hyperconvex domain of $\mathbb{C}^n$, that means it is open, bounded, connected and there exists a negative m-subharmonic function h such that for all $c < 0$, the set $\{z \in \Omega,\ h(z) < c\}$ is relatively compact in Ω and T be an m-positive closed current of bidimension (q,q) $(1 \leq q \leq m \leq n)$. In 2013, Lu [15] studied the m-Hessian operator in the set of m-subharmonic functions. He introduced the classes $\mathcal{E}_{0,m}(\Omega)$, $\mathcal{F}_m(\Omega)$ and $\mathcal{E}_{p,m}(\Omega)$ (where $p \geq 1$) which coincide, in the case $m = n$, with the well-known classes introduced by Cegrell in [3, 4]. After proving that the Hessian operator is well defined on those classes, Lu shows the continuity of this operator under decreasing sequence and that all functions belong to those classes are quasicontinuous with respect to the capacity Cap_m constructed by the Hessian measure.
In this paper, we study the Monge-Ampère operator with respect to an m-positive closed current in the set of m-subharmonic functions. So it contains three sections organized as follows:

In the first section, we recall the definition of the m-positivity of currents and m-subharmonic functions defined on Ω as well as their basic properties.

Then we associate to every m-positive closed current T the classes $\mathcal{F}_m^T(\Omega)$ and $\mathcal{E}_{p,m}^T(\Omega)$. In the classical case $T = (dd^c|z|)^{n-m}$ those classes coincide with Lu classes [15] and with Cegrell [3, 4] classes when $T = 1$ and $m = n$. Following Cegrell [3] technics and Lu [15] convergence theorem we prove first that the Monge-Ampère operator is well defined on $\mathcal{E}_{p,m}^T(\Omega)$ (See Theorem 4). Then we are interested to an approximation theorem cited in the case $m = n$ by Hai and Dung [11] but with incomplete proof. Here we give a complete proof in the general case $1 \leq m \leq n$.

In the second section, we introduce a new capacity $C_{m,T}$ inspired from the Monge-Ampère measure which allow us to study the quasicontinuity of m-subharmonic functions with respect to this capacity. This problem has been studied first by Bedford and Taylor [1] who proved that every bounded plurisubharmonic function is continuous outside a subset of small capacity. This result was extended by Dabbek and Elkhadra [8] with respect to the capacity C_T and by Lu with respect to the capacity Cap_m. In our paper, we give first an estimate of the sub-level $\{u < s\}$ for all $u \in \mathcal{E}_{p,m}^T(\Omega)$ (resp. $u \in \mathcal{F}_m^T(\Omega)$) and $s < 0$. As a consequence of this estimate, we prove that all functions belong to the introduced classes are $C_{m,T}$-quasicontinuous. Namely we prove the following statement:

THEOREM 1. *Let $u \in \mathcal{F}_m^T(\Omega)$ (resp. $u \in \mathcal{E}_{p,m}^T(\Omega)$). Then for all $\varepsilon > 0$, there exists an open subset O_ε such that $C_{m,T}(O_\varepsilon, \Omega) < \varepsilon$ and $u_{|\Omega\setminus O_\varepsilon}$ is continuous.*

The last section is devoted to extend Xing inequality and domination principle to the classes $\mathcal{E}_{m,p}^T(\Omega)$ and $\mathcal{F}_m^T(\Omega)$ using fundamental results proved in Section 2. We essentially extend The Xing comparison principle to the classe $\mathcal{E}_{m,p}^T(\Omega)$. More precisely we prove the following result

THEOREM 2. *Let $0 < p \leq 1$ and $u, v \in \mathcal{E}_p^T(\Omega)$ such that the measure $(dd^c v)^q \wedge T$ has no mass on $(m,T)-$pluripolar sets, then*

$$\int_{\{v<u\}} (dd^c u)^q \wedge T \leq \int_{\{v<u\}} (dd^c v)^q \wedge T$$

2. ENERGY m-SUBHARMONIC CLASSES

In the first part of this section, we recall the notion of m-positivity forms and m-subharmonicity defined by Blocki in [2].

Definition 1. A real form α of bidegree $(1,1)$ in a domain Ω of $\mathbb{C}^n$ is said to be m-positive if at every point of Ω one has

$$\alpha^j \wedge \beta^{n-j} \geq 0, \quad \forall j = 1, \cdots, m$$

where $\beta := dd^c|z|^2$.

Let notice that, in the case $m = n$, the m-positivity notion defined below coincide with the standard notion of positivity. This is not the case when $m < n$ as we can see in the following example.

Example 1. The $(1,1)-$form $\alpha := 2idz_1 \wedge idz_1 + 3id\overline{z}_2 \wedge id\overline{z}_2 - d\overline{z}_3 \wedge id\overline{z}_3$ in $\mathbb{C}^3$ is 2-positive but not positive.

Definition 2. 1. A function $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ is called m-subharmonic if it is subharmonic and

$$dd^c u \wedge \beta^{n-m} \wedge \alpha_1 \wedge \cdots \wedge \alpha_{m-1} \geq 0$$

for all m-positive forms $\alpha_1, \cdots, \alpha_{m-1}$.

2. A current T of bidimension (p,p), with $1 \leq p \leq m$, is called m-positive if

$$\alpha_1 \wedge \cdots \wedge \alpha_p \wedge T \geq 0.$$

for every m-positive $(1,1)$-forms $\alpha_1, \cdots, \alpha_p$.

The class of m-subharmonic (resp. plurisubharmonic) functions in Ω will be denoted by $SH_m(\Omega)$ (resp. $PSH(\Omega)$).

We list below some basic properties of m-subharmonicity. For the proof, one can refers to [15].

Proposition 1. 1. Let $1 \leq p \leq m$. If $\alpha_1, \cdots, \alpha_p$ are m-positive $(1,1)$-forms then $\alpha_1 \wedge \cdots \wedge \alpha_p \wedge \beta^{n-m} \geq 0$.

2. If $u \in \mathcal{C}^2(\Omega)$ then: $u \in SH_m(\Omega)$ if and only if the form $dd^c u$ is m-positive on Ω.

3. If $u \in SH_m(\Omega)$ then the current $dd^c u \wedge \beta^{n-m}$ is m-positive.

4. If $u, v \in SH_m(\Omega)$ then $\lambda u + \mu v \in SH_m(\Omega), \forall \lambda, \mu > 0$.

5. $PSH(\Omega) = SH_n(\Omega) \subsetneq \cdots \subsetneq SH_m(\Omega) \subsetneq \cdots \subsetneq SH_1(\Omega) = SH(\Omega)$.

6. If u is m-subharmonic on Ω then the standard regularizations $u * \chi_\epsilon$ are also m-subharmonic on $\Omega_\epsilon := \{x \in \Omega \ / \ d(x, \partial\Omega) > \epsilon\}$.

7. If $(u_i)_j$ is a decreasing sequence of m-subharmonic functions then $u := \lim u_j$ is either m-subharmonic or identically equal to $-\infty$.

Remark 1. 1. The first notion of Definition 2 was introduced by Lu [15] and one can prove, using the third assertion of Proposition 1, that: for all $1 \le s < r \le m$, every $s-$positive current is $r-$positive.

2. Recently Dhouib and Elkhadra [6] gave a new notion of m-positivity of current. This notion generalizes classic positivity defined by Lelong since 1967, but the first statement of this remark statement does not holds.

3. If T is m-positive in the sense of Dhouib and Elkhadra, then the current $T \wedge \beta^{n-m}$ is m-positive in the sense of Lu.

In the hole of this paper, we will use the Definition 2 introduced by Lu.

2.1. The class $\mathcal{E}^T_{p,m}(\Omega)$

Throughout this paper, we denote by T an m-positive closed current of bidimension (q,q) defined on an m-hyperconvex domain Ω of $\mathbb{C}^n$. We introduce the class $\mathcal{E}^T_{0,m}(\Omega)$ associated to T as follows:

$$\mathcal{E}^T_{0,m}(\Omega) := \Big\{\varphi \in SH^-_m(\Omega) \cap L^\infty(\Omega);\ \lim_{z \to \partial\Omega \cap Supp\ T} \varphi(z) = 0, \\ \int_\Omega (dd^c\varphi)^q \wedge T < +\infty\Big\}.$$

Remark 2. 1. In the trivial case $T = 1$, the class $\mathcal{E}^T_{0,m}(\Omega)$ coincides with $\mathcal{E}_{0,m}(\Omega)$ introduced by Lu [15].

2. The class $\mathcal{E}^T_{0,m}(\Omega)$ coincides with Cegrell [3] one when $m = n$ and $T = 1$.

3. If the current T is defined on a neighborhood of Ω, then $\mathcal{E}^T_{0,m}(\Omega)$ contains all bounded functions of $SH^-_m(\Omega)$.

4. A continuous function in $\mathcal{E}^T_{0,m}(\Omega)$ can be seen as a test function. More precisely it is easy to prove that:

$$\mathcal{D}'(\Omega) = \mathcal{E}^T_{0,m}(\Omega) \cap \mathcal{C}(\overline{\Omega}) - \mathcal{E}^T_{0,m}(\Omega) \cap \mathcal{C}(\overline{\Omega}).$$

Assume in the hole of this paper that $\mathcal{E}^T_{0,m}(\Omega) \neq \{0\}$.
In this section, we introduce two new energy classes $\mathcal{E}^T_{p,m}(\Omega)$ and $\mathcal{F}^T_m(\Omega)$, similar to Cegrell's ones and we will prove that the Monge-Ampère operator is well defined on them.

Definition 3. For every real $p \geq 1$ we define $\mathcal{E}^T_{p,m}(\Omega)$ as the set:

$$\mathcal{E}^T_{p,m}(\Omega) := \Big\{ \varphi \in SH^-_m(\Omega);\ \exists\ \mathcal{E}^T_{0,m}(\Omega) \ni \varphi_j \searrow \varphi, \\ \sup_{j\geq 1} \int_\Omega (-\varphi_j)^p (dd^c\varphi_j)^q \wedge T < +\infty \Big\}.$$

When the sequence $(\varphi_j)_j$ associated to φ could be chosen such that

$$\sup_{j\geq 1} \int_\Omega (dd^c\varphi_j)^q \wedge T < +\infty,$$

we say that $\varphi \in \mathcal{F}^T_{p,m}(\Omega)$.

We cite belong useful properties of the different introduced classes. Those properties generalize well known ones proved by Hai and Dung [11] (in the particular case $m = n$) and Cegrell in the case $T = 1$ and $m = n$. So it can be proved using the same technics.

Propertiess. 1. If $\psi \in SH^-_m(\Omega)$ and $\varphi \in \mathcal{E}^T_{0,m}(\Omega)$ then the function $\max(\varphi, \psi) \in \mathcal{E}^T_{0,m}(\Omega)$.

2. The class $\mathcal{E}^T_{0,m}(\Omega)$ is a convex cone.

3. $\mathcal{E}^T_{0,m}(\Omega) \subset \mathcal{F}^T_{p,m}(\Omega) \subset \mathcal{E}^T_{p,m}(\Omega)$.

4. $\mathcal{F}^T_{p_1,m}(\Omega) \subset \mathcal{F}^T_{p_2,m}(\Omega)$ for all $p_2 \leq p_1$.

The following result was proved in the classic case $m = n$ by Dabbek and ElKhadra [8] and will be useful to prove some properties of our classes.

Theorem 3 (See [8]). *Suppose that $u, v \in \mathcal{E}^T_{0,m}(\Omega)$. If $p \geq 1$ then for every $0 \leq s \leq q$ one has*

$$\int_\Omega (-u)^p (dd^c u)^s \wedge (dd^c v)^{q-s} \wedge T \\ \leq D_{s,p} \left(\int_\Omega (-u)^p (dd^c u)^q \wedge T \right)^{\frac{p+s}{p+q}} \left(\int_\Omega (-v)^p (dd^c v)^q \wedge T \right)^{\frac{q-s}{p+q}}$$

where $D_{s,1} = 1$ and $D_{s,p} = p^{\frac{(p+s)(q-s)}{p-1}}$, $p > 1$.

Proof. The inequality can be proved using same technics as in [8] where authors have also proved that $D_{s,1} = e^{\{(1+j)(p-j)\}}$. Here we prove that $D_{s,1}$ can be equal to 1. We will proceed by induction on the dimension of T.

1. Assume that T is of bidimension $(1,1)$. The result is obvious when $s=1$. For $s=0$, using Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}\int_\Omega(-u)dd^cv\wedge T &= \int_\Omega du\wedge d^cv\wedge T\\ &\leq \left(\int_\Omega du\wedge d^cu\wedge T\right)^{\frac{1}{2}}\left(\int_\Omega dv\wedge d^cv\wedge T\right)^{\frac{1}{2}}\\ &= \left(\int_\Omega(-u)dd^cu\wedge T\right)^{\frac{1}{2}}\left(\int_\Omega(-v)dd^cv\wedge T\right)^{\frac{1}{2}}.\end{aligned}$$

This prove the result.

2. Assume now by induction that the result is true for all current T of bidimension (q,q) $(q\leq n-1)$. Let T be an m-positive closed current of bidimension $(q+1,q+1)$. Assume also that $s=q$. Since $dd^cu\wedge T$ is an m-positive closed current of bidimmension (q,q), so we obtain that

$$\begin{aligned}&\int_\Omega(-u)(dd^cu)^q\wedge dd^cv\wedge T=\int_\Omega(-u)(dd^cu)^{q-1}\wedge dd^cv\wedge(dd^cu\wedge T)\\ &\leq\left(\int_\Omega(-u)(dd^cu)^q\wedge(dd^cu\wedge T)\right)^{\frac{q}{q+1}}\left(\int_\Omega(-v)(dd^cv)^q\wedge(dd^cu\wedge T)\right)^{\frac{1}{q+1}}\\ &\times\left(\int_\Omega(-u)(dd^cu)^{q+1}\wedge T\right)^{\frac{q}{q+1}}\\ &\leq\left[\left(\int_\Omega(-v)(dd^cv)^{q+1}\wedge T\right)^{\frac{q}{q+1}}\left(\int_\Omega(-u)(dd^cu)^q\wedge dd^cv\wedge T\right)^{\frac{1}{q+1}}\right]^{\frac{1}{q+1}}.\end{aligned}$$

It follows that

$$\begin{aligned}\int_\Omega(-u)(dd^cu)^q\wedge dd^cv\wedge T &\leq \left(\int_\Omega(-u)(dd^cu)^{q+1}\wedge T\right)^{\frac{q+1}{q+2}}\\ &\times\left(\int_\Omega(-v)(dd^cv)^{q+1}\wedge T\right)^{\frac{1}{q+2}}.\end{aligned}$$

So we can deduce the case $s=0$:

$$\begin{aligned}&\int_\Omega(-u)(dd^cv)^{q+1}\wedge T=\int_\Omega(-u)(dd^cv)^q\wedge(dd^cv\wedge T)\\ &\leq\left(\int_\Omega(-u)(dd^cu)^q\wedge(dd^cv\wedge T)\right)^{\frac{1}{q+1}}\left(\int_\Omega(-v)(dd^cv)^q\wedge(dd^cv\wedge T)\right)^{\frac{q}{q+1}}\\ &\leq\left(\int_\Omega(-u)(dd^cu)^{q+1}\wedge T\right)^{\frac{q+1}{q+2}\times\frac{1}{q+1}}\left(\int_\Omega(-v)(dd^cv)^{q+1}\wedge T\right)^{\frac{1}{(q+1)(q+2)}+\frac{q}{q+1}}\\ &\leq\left(\int_\Omega(-u)(dd^cu)^{q+1}\wedge T\right)^{\frac{1}{q+2}}\left(\int_\Omega(-v)(dd^cv)^{q+1}\wedge T\right)^{\frac{q+1}{q+2}}.\end{aligned}$$

If we assume now that $0 < s < q$, so by using previous inequalities we obtain

$$\int_\Omega (-u)(dd^c u)^s \wedge (dd^c v)^{q-s+1} \wedge T = \int_\Omega (-u)(dd^c u)^s \wedge (dd^c v)^{q-s} \wedge (dd^c v \wedge T)$$
$$\leq \left(\int_\Omega (-u)(dd^c u)^q \wedge (dd^c v \wedge T)\right)^{\frac{s+1}{q+1}} \left(\int_\Omega (-v)(dd^c v)^q \wedge (dd^c v \wedge T)\right)^{\frac{q-s}{q+1}}$$
$$\leq \left[\left(\int_\Omega (-u)(dd^c u)^{q+1} \wedge T\right)^{\frac{q+1}{q+2}} \left(\int_\Omega (-v)(dd^c v)^{q+1} \wedge T\right)^{\frac{1}{q+2}}\right]^{\frac{s+1}{q+1}}$$
$$\times \left(\int_\Omega (-v)(dd^c v)^q \wedge (dd^c v \wedge T)\right)^{\frac{q-s}{q+1}}$$
$$= \left(\int_\Omega (-u)(dd^c u)^{q+1} \wedge T\right)^{\frac{s+1}{q+2}} \left(\int_\Omega (-v)(dd^c v)^{q+1} \wedge T\right)^{\frac{q+1-s}{q+2}}.$$

□

We extend now some properties of $\mathcal{E}_0^T(\Omega)$ to our classes.

Proposition 2. 1. The classes $\mathcal{E}_{p,m}^T(\Omega)$ and $\mathcal{F}_{p,m}^T(\Omega)$ are convex cones.

2. For all $u \in \mathcal{E}_{p,m}^T(\Omega)$ (resp. $\mathcal{F}_{p,m}^T(\Omega)$) and $v \in SH_m^-(\Omega)$, the function $w := \max(u, v)$ is in $\mathcal{E}_{p,m}^T(\Omega)$ (resp. in $\mathcal{F}_{p,m}^T(\Omega)$).

Proof. 1. It is easy to check that for all $\lambda > 0$ and $u \in \mathcal{E}_{p,m}^T(\Omega)$ one has $\lambda u \in \mathcal{E}_{p,m}^T(\Omega)$. So it suffices to prove that $u + v \in \mathcal{E}_{p,m}^T(\Omega)$ for every $u, v \in \mathcal{E}_{p,m}^T(\Omega)$.

Let $(u_j)_j$ and $(v_j)_j$ be two sequences that decrease to u and v respectively as in Definition 3. To estimate

$$\int_\Omega (-u_j - v_j)^p (dd^c(u_j + v_j))^q \wedge T$$

it is enough, by Minkowsky Inequality, to estimate the following terms:

$$\int_\Omega (-u_j)^p (dd^c u_j)^s \wedge (dd^c v_j)^{q-s} \wedge T$$

and

$$\int_\Omega (-v_j)^p (dd^c u_j)^s \wedge (dd^c v_j)^{q-s} \wedge T$$

for all $0 < s < q$. Now by Theorem 3, we can estimate last terms by

$$\int_\Omega (-u_j)^p (dd^c u_j)^q \wedge T \quad \text{and} \quad \int_\Omega (-v_j)^p (dd^c v_j)^q \wedge T.$$

As these sequences are uniformly bounded by the definition of $\mathcal{E}^T_{p,m}(\Omega)$, the first assertion follows.

2. Let $(u_j)_j$ be a sequence that decreases to u as in Definition 3 and take $w_j := \max(u_j, v)$. It is clear that the sequence (w_j) decreases to w. So it's enough to prove that

$$\sup_j \int_\Omega (-w_j)^p (dd^c w_j)^q \wedge T < +\infty.$$

Thanks to Theorem 3, one has

$$\int_\Omega (-w_j)^p (dd^c w_j)^q \wedge T \leq \int_\Omega (-u_j)^p (dd^c w_j)^q \wedge T$$
$$\leq D_{0,p} \left(\int_\Omega (-u_j)^p (dd^c u_j)^q \wedge T \right)^{\frac{p}{p+q}} \left(\int_\Omega (-w_j)^p (dd^c w_j)^q \wedge T \right)^{\frac{q}{p+q}}.$$

Therefore

$$\int_\Omega (-w_j)^p (dd^c w_j)^q \wedge T \leq D_{0,p}^{\frac{p+q}{p}} \int_\Omega (-u_j)^p (dd^c u_j)^q \wedge T.$$

The right-hand side is uniformly bounded because $u \in \mathcal{E}^T_{p,m}(\Omega)$. So the result follows. $\square$

To deal well with the Monge-Ampère operator $(dd^c.)^q \wedge T$ on our classes, we will prove first that this operator is well defined.

THEOREM 4. *Let $u \in \mathcal{E}^T_{p,m}(\Omega)$ and $(u_j)_j$ be a sequence of m-subharmonic functions that decreases to u as in Definition 3. Then the sequence $((dd^c u_j)^q \wedge T))_j$ converges weakly to a positive measure μ and this limit is independent of the choice of the sequence $(u_j)_j$. We set $(dd^c u)^q \wedge T := \mu$.*

Proof. Let $0 \leq \chi \in \mathcal{D}(\Omega)$, $\delta = \sup\{u_1(z);\ z \in Supp\chi\}$, $\varepsilon > 0$ and $u_r(z) := \int_{\mathbb{B}} u_1(z + r_1\xi) dV(\xi)$ where dV is the normalized Lebesgue measure on the unit ball $\mathbb{B}$.

Then there exists $r_1 > 0$ such that $r_1 < dist(\{u_1 < \frac{\delta}{2}\}, \Omega^c)$ and for all $r \leq r_1$ one has

$$\left| \int_\Omega \chi (dd^c u_r)^q \wedge T - \chi (dd^c u_1)^q \wedge T \right| < \varepsilon$$

There exists also $r_2 < r_1$ such that $r_2 < dist(\{u_2 < \frac{\delta}{2}\}, \Omega^c)$ and $\forall r \leq r_2$

$$\left| \int_\Omega \chi (dd^c u_r)^q \wedge T - \chi (dd^c u_2)^q \wedge T \right| < \varepsilon.$$

Thus we construct a sequence $(r_j)_j$ such that $0 < r_j < r_{j-1}$,

$$r_j < dist(\{u_j < \frac{\delta}{2}\}, \Omega^c)$$

and

$$\left|\int_\Omega \chi(dd^c u_{r_j})^q\wedge T-\chi(dd^c u_j)^q\wedge T\right|<\varepsilon$$

where

$$u_{r_j}(z):=\int_{\mathbb{B}} u_j(z+r_j\xi)dV(\xi).$$

The function u_{r_j} is continuous and m-subharmonic on $\{u_j<\frac{\delta}{2}\}$ satisfying $u_j\le u_{r_j}$ on Ω. If we take $\widetilde{u_j}=\max(u_{r_j}+\delta,2u_j)$, then the sequence $(\widetilde{u_j})_j$ decreases to an m-subharmonic function $\widetilde{u}$ and $\widetilde{u_j}\in\mathcal{E}_0^T(\Omega)$ by Proposition 2. Furthermore, the sequence $(\widetilde{u_j})_j$ satisfies

$$\sup_{j\ge1}\int_\Omega(-\widetilde{u_j})^p(dd^c\widetilde{u_j})^q\wedge T<+\infty.$$

So it remains to prove that

$$\lim_{j\to+\infty}\int_\Omega \chi(dd^c\widetilde{u_j})^q\wedge T$$

exists.

Let h be an exhaustion function in $\mathcal{E}_{0,m}^T(\Omega)$. Then

$$\begin{aligned}\int_\Omega(-\widetilde{u})^p(dd^ch)^q\wedge T&=\lim_{j\to+\infty}\int_\Omega(-\widetilde{u_j})^p(dd^ch)^q\wedge T\\ &\le D_{0,p}\sup_{j\ge1}\left(\int_\Omega(-\widetilde{u_j})^p(dd^c\widetilde{u_j})^q\wedge T\right)^{\frac{p}{p+q}}\left(\int_\Omega(-h)^p(dd^ch)^q\wedge T\right)^{\frac{q}{p+q}}<+\infty.\end{aligned}$$

Thanks to Lu [14], the sequence of measures $(dd^c\max(\widetilde{u_j},-k))^q\wedge T$ converges weakly for every k. So it is enough to control

$$\left|\int \chi(dd^c u_{r_j})^q\wedge T-\chi(dd^c\max(\widetilde{u_j},-k))^q\wedge T\right|.$$

Since $\widetilde{u_j}=u_{r_j}+\delta$ on $\{u_j\le\frac{\delta}{2}\}$ and $Supp\chi\subset\{u_j\le\frac{\delta}{2}\}$, then $\widetilde{u_j}$ is continuous on a neighborhood of $Supp\chi$. It follows that

$$\begin{aligned}&\left|\int \chi(dd^c u_{r_j})^q\wedge T-\chi(dd^c\max(\widetilde{u_j},-k))^q\wedge T\right|\\ =&\left|\int \chi(dd^c\widetilde{u_j})^q\wedge T-\chi(dd^c\max(\widetilde{u_j},-k))^q\wedge T\right|\\ =&\left|\int_{\{\widetilde{u}\le-k\}}\chi(dd^c\widetilde{u_j})^q\wedge T+\int_{\{\widetilde{u}>-k\}}\chi(dd^c\widetilde{u_j})^q\wedge T\right.\\ &\left.-\int_{\{\widetilde{u}\le-k\}}\chi(dd^c\max(\widetilde{u_j},-k))^q\wedge T-\int_{\{\widetilde{u}>-k\}}\chi(dd^c\max(\widetilde{u_j},-k))^q\wedge T\right|\end{aligned}$$

$$\begin{aligned}
&\leq \int_{\{\widetilde{u}\leq -k\}}\chi(dd^c\widetilde{u_j})^q\wedge T+\int_{\{\widetilde{u}\leq -k\}}\chi(dd^c\max(\widetilde{u_j},-k))^q\wedge T\\
&\leq \frac{\sup\chi}{k^p}\int_{\{-\widetilde{u}\geq k\}}k^p\left[(dd^c\widetilde{u_j})^q\wedge T+(dd^c\max(\widetilde{u_j},-k))^q\wedge T\right]\\
&\leq \frac{\sup\chi}{k^p}\int_{\Omega}(-\widetilde{u})^p(dd^c\widetilde{u_j})^q\wedge T+(-\max(\widetilde{u_j},-k))^p dd^c\max(\widetilde{u_j},-k))^q\wedge T\\
&\leq C\frac{\sup\chi}{k^p}\sup_{m\geq 1}\int_{\Omega}(-\widetilde{u}_m)^p((dd^c\widetilde{u_m})^q\wedge T.
\end{aligned}$$

The first inequality is due to the fact that $-k<\widetilde{u}\leq\widetilde{u}_j$. For the last inequality one has

$$\int_{\Omega}(\max(\widetilde{u_j},-k))^p(dd^c\max(\widetilde{u_j},-k))^q\wedge T\leq D_{0,p}^{\frac{p+q}{p}}\int_{\Omega}(-\widetilde{u_j})^p(dd^c\widetilde{u_j})^q\wedge T<+\infty$$

and for all $j\geq 0$

$$\begin{aligned}
&\int_{\Omega}(-\widetilde{u})^p(dd^c\widetilde{u}_j)^q\wedge T=\lim_{m\to+\infty}\int_{\Omega}(-\widetilde{u_m})^p(dd^c\widetilde{u}_j)^q\wedge T\\
&\leq D_{0,p}\sup_{m\geq 1}\left(\int_{\Omega}(-\widetilde{u_m})^p(dd^c\widetilde{u_m})^q\wedge T\right)^{\frac{p}{p+q}}\left(\int_{\Omega}(-\widetilde{u}_j)^p(dd^c\widetilde{u}_j)^q\wedge T\right)^{\frac{q}{p+q}}<+\infty.
\end{aligned}$$

No by tending k to $+\infty$, we obtain that the sequence $((dd^cu_{r_j})^q\wedge T)_j$ converges which implies the convergence of the sequence $((dd^cu_j)^q\wedge T)_j$. The result follows.

Let prove now that the convergence is independent of the chosen sequence. For this let $(u_j)_j$ and $(v_j)_j$ be a sequence of m-subharmonic functions that decreases to u as in Definition 3 and take $h\in\mathcal{E}_{0,m}^T(\Omega)$. We have

$$\begin{aligned}
\int_{\Omega}h(dd^cu_j)^q\wedge T&=\int_{\Omega}u_j(dd^cu_j)^{q-1}\wedge dd^ch\wedge T\\
&\geq\int_{\Omega}u(dd^cu_j)^{q-1}\wedge dd^ch\wedge T\\
&=\lim_{k_1\longrightarrow+\infty}\int_{\Omega}v_{k_1}(dd^cu_j)^{q-1}\wedge dd^ch\wedge T.\\
&=\lim_{k_1\longrightarrow+\infty}\int_{\Omega}u_j(dd^cv_{k_1})\wedge(dd^cu_j)^{q-2}\wedge dd^ch\wedge T.\\
&=\lim_{k_1\longrightarrow+\infty}\int_{\Omega}u(dd^cv_{k_1})\wedge(dd^cu_j)^{q-2}\wedge dd^ch\wedge T.\\
&=\lim_{k_1\longrightarrow+\infty}\lim_{k_2\longrightarrow+\infty}\int_{\Omega}v_{k_2}(dd^cv_{k_1})\wedge(dd^cu_j)^{q-2}\wedge dd^ch\wedge T.\\
&\geq\lim_{k_1,k_2,\dots,k_q\longrightarrow+\infty}\int_{\Omega}hdd^cv_{k_1}\wedge\dots dd^cv_{k_q}\wedge T.
\end{aligned}$$

$$\geq \lim_{k\longrightarrow+\infty}\int_\Omega h(dd^cv_k)^q\wedge T.$$

The result follows since $(u_j)_j$ and $(v_j)_j$ have a symmetrical role. $\square$

In the case $p=1$ one has the following convergence theorem which has been established by Cegrell [3] in the case $T=1$ and $m=n$.

Proposition 3. *Let $u\in\mathcal{E}^T_{1,m}(\Omega)$ and $(u_j)_j$ is a decreasing sequence to u as in Definition 3, then the sequence $(\int_\Omega u_j(dd^cu_j)^q\wedge T)_j$ decreases to $\int_\Omega u(dd^cu)^q\wedge T$.*

Proof. Since $u_j\in\mathcal{E}^T_{0,m}(\Omega)$ so one has

$$\int_\Omega u_{j+1}(dd^cu_{j+1})^q\wedge T \quad\leq\int_\Omega u_j(dd^cu_{j+1})^q\wedge T \quad\leq\int_\Omega u_j(dd^cu_j)^q\wedge T$$

This prove that $\left(\int_\Omega u_j(dd^cu_j)^q\wedge T\right)_j$ is a decreasing sequence.

Let us prove that

$$\lim_{j\to+\infty}\int_\Omega u_j(dd^cu_j)^q\wedge T=\int_\Omega u(dd^cu)^q\wedge T.$$

For every $k\geq j$ and $\varepsilon>0$, one has

$$\begin{aligned}&\int_\Omega -u_j(dd^cu_j)^q\wedge T\\ &\leq\int_\Omega -u_j(dd^cu_k)^q\wedge T\\ &=\int_{\{u_j\geq-\varepsilon\}}-u_j(dd^cu_k)^q\wedge T+\int_{\{u_j<-\varepsilon\}}-u_j(dd^cu_k)^q\wedge T\end{aligned}$$

and

$$\begin{aligned}&\int_{\{u_j\geq-\varepsilon\}}-u_j(dd^cu_k)^q\wedge T\\ &=\int_{\{u_j\geq-\varepsilon\}}-\max(u_j,-\varepsilon)(dd^cu_k)^q\wedge T\\ &\leq\left(\int_\Omega-\max(u_j,-\varepsilon)(dd^c\max(u_j,-\varepsilon))^q\wedge T\right)^{\frac{1}{q+1}}\left(\int_\Omega -u_k(dd^cu_k)^q\wedge T\right)^{\frac{q}{q+1}}\\ &\leq\left(\varepsilon\int_\Omega(dd^cu_j)^q\wedge T\right)^{\frac{1}{q+1}}\alpha^{\frac{q}{q+1}}\end{aligned}$$

This goes to 0 when $\varepsilon\to0$. By Theorem 4 we obtain

$$\limsup_{k\to+\infty}\int_{\{u_j<-\varepsilon\}}-u_j(dd^cu_k)^q\wedge T\leq\int_\Omega-u_j(dd^cu)^q\wedge T.$$

Now since $-u_j$ is lower semi-continuous then

$$\liminf_{k\to+\infty}\int_\Omega -u_j(dd^cu_k)^q\wedge T\geq\int_\Omega -u_j(dd^cu)^q\wedge T.$$

Hence for all j,

$$\lim_{k\to+\infty}\int_\Omega u_j(dd^cu_k)^q\wedge T=\int_\Omega u_j(dd^cu)^q\wedge T.$$

It follows that

$$\begin{aligned}&\lim_{j\to+\infty}\int_\Omega u_j(dd^cu_j)^q\wedge T\\ &\geq\lim_{j\to+\infty}\lim_{k\to+\infty}\int_\Omega u_j(dd^cu_k)^q\wedge T=\int_\Omega u(dd^cu)^q\wedge T\\ &\geq\limsup_{k\to+\infty}\int_\Omega u(dd^cu_k)^q\wedge T=\limsup_{k\to+\infty}\lim_{j\to+\infty}\int_\Omega u_j(dd^cu_k)^q\wedge T\\ &\geq\lim_{j\to+\infty}\int_\Omega u_j(dd^cu_j)^q\wedge T.\end{aligned}$$

Thus

$$\lim_{j\to+\infty}\int_\Omega u_j(dd^cu_j)^q\wedge T=\int_\Omega u(dd^cu)^q\wedge T. \tag{2.1}$$

As $(v_k)_k$ decreases to u then $v_k\in\mathcal{E}_1^T(\Omega)$. It follows that

$$\int_\Omega \max(u_j,v_k)(dd^c\max(u_j,v_k))^q\wedge T\geq\int_\Omega u_j(dd^cu_j)^q\wedge T\geq-\alpha. \tag{2.2}$$

Moreover, $(\max(u_j,v_k))_{j\in\mathbb{N}}\subset\mathcal{E}_{0,m}^T(\Omega)$ and decreases to v_k so thanks to Equality (2.1),

$$\lim_{j\to+\infty}\int_\Omega \max(u_j,v_k)(dd^c\max(u_j,v_k))^q\wedge T=\int_\Omega v_k(dd^cv_k)^q\wedge T. \tag{2.3}$$

By tending $j\to+\infty$, Inequality (2.2), Equalities (2.1) and (2.3) give

$$\int_\Omega v_k(dd^cv_k)^q\wedge T\geq\int_\Omega u(dd^cu)^q\wedge T.$$

Thus

$$\liminf_{k\to+\infty}\int_\Omega v_k(dd^cv_k)^q\wedge T\geq\int_\Omega u(dd^cu)^q\wedge T. \tag{2.4}$$

With the same reason, as $(\max(u_j,v_k))_{k\in\mathbb{N}}$ decreases to u_j then

$$\int_\Omega u_j(dd^cu_j)^q\wedge T\geq\limsup_{k\to+\infty}\int_\Omega v_k(dd^cv_k)^q\wedge T.$$

Hence

$$\limsup_{k\to+\infty}\int_\Omega v_k(dd^cv_k)^q\wedge T\leq\int_\Omega u(dd^cu)^q\wedge T. \tag{2.5}$$

The result follows from Inequalities (2.4) and (2.5). $\square$

2.2. The class $\mathcal{F}_m^T(\Omega)$

Let us recall the following well-known theorem.

THEOREM 5 (See [11]). *Suppose that Ω is an $n-$hyperconvex domain, u is a negative plurisubarmonic function and T a positive closed current of bidimension (q,q) such that $\lim\limits_{z\longrightarrow\partial\Omega} u(z)=0$ and $\int_\Omega (dd^cu)^q\wedge T<+\infty$. Then there exists a sequence $u_j\in\mathcal{E}_{0,n}^T(\Omega)$ that decreases to u.*

The previous theorem was proved in the case $m=n$ by Hai and Dung [11, Th.5.1] but their proof was incomplete since they applied a comparison type theorem, which is valid only in the case of bounded plurisubarmonic functions, to the class $\mathcal{F}_n^T(\Omega)$ that contains unbounded functions. So the last part of this section will be devoted to give another proof of this theorem in more general case (See Theorem 6 below).

Definition 4. We say that $u\in\mathcal{F}_m^T(\Omega)$ if there exists a sequence $(u_j)_j\subset\mathcal{E}_{0,m}^T(\Omega)$ which decreases to u such that

$$\sup_j\int_\Omega (dd^cu_j)^q\wedge T<+\infty.$$

We say that $u\in\mathcal{E}_m^T(\Omega)$ if for all $z\in\Omega$ there exist a neighborhood ω of z and a function $v\in\mathcal{F}_m^T(\Omega)$ such that $u=v$ on ω.

As a consequence, for every $p\geq 1$ one has $\mathcal{F}_{p,m}^T(\Omega)\subset\mathcal{F}_m^T(\Omega)\subset\mathcal{E}_m^T(\Omega)$ and $\mathcal{F}_{p,m}^T(\Omega)\subset\mathcal{E}_{p,m}^T(\Omega)$ but there is no relationship between $\mathcal{E}_{p,m}^T(\Omega)$ and $\mathcal{E}_m^T(\Omega)$. Using same technics as in [6], one can prove that the Monge-Ampère operator is well defined on the class $\mathcal{E}_m^T(\Omega)$. Moreover by repeating the proof of Proposition 5.16 in [7] with T instead of $(dd^c|z|^2)^p$ one can show the following statement

PROPOSITION 4. *Let $u^1,...,u^q\in\mathcal{F}_m^T(\Omega)$ and $h\in\mathcal{E}_{0,m}^T(\Omega)$. For all $1\leq s\leq q$ we denote by $(u_j^s)_j$ the corresponding sequence that decrease to u^s as in Definition 4, then*

$$\lim_{j\longrightarrow+\infty}\int_\Omega h dd^cu_j^1\wedge dd^cu_j^2\cdots\wedge dd^cu_j^q\wedge T=\int_\Omega h dd^cu^1\wedge dd^cu^2\cdots\wedge dd^cu^q\wedge T.$$

To establish a proof of Theorem 5, we will prove first some intermediate lemmas.

LEMMA 1. *Let $u,v\in\mathcal{F}_m^T(\Omega)$. Assume that there exists an open subset U of Ω such that $u=v$ near ∂U. Then*

$$\int_U (dd^cu)^q\wedge T=\int_U (dd^cv)^q\wedge T.$$

Proof. Let u_ε and v_ε be the usual regularization of u and v respectively. Choose $U' \subset\subset U$ such that $u = v$ near $\partial U'$. If $\varepsilon > 0$ is small enough, one has $u_\varepsilon = v_\varepsilon$ near $\partial U'$ and if we take $\chi \in \mathcal{D}(U')$ with $\chi = 1$ near $\{u_\varepsilon \neq v_\varepsilon\}$ then $dd^c\chi = 0$ on $\{u_\varepsilon \neq u_\varepsilon\}$. So we obtain

$$\begin{aligned}\int_\Omega \chi(dd^c u_\varepsilon)^q \wedge T &= \int_\Omega u_\varepsilon dd^c\chi \wedge (dd^c u_\varepsilon)^{q-1} \wedge T \\ &= \int_\Omega v_\varepsilon dd^c\chi \wedge (dd^c u_\varepsilon)^{q-1} \wedge T \\ &= \int_\Omega \chi(dd^c v_\varepsilon)^q \wedge T.\end{aligned}$$

By a proof similar to that of Proposition 5 in [6], one can prove that $(dd^c u_\varepsilon)^q \wedge T$ and $(dd^c v_\varepsilon)^q \wedge T$ converge respectively to $(dd^c u)^q \wedge T$ and $(dd^c v)^q \wedge T$. Hence

$$\int_\Omega \chi(dd^c u)^q \wedge T = \int_\Omega \chi(dd^c v)^q \wedge T.$$

□

Proposition 5. *For $u, v \in \mathcal{F}_m^T(\Omega)$ such that $u \le v$ on Ω one has*

$$\int_\Omega (dd^c v)^q \wedge T \le \int_\Omega (dd^c u)^q \wedge T.$$

Proof. Let $(u_j)_j$ and $(v_j)_j$ be the corresponding decreasing sequences to u and v respectively as in Definition 4. Replace v_j by $\max(u_j, v_j)$, we can assume that $u_j \le v_j$ for all $j \in \mathbb{N}$. For $h \in \mathcal{E}_{0,m}^T(\Omega)$ and $\varepsilon > 0$ we have

$$\begin{aligned}\int_\Omega -h(dd^c v_j)^q \wedge T &\le \int_\Omega -h(dd^c u_j)^q \wedge T \\ &\le \int_\Omega -h(dd^c u)^q \wedge T + \limsup_{j\to+\infty} \int_{\{h>-\varepsilon\}} -h(dd^c u_j)^q \wedge T \\ &\le \int_\Omega -h(dd^c u)^q \wedge T + \varepsilon \limsup_{j\to+\infty} \int_\Omega (dd^c u_j)^q \wedge T.\end{aligned}$$

Now if we let ε go to 0 we get

$$\int_\Omega -h(dd^c v)^q \wedge T \le \int_\Omega -h(dd^c u)^q \wedge T$$

The result follows by choosing h decreasing to -1. □

Theorem 6. *For all $\varphi \in \mathcal{F}_m^T(\Omega)$, there exists a sequence $(\varphi_j)_j \subset \mathcal{E}_0^T(\Omega) \cap \mathcal{C}(\overline{\Omega})$ that decreases to φ.*

Proof. We refer to Lu [15, Th.3.1] for the construction of the sequence $(\varphi_j)_j$. It remains to show that

$$\int_\Omega (dd^c\varphi_j)^q \wedge T < +\infty.$$

As $\varphi_j \geq \varphi$ then by Proposition 5 one has

$$\int_\Omega (dd^c\varphi_j)^q \wedge T \leq \int_\Omega (dd^c\varphi)^q \wedge T < +\infty.$$

□

3. QUASICONTINUITY OF m-SUBHARMONIC FUNCTIONS

This aim of this section is to prove the quasicontinuity of m-subharmonic functions that belong to the classes $\mathcal{F}_m^T(\Omega)$ and $\mathcal{E}_{p,m}^T(\Omega)$ with respect to a suitable capacity.

Definition 5. The m-capacity associated to T denoted by $C_{m,T}$ is defined as:

$$C_{m,T}(K,\Omega) = C_{m,T}(K) = \sup\left\{\int_K (dd^c v)^q \wedge T,\ v \in SH_m(\Omega,[-1,0])\right\}$$

for all compact subset K of Ω. If E is a subset of Ω, we define

$$C_{m,T}(E,\Omega) = \sup\{C_{m,T}(K),\ K \text{ compact subset of } E\}.$$

This capacity coincides with the standard m-capacity defined by Lu [15] in the case $q = m$ and $T = (dd^c|z|^2)^{n-m}$, with the capacity C_T introduced by Dabbek and Elkhadra in the case $m = n$ and with the standard Bedford and Taylor capacity in the classic case $m = n$ and $T = 1$. We cite below some basics properties of the introduced capacity which can be proved as in [8].

Proposition 6. 1. If $E_1 \subset E_2$ then $C_{m,T}(E_1,\Omega) \leq C_{m,T}(E_2,\Omega)$.

2. If $(E_j)_j$ is a sequence of subsets that belong to Ω then

$$C_{m,T}\left(\bigcup_j E_j,\Omega\right) \leq \sum_j C_{m,T}(E_j,\Omega).$$

Definition 6. A subset E of Ω is said to be (m,T)-pluripolar if

$$C_{m,T}(E,\Omega) = 0.$$

A m-subharmonic function u is said to be quasi-continuous with respect to the capacity $C_{m,T}$, if for every $\varepsilon > 0$, there exists an open subset O_ε such that $C_{m,T}(O_\varepsilon,\Omega) < \varepsilon$ and $u_{|\Omega\setminus O_\varepsilon}$ is continuous.

In the trivial case $T = (dd^c|z|^2)^{n-m}$, Lu [15] proved that every m-subharmonic function is quasicontinuous with respect to the capacity Cap_m. In the general case Dhouib and Elkhadra proved the quasicontinuity of every *bounded* m-subharmonic function (see Theorem 1 in [6]). Such result is not true in the general case as we can see in the following example.

Example 2. If Ω is the polydisc of $\mathbb{C}^3$, $T := [z_1 = 0] \wedge dd^c|z|^2$ and $u(z_1, z_2) = \log|z_1|$. The current T is 2-positive, $C_{m,T}(SuppT) > 0$ but the function u is not continuous on the support of T.

The previous example proves that a condition on the m-subharmonic function is needed to be quasicontinuous with respect to Capacity $C_{m,T}$. In the following we prove that: belonging to one of the introduced classes is sufficient for any m-subharmonic function to be quasicontinuous. To prove this we establish first the following estimate.

PROPOSITION 7. *For all $u \in \mathcal{F}_m^T(\Omega)$ and $s > 0$ one has*

$$s^q C_{m,T}(\{u \le -s\}, \Omega) \le \int_\Omega (dd^c u)^q \wedge T.$$

In particular, the set $\{u = -\infty\}$ is (m, T)-pluripolar.

Proof. Let $(u_j)_j \subset \mathcal{E}_{0,m}^T(\Omega)$ be a decreasing sequence to u on Ω as in Definition 4. Take $s > 0$, $v \in SH(\Omega, [-1, 0])$ and K a compact subset in $\{u_j \le -s\}$. Thanks to the comparison principle (for bounded m-subharmonic functions [6]), we have

$$\begin{aligned}\int_K (dd^c v)^q \wedge T &\le \int_{\{s^{-1}u_j < v\}} (dd^c v)^q \wedge T \le \frac{1}{s^q} \int_{\{s^{-1}u_j < v\}} (dd^c u_j)^q \wedge T \\ &\le \frac{1}{s^q} \int_\Omega (dd^c u_j)^q \wedge T\end{aligned}$$

By proposition 5, It follows that

$$C_{m,T}(\{u_j \le -s\}, \Omega) \le \frac{1}{s^q} \int_\Omega (dd^c u_j)^q \wedge T \le \frac{1}{s^q} \int_\Omega (dd^c u)^q \wedge T.$$

Now if we let j goes to infinity, we obtain

$$C_{m,T}(\{u \le -s\}, \Omega) \le \frac{1}{s^q} \int_\Omega (dd^c u)^q \wedge T.$$

□

In the case of functions in $\mathcal{E}_{p,m}^T(\Omega)$, we get another estimate. More precisely we have the following result

Proposition 8. *Let $u \in \mathcal{E}^T_{p,m}(\Omega)$ and $(u_j)_j \subset \mathcal{E}^T_{0,m}(\Omega)$ decreases to u on Ω as in Definition 3. Then for every $s > 0$ one has*

$$s^{p+q} C_{m,T}(\{u \leq -2s\}, \Omega) \leq \sup_{j\geq 1} \int_\Omega (-u_j)^p (dd^c u_j)^q \wedge T.$$

In particular, the set $\{u = -\infty\}$ is (m, T)-pluripolar.

Proof. Let $s > 0$, $v \in PSH(\Omega, [-1, 0])$. Thanks to comparison principle (for bounded m-subharmonic functions), we have

$$\begin{aligned}\int_{\{u_j\leq -2s\}} (dd^c v)^q \wedge T &\leq \int_{\{u_j < -s+sv\}} (dd^c v)^q \wedge T \\ &\leq \frac{1}{s^q} \int_{\{s^{-1}u_j < -1+v\}} (dd^c u_j)^q \wedge T \leq \frac{1}{s^{p+q}} \int_\Omega (-u_j)^p (dd^c u_j)^q \wedge T\end{aligned}$$

It follows that

$$C_{m,T}(\{u_j \leq -2s\}, \Omega) \leq \frac{1}{s^{p+q}} \sup_{m\geq 1} \int_\Omega (-u_m)^p (dd^c u_m)^q \wedge T.$$

By tending j to infinity, we obtain

$$C_{m,T}(\{u \leq -2s\}, \Omega) \leq \frac{1}{s^{p+q}} \sup_{m\geq 1} \int_\Omega (-u_m)^p (dd^c u_m)^q \wedge T.$$

□

Corollary 1. *Every $u \in \mathcal{F}^T_m(\Omega)$ (resp. $\mathcal{E}^T_{p,m}(\Omega)$) is $C_{m,T}$-quasi-continuous.*

Proof. Let $u \in \mathcal{F}^T_m(\Omega)$ (resp. $\mathcal{E}^T_{p,m}(\Omega)$) and $\varepsilon > 0$. Denote by $B_u(t) := \{z \in \Omega;\ u(z) < t\}$, $t \leq 0$. By Proposition 7 (resp. Proposition 8), there exists $s_\varepsilon \geq 1$ such that $C_{m,T}(B_u(-s_\varepsilon), \Omega) < \frac{\varepsilon}{2}$. The function $u_\varepsilon := \max(u, -s_\varepsilon)$ is bounded on Ω so thanks to Dhouib and Elkhadra theorem [6], there exists an open subset $\mathcal{O}$ in Ω such that $C_{m,T}(\mathcal{O}, \Omega) < \frac{\varepsilon}{2}$ and u_ε is continuous on $\Omega \setminus \mathcal{O}$. The result follows by taking $\mathcal{O}_\varepsilon = \mathcal{O} \cup B_u(-s_\varepsilon)$. □

4. XING INEQUALITY IN THE CLASS $\mathcal{F}^T_M(\Omega)$

It is well known that Xing inequality have a crucial role in the m-potential theory so we end this paper by extending some well-known inequalities (see [16, 17] for more details), to $\mathcal{F}^T_m(\Omega)$. We start by citing the following lemma which generalize a result proven by Hai and Dung [11] so it can be shown using same technics.

Lemma 2 (See [11]). *Let S be an m-positive closed current of bidimension $(1,1)$ on Ω and $u,v \in SH_m(\Omega) \cap L^\infty(\Omega)$. Assume that $u \leq v$ on Ω and*

$$\lim_{z \to \partial\Omega} [u(z) - v(z)] = 0.$$

Then one has

$$\int_\Omega (v-u)^k dd^c w \wedge S \leq k \int_\Omega (1-w)(v-u)^{k-1} dd^c u \wedge S$$

for all $k \geq 1$ and $w \in SH_m(\Omega, [0,1])$.

Lemma 3. *Let $u,v \in SH_m(\Omega) \cap L^\infty(\Omega)$ such that $u \leq v$ on Ω and*

$$\lim_{z \to \partial\Omega} [u(z) - v(z)] = 0.$$

Then one has

$$\begin{aligned} \frac{1}{q!} \int_\Omega (v-u)^q dd^c w_1 \wedge \ldots \wedge dd^c w_q \wedge T + \int_\Omega (r - w_1)(dd^c v)^q \wedge T \\ \leq \int_\Omega (r - w_1)(dd^c u)^q \wedge T \end{aligned}$$

for every $r \geq 1$ and $w_1, \ldots, w_q \in SH_m(\Omega, [0,1])$.

Proof. Let $K \subset\subset \Omega$. We prove the statement firstly when $u = v$ on $\Omega \setminus K$. In this case it suffices to use Lemma 2 to obtain that

$$\begin{aligned} &\int_\Omega (v-u)^q dd^c w_1 \wedge \ldots \wedge dd^c w_q \wedge T \\ &\leq q \int_\Omega (v-u)^{q-1} dd^c w_1 \wedge \ldots \wedge dd^c w_{q-1} \wedge dd^c u \wedge T \\ &\vdots \\ &\leq q! \int_\Omega (v-u) dd^c w_1 \wedge (dd^c u)^{q-1} \wedge T \\ &\leq q! \int_\Omega (w_1 - r) dd^c (v-u) \wedge \left(\sum_{i=0}^{q-1} (dd^c u)^i \wedge (dd^c v)^{q-i-1} \right) \wedge T \\ &= q! \int_\Omega (r - w_1) dd^c (u-v) \wedge \left(\sum_{i=0}^{q-1} (dd^c u)^i \wedge (dd^c v)^{q-i-1} \right) \wedge T \\ &= q! \int_\Omega (r - w_1)((dd^c u)^q - (dd^c v)^q) \wedge T. \end{aligned}$$

In the general case, for every $\varepsilon > 0$ we set $v_\epsilon = \max(u, v - \varepsilon)$. Then $v_\epsilon \nearrow v$ on Ω and satisfies $v_\epsilon = u$ on $\Omega \setminus K$ for some $K \subset\subset \Omega$. Hence

$$\frac{1}{q!} \int_\Omega (v_\varepsilon - u)^q dd^c w_1 \wedge \ldots \wedge dd^c w_q \wedge T + \int_\Omega (r - w_1)(dd^c v_\varepsilon)^q \wedge T$$

$$\leq \int_\Omega (r-w_1)(dd^c u)^q \wedge T.$$

Since $v_\varepsilon - u \nearrow v-u$, the family of measures $(dd^c v_\varepsilon)^q \wedge T$ converges weakly to $(dd^c v)^q \wedge T$ as $\varepsilon \searrow 0$ and the function $r-w_1$ is lower semicontinuous then, by letting $\varepsilon \searrow 0$, we obtain the desired inequality. □

PROPOSITION 9. *Let $r \geq 1$, $u, v \in \mathcal{F}_m^T(\Omega)$ and $w \in r + \mathcal{E}_{0,m}^T(\Omega)$. Assume that $u \leq v$ on Ω, then*

$$\frac{1}{q!}\int_\Omega (v-u)^q (dd^c w)^q \wedge T + \int_\Omega (r-w)(dd^c v)^q \wedge T \leq \int_\Omega (r-w)(dd^c u)^q \wedge T.$$

Proof. Let $u, v \in \mathcal{F}_m^T(\Omega)$ and $u_m, v_j \in \mathcal{E}_0^T(\Omega)$ which decrease to u and v respectively as in Definition 4. Replace v_j by $\max(u_j, v_j)$ we may assume that $u_j \leq v_j$ for $j \geq 1$. By lemma 3 we have for $m \geq j \geq 1$

$$\frac{1}{q!}\int_\Omega (v_j-u_m)^q (dd^c w)^q \wedge T + \int_\Omega (r-w)(dd^c v_j)^q \wedge T \leq \int_\Omega (r-w)(dd^c u_m)^q \wedge T.$$

Now since $w \in r + \mathcal{E}_{0,m}^T(\Omega)$, so by taking j goes to $+\infty$ we get:

$$\frac{1}{q!}\int_\Omega (v-u_m)^q (dd^c w)^q \wedge T + \int_\Omega (r-w)(dd^c v)^q \wedge T \leq \int_\Omega (r-w)(dd^c u_m)^q \wedge T.$$

Hence by tending $m \to +\infty$, we obtain the result. □

THEOREM 7. *Let $u, w_1, ..., w_{q-1} \in \mathcal{E}_{p,m}^T(\Omega)$ and $v \in SH_m^-(\Omega)$. If we set $S = dd^c w_1 \wedge ... \wedge dd^c w_{q-1}$ then*

$$dd^c \max(u,v) \wedge T \wedge S_{|\{u>v\}} = dd^c u \wedge T \wedge S_{|\{u>v\}}.$$

Proof. First step: $v \equiv a < 0$.

Let $u_j, w_{k,j} \in \mathcal{E}_0^T(\Omega) \cap \mathcal{C}(\overline{\Omega})$ such that $(u_j)_j$ decreases to u and $(w_{k,j})_j$ decreases to w_k for each $1 \leq k \leq q-1$. Since $\{u_j > a\}$ is open, one has

$$dd^c \max(u_j, a) \wedge T \wedge S^j_{|\{u_j>a\}} = dd^c u_j \wedge T \wedge S^j_{|\{u_j>a\}}$$

where $S^j = dd^c w_{1,j} \wedge ... \wedge dd^c w_{q-1,j}$. As $\{u > a\} \subset \{u_j > a\}$ we obtain

$$dd^c \max(u_j, a) \wedge T \wedge S^j_{|\{u>a\}} = dd^c u_j \wedge T \wedge S^j_{|\{u>a\}}.$$

It follows from [11] that

$$\max(u-a,0) dd^c \max(u_j,a) \wedge T \wedge S^j \underset{j\to+\infty}{\longrightarrow} \max(u-a,0) dd^c \max(u,a) \wedge T \wedge S$$

$$\max(u-a,0) dd^c u_j \wedge T \wedge S^j \underset{j\to+\infty}{\longrightarrow} \max(u-a,0) dd^c u \wedge T \wedge S.$$

By Lemma 4.2 in [12]

$$\max(u-a,0)[dd^c\max(u,a)\wedge T\wedge S-dd^cu\wedge T\wedge S]=0.$$

So

$$dd^c\max(u,a)\wedge T\wedge S=dd^cu\wedge T\wedge S\quad on\ \{u>a\}.$$

Second step: $v\in PSH^-(\Omega)$.

Since $\{u>v\}=\cup_{a\in\mathbb{Q}^-}\{u>a>v\}$, it suffices to show that

$$dd^c\max(u,v)\wedge T\wedge S=dd^cu\wedge T\wedge S\quad on\ \{u>a>v\}$$

for all $a\in\mathbb{Q}^-$. As $\max(u,v)\in\mathcal{F}^T(\Omega)$ then by the first step, we have

$$\begin{aligned}dd^c\max(u,v)\wedge T\wedge S_{|\{\max(u,v)>a\}}&=dd^c\max(\max(u,v),a)\wedge T\wedge S_{|\{\max(u,v)>a\}}\\&=dd^c\max(u,v,a)\wedge T\wedge S_{|\{\max(u,v)>a\}}\\dd^cu\wedge T\wedge S_{|\{u>a\}}&=dd^c\max(u,a)\wedge T\wedge S_{|\{v>a\}}.\end{aligned}$$

The fact that $\max(u,v,a)=\max(u,a)$ on the open set $\{a>v\}$ gives

$$dd^c\max(u,v,a)\wedge T\wedge S_{|\{a>v\}}=dd^c\max(u,a)\wedge T\wedge S_{|\{a>v\}}.$$

As $\{u>a>v\}$ is contained in $\{u>a\}$, in $\{\max(u,v)>a\}$ and in $\{a>v\}$, then by combining the last equalities we obtain

$$dd^c\max(u,v)\wedge T\wedge S_{|\{u>a>v\}}=dd^c\max(u,a)\wedge T\wedge S_{|\{u>a>v\}}.\qquad\square$$

By repeating line by line the same proof as in [15], on can deduce from the previous theorem that integration by part is allowed in $\mathcal{E}^T_{p,m}(\Omega)$ and that proposition 10 can be extended to this class. Namely we have the following result.

COROLLARY 2. *Let $u,w_1,...,w_{q-1}\in\mathcal{E}^T_{p,m}(\Omega)$ and $S=dd^cw_1\wedge...\wedge dd^cw_{q-1}\wedge T$. Then*

$$\int_\Omega vdd^cu\wedge T=\int_\Omega udd^cv\wedge T.$$

Moreover if we assume that $u\le v$ on Ω, then for all $p>0$ and $h\in\mathcal{E}^T_{0,m}(\Omega)\cap\overline{\mathcal{C}}(\Omega)$

$$\int_\Omega(-h)(dd^cv)^q\wedge T\le\int_\Omega(-h)(dd^cu)^q\wedge T.$$

Now we prove a similar Xing pinciple in the class $\mathcal{E}^T_{p,m}(\Omega)$:

THEOREM 8. *Let $0<p\le1$ and $u,v\in\mathcal{E}^T_p(\Omega)$ such that the measure $(dd^cv)^q\wedge T$ has no mass on $(m,T)-$pluripolar sets then*

$$\int_{\{v<u\}}(dd^cu)^q\wedge T\le\int_{\{v<u\}}(dd^cv)^q\wedge T$$

Proof. Let $h \in \mathcal{E}_{0,m}^T(\Omega) \cap \overline{\mathcal{C}}(\Omega)$ and $\mu := (dd^c v)^q \wedge T$. The measure μ has no mass on $(m,T)-$pluripolar sets. Since μ is a Borel measure then the set $E_\mu := \{t > 0, \mu(u = tv) = 0\}$ is at most countable. Thus for almost every $t > 0$, one has

$$\int_{\{u=tv\}} (-h)(dd^c v)^q \wedge T = 0.$$

So without loss of generality, one can only treat the case $\int_{\{u=v\}} (-h)(dd^c v)^q \wedge T = 0$. Using Theorem 7, we get

$$1\!\!1_{\{v<u\}}(dd^c u)^q \wedge T = 1\!\!1_{\{v<u\}}(dd^c \max(u,v))^q \wedge T.$$

Now by corollary 2 one has

$$\int_\Omega (-h)(dd^c \max(u,v))^q \wedge T \leq \int_\Omega (-h)(dd^c u)^q \wedge T.$$

It follows that

$$\int_{\{v<u\}} (-h)(dd^c u)^q \wedge T = \int_{\{v<u\}} (-h)(dd^c \max(u,v))^q \wedge T$$
$$\leq \int_\Omega (-h)(dd^c \max(u,v))^q \wedge T + \int_{\{u<v\}} (-h)(dd^c \max(u,v))^q \wedge T$$
$$\leq \int_\Omega (-h)(dd^c v)^q \wedge T + \int_{\{u<v\}} (-h)(dd^c v)^q \wedge T = \int_{\{u<v\}} (-h)(dd^c v)^q \wedge T.$$

As $0 < p \leq 1$, the above terms are finite. Hence the result follows by letting h goes to -1. □

Proposition 10. *Let $u, v \in \mathcal{F}_m^T(\Omega)$, $h \in \mathcal{E}_{0,m}^T(\Omega)$, q and s two natural integer satisfying $q + s = p$. Then*

$$\int_\Omega -h(dd^c u)^q \wedge (dd^c v)^s \wedge T \leq \left(\int_\Omega -h(dd^c u)^p \wedge T\right)^{\frac{q}{p}} \left(\int_\Omega -h(dd^c v)^p \wedge T\right)^{\frac{s}{p}}.$$

Proof. Thanks to Proposition 4 it suffices to prove the result in the case $u, v \in \mathcal{E}_{0,m}^T(\Omega)$. Let $u, v \in \mathcal{E}_{0,m}^T(\Omega)$ and $R := (dd^c u)^r \wedge (dd^c v)^t \wedge T$ where r and t are two natural integer such that $r + t = p - 2$. Using integration by part in the case $q = s = 1$, we obtain

$$\int_\Omega -hdd^c u \wedge dd^c v \wedge R = \int_\Omega -udd^c h \wedge dd^c v \wedge R = \int_\Omega du \wedge d^c v \wedge dd^c h \wedge R.$$

Since $(u,v) \longmapsto \int_\Omega du \wedge d^c v \wedge dd^c h \wedge R$ is a positive bilinear symmetric form on $\mathcal{C}^\infty(\Omega) \times \mathcal{C}^\infty(\Omega)$, then by Cauchy Schwartz inequality on has

$$\int_\Omega -hdd^c u \wedge dd^c v \wedge R \leq \left(\int_\Omega -h(dd^c u)^2 \wedge R\right)^{\frac{1}{2}} \left(\int_\Omega -h(dd^c v)^2 \wedge R\right)^{\frac{1}{2}}.$$

The last inequality is still true for $u, v \in SH_m^-(\Omega) \cap L^\infty(\Omega)$ by regularization. Assume now that the inequality of the proposition hold for $q + s = m < p$, $m \geq 2$, that means

$$\int_\Omega -h(dd^c u)^q \wedge (dd^c v)^s \wedge R \leq \left(\int_\Omega -h(dd^c u)^{q+s} \wedge R\right)^{\frac{q}{q+s}} \left(\int_\Omega -h(dd^c v)^{q+s} \wedge R\right)^{\frac{s}{q+s}}.$$

Let prove it for $q + s = m + 1$. Consider $q' + s' = m$ and $S = dd^c u \wedge R$, so

$$\begin{aligned}
&\int_\Omega -h(dd^c u)^{q'+s'} \wedge dd^c v \wedge R \\
&= \int_\Omega -h(dd^c u)^{q'+s'-1} \wedge dd^c v \wedge S \\
&\leq \left(\int_\Omega -h(dd^c u)^{q'+s'} \wedge S\right)^{\frac{q'+s'-1}{q'+s'}} \left(\int_\Omega -h(dd^c v)^{q'+s'} \wedge S\right)^{\frac{1}{q'+s'}} \\
&= \left(\int_\Omega -h(dd^c u)^{q'+s'+1} \wedge R\right)^{\frac{q'+s'-1}{q'+s'}} \left(\int_\Omega -h(dd^c v)^{q'+s'} \wedge dd^c u \wedge R\right)^{\frac{1}{q'+s'}} \\
&\leq \left(\int_\Omega -h(dd^c u)^{q'+s'+1} \wedge R\right)^{\frac{q'+s'-1}{q'+s'}} \\
&\times \left[\left(\int_\Omega -h(dd^c v)^{q'+s'+1} \wedge R\right)^{\frac{q'+s'-1}{q'+s'}} \left(\int_\Omega -h(dd^c u)^{q'+s'+1} \wedge R\right)^{\frac{1}{q'+s'}}\right]^{\frac{1}{q'+s'}}.
\end{aligned}$$

Hence we obtain that

$$\begin{aligned}
&\int_\Omega -h(dd^c u)^{q'+s'} \wedge dd^c v \wedge R \\
&\leq \left(\int_\Omega -h(dd^c u)^{q'+s'+1} \wedge R\right)^{\frac{q'+s'}{q'+s'+1}} \left(\int_\Omega -h(dd^c v)^{q'+s'+1} \wedge R\right)^{\frac{1}{q'+s'+1}}.
\end{aligned}$$

It follows that:

$$\begin{aligned}
&\int_\Omega -h(dd^c v)^{q'+1} \wedge (dd^c u)^{s'} \wedge R = \int_\Omega -h(dd^c v)^{q'} \wedge (dd^c u)^{s'} \wedge dd^c v \wedge R \\
&\leq \left(\int_\Omega -h(dd^c v)^{q'+s'} \wedge dd^c v \wedge R\right)^{\frac{q'}{q'+s'}} \left(\int_\Omega -h(dd^c u)^{q'+s'} \wedge dd^c v \wedge R\right)^{\frac{s'}{q'+s'}} \\
&\leq \left(\int_\Omega -h(dd^c u)^{q'+s'+1} \wedge R\right)^{\frac{s'}{q'+s'+1}} \left(\int_\Omega -h(dd^c v)^{q'+s'+1} \wedge R\right)^{\frac{q'+1}{q'+s'+1}}.
\end{aligned}$$

□

The previous proposition was proved by Hai and Dung [11] in the classic case $m = n$. As a consequence, we will prove the following theorem

THEOREM 9. *Let $u_1, u_2, \cdots, u_p \in \mathcal{F}^{m,T}(\Omega)$ and $h \in \mathcal{E}_0^{m,T}(\Omega)$ then one has:*

$$\int_\Omega -hdd^c u_1 \wedge \cdots \wedge dd^c u_p \wedge T \leq \left(\int_\Omega -h(dd^c u_1)^p \wedge T\right)^{\frac{1}{p}} \cdots \left(\int_\Omega -h(dd^c u_p)^p \wedge T\right)^{\frac{1}{p}}.$$

Proof. It suffices to prove the result in the case $u_1, \cdots, u_p \in \mathcal{E}_{0,m}^T(\Omega)$. Using Proposition 10 we have:

$$\int_\Omega -hdd^c u_1 \wedge (dd^c u_2)^{p-1} \wedge T \leq \left(\int_\Omega -h(dd^c u_1)^p \wedge T\right)^{\frac{1}{p}} \left(\int_\Omega -h(dd^c u_2)^p \wedge T\right)^{\frac{p-1}{p}}.$$

Let us prove first the theorem for $u_2 = \cdots = u_p = u$. Assume that the theorem hold for $u_{s+1} = \cdots = u_p = u$, that means:

$$\begin{aligned}&\int_\Omega -hdd^c u_1 \wedge \cdots \wedge dd^c u_s \wedge (dd^c u)^{p-s} \wedge T \\ &\leq \left(\int_\Omega -h(dd^c u_1)^p \wedge T\right)^{\frac{1}{p}} \cdots \left(\int_\Omega -h(dd^c u_s)^p \wedge T\right)^{\frac{p-s}{p}} \left(\int_\Omega -h(dd^c u)^p \wedge T\right)^{\frac{p-s}{p}}.\end{aligned}$$

Let us prove it for $u_{s+2} = \cdots = u_p = u$. Take $R = dd^c u_1 \cdots \wedge dd^c u_s \wedge T$, then

$$\begin{aligned}&\int_\Omega -hdd^c u_1 \wedge \cdots \wedge dd^c u_{s+1} \wedge (dd^c u)^{p-s-1} \wedge T = \int_\Omega -hdd^c u_{s+1} \wedge (dd^c u)^{p-s-1} \wedge R \\
&\leq \left(\int_\Omega -h(dd^c u_{s+1})^{p-s} \wedge R\right)^{\frac{1}{p-s}} \left(\int_\Omega -h(dd^c u)^{p-s} \wedge R\right)^{\frac{p-s-1}{p-s}} \\
&\leq \Bigg[\left(\int_\Omega -h(dd^c u_1)^p \wedge T\right)^{\frac{1}{p}} \cdots \left(\int_\Omega -h(dd^c u_s)^p \wedge T\right)^{\frac{1}{p}} \\
&\qquad \times \left(\int_\Omega -h(dd^c u_{s+1})^p \wedge T\right)^{\frac{p-s}{p}}\Bigg]^{\frac{1}{p-s}} \Bigg[\left(\int_\Omega -h(dd^c u_1)^p \wedge T\right)^{\frac{1}{p}} \cdots \\
&\qquad \times \left(\int_\Omega -h(dd^c u_s)^p \wedge T\right)^{\frac{1}{p}} \left(\int_\Omega -h(dd^c u)^p \wedge T\right)^{\frac{p-s}{p}}\Bigg]^{\frac{p-s-1}{p-s}} \\
&\leq \left(\int_\Omega -h(dd^c u_1)^p \wedge T\right)^{\frac{1}{p}} \cdots \left(\int_\Omega -h(dd^c u_{s+1})^p \wedge T\right)^{\frac{1}{p}} \\
&\qquad \times \left(\int_\Omega -h(dd^c u)^p \wedge T\right)^{\frac{p-s-1}{p}}.\end{aligned}$$

$\square$

Acknowledgments. Authors extend their appreciation to the Deanship of Scientific Research at Jouf University for funding this work through research Grant no. DSR-2021-03-03135.

REFERENCES

[1] E. Bedford and B.A.Taylor, *A new capacity for plurisubharmonic functions* Acta. Math. **149** (1982), 1–40.

[2] Z. Blocki, *Estimates for the complex Monge-Ampère operator*. Bull. Polish Acad. Sci. Math. **41** (1993), *2*, 151–157.

[3] U. Cegrell, *Pluricomplex energy*. Acta. Math. **180** (1998), 187–217.

[4] U. Cegrell, *The general definition of the comlex Monge-Ampère operator*. Ann. Inst. Fourier (Grenoble) **54** (2004), 159—179.

[5] U. Cegrell, *A general Dirichlet problem for the complex Monge-Ampr̀e operator*. Ann. Polon. Math. **94** (2008), *2*, 131–147.

[6] A. Dhouib and F. Elkhadhra, *m-Potential theory associated to a positive closed current in the class of m-sh functions*. Complex Variables and Elliptic Equations **61** (2016), 875–901.

[7] S. Dinew and S.Kolodziej, *A priori estimates for complex Hessian equations*. Anal. PDE. **7** (2014), 227–244.

[8] K. Dabbek and F. Elkhadhra, *Capacité associée à un courant positif fermé*. Documenta Math. **11** (2006), 469-486.

[9] J. P. Demailly, *Complex analytic and differential geometry*. Open book available at http://www-fourier.ujf-grenoble.fr/∼demailly/books.html

[10] F. Elkhadhra, *Lelong-Demailly numbers in terms of capacity and weak convergence for closed positive currents*. Acta Math. Scientia **33B** (2013), *6*, 1652–1666.

[11] L. M. Hai and N. T. Dung, *Local T-pluripolarity and T-pluripolarity of a subset and some Cegrell's pluricomplex energy classes associated to a positive closed current*. Vietnam J. of Math. **37** (2009), *2&3*, 1–19.

[12] N. V. Khue and H. H. Pham, *A comparison principle for the complex Monge-Ampère Operator in Cegrell's classes and applications*. Trans. Amer. Math. Soc. **361** (2009), 5539–5554 .

[13] S. Kolodziej, *The complex Monge-Ampère equation and pluripotential theory*. Mem. Amer. Math. Soc. **178** (2005), *840*, 1–64.

[14] H. Lu, *A variational approach to complex Hessian equations in* $\mathbb{C}^n$. J. Math. Anal. Appl. **431** (2015), 228–259.

[15] H. Lu, *Equations Hessiennes complexes*. Thèse de l'Université Toulouse III (UT3 Paul Sabatier), 2012.

[16] Y. Xing, *Continuity of the complex Monge-Ampère operator*. Proc. Amer. Math. Soc. **124** (1996), 457–467.

[17] Y. Xing, *Complex Monge-Ampère measures of pluriharmonic functions with bounded values near the boundary*. Cand. J. Math. **52** (2000), 1085–1100.

Received July 30, 2018

Jouf University
Mathematics Departments, College of Science
P.O. Box: 2014, Sakaka, Saudia Arabia
`jmhbil@ju.edu.sa`