MORPHISMS THAT FACTOR THROUGH A COTORSION MODULE

LIXIN MAO

Communicated by Constantin Năstăsescu

A morphism $f : M \to N$ of left R-modules is said to be a twist morphism if the induced morphism $\mathrm{Ext}_R^1(F, f) : \mathrm{Ext}_R^1(F, M) \to \mathrm{Ext}_R^1(F, N)$ is 0 for every flat left R-module F. We prove that $f : M \to N$ is a twist morphism if and only if $\mathrm{Ext}_R^1(L, f) : \mathrm{Ext}_R^1(L, M) \to \mathrm{Ext}_R^1(L, N)$ takes values in the subgroup consisting of flat-pure exact sequences for any left R-module L if and only if f factors through a cotorsion left R-module. Some applications are given.

AMS 2020 Subject Classification: 16D90, 16E30, 18G25.

Key words: twist morphism, cotorsion module, precover, preenvelope.

1. INTRODUCTION

Ideal approximation theory has been recently introduced and developed by Fu, Guil Asensio, Herzog and Torrecillas in [7]. This theory is a generalization of the classical theory of covers and envelopes (approximation theory) initiated by Enochs, Auslander and Smalø [1, 2] since it need to be set forth in terms of morphisms instead of objects. An important instance of morphisms in ideal approximation theory is phantom morphisms. Herzog called a morphism $f : M \to N$ of left R-modules a *phantom morphism* [10] if the induced morphism $\mathrm{Tor}_1^R(A, f) : \mathrm{Tor}_1^R(A, M) \to \mathrm{Tor}_1^R(A, N)$ is 0 for every (finitely presented) right R-module A. In particular, he considered the trivial phantom morphisms, *i.e.*, morphisms that factor through a flat module. Similarly, Herzog called a morphism $g : M \to N$ of left R-modules an Ext-*phantom morphism* [11] if the induced morphism $\mathrm{Ext}_R^1(B, g) : \mathrm{Ext}_R^1(B, M) \to \mathrm{Ext}_R^1(B, N)$ is 0 for every finitely presented left R-module B. He also investigated the trivial Ext-phantom morphisms, *i.e.*, morphisms that factor through an FP-injective module or an injective module.

On the other hand, the right orthogonal class of the class of flat modules is called the class of *cotorsion modules* [3]. Wakamatsu's Lemma [16, Lemma 2.1.1] implies that the kernel of a flat cover is cotorsion. It seems to be a natural question to study those morphisms that factor through a cotorsion module. In this paper, we first introduce the concept of twist morphisms. We call a morphism $f : M \to N$ of left R-modules a *twist morphism* if the induced morphism $\mathrm{Ext}^1_R(F, f) : \mathrm{Ext}^1_R(F, M) \to \mathrm{Ext}^1_R(F, N)$ is 0 for every flat left R-module F. It is proven that $f : M \to N$ is a twist morphism if and only if $\mathrm{Ext}^1_R(L, f) : \mathrm{Ext}^1_R(L, M) \to \mathrm{Ext}^1_R(L, N)$ takes values in the subgroup consisting of flat-pure exact sequences for any left R-module L if and only if f factors through a cotorsion left R-module. As a consequence, we characterize left perfect rings and von Neumann regular rings in terms of twist morphisms. In addition, we study preenvelopes and precovers by twist morphisms under change of rings.

Throughout this paper, all rings are associative with identity and all modules are unitary. For a ring R, we write R-Mod (resp. Mod-R) for the category of left (resp. right) R-modules. ${}_RM$ (resp. M_R) denotes a left (resp. right) R-module. $E(M)$ stands for the injective envelope of M. The character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of M is denoted by M^+.

2. FLAT-PURE EXACT SEQUENCES AND TWIST MORPHISMS

According to Zhu and Ding [17], an exact sequence $0 \to A \stackrel{\iota}{\to} B \stackrel{\pi}{\to} C \to 0$ in R-Mod is called *flat-pure* if the sequence $0 \to \mathrm{Hom}_R(F, A) \stackrel{\iota_*}{\to} \mathrm{Hom}_R(F, B) \stackrel{\pi_*}{\to} \mathrm{Hom}_R(F, C) \to 0$ is exact for any flat left R-module F. In this case, ι is called a *flat-pure monomorphism* and π is called a *flat-pure epimorphism*.

Let $\mathcal{D}$ be a class of R-modules and M an R-module. Recall that a homomorphism $\phi : M \to D$ with $D \in \mathcal{D}$ is a *$\mathcal{D}$-preenvelope of M* [2, 4] if for any homomorphism f: $M \to D'$ with $D' \in \mathcal{D}$, there is a homomorphism $g : D \to D'$ such that $g\phi = f$. Moreover, if the only such g are automorphisms of D when $D' = D$ and $f = \phi$, then the $\mathcal{D}$-preenvelope ϕ is called a *$\mathcal{D}$-envelope of M*. Dually we have the notions of a $\mathcal{D}$-precover and a $\mathcal{D}$-cover. Obviously, $\alpha : M \to N$ is a flat precover of N in R-Mod if and only if M is a flat left R-module and $\alpha : M \to N$ is a flat-pure epimorphism.

Recall that a left R-module C is *cotorsion* [3] if $\mathrm{Ext}^1_R(F, C) = 0$ for any flat left R-module F. It is well known that every module has a cotorsion envelope (see [8, Theorem 4.1.1]). In what follows, we always denote the cotorsion envelope of M by $M \stackrel{\lambda}{\to} C(M)$.

PROPOSITION 2.1. *The following are equivalent for a left R-module M:*

1. *M is a cotorsion left R-module.*
2. *Every exact sequence $0 \to M \to N \to L \to 0$ in R-Mod is flat-pure.*

3. *The exact sequence* $0 \to M \stackrel{\iota}{\to} E(M) \to H \to 0$ *in* R*-Mod is flat-pure.*

4. *The exact sequence* $0 \to M \stackrel{\lambda}{\to} C(M) \to G \to 0$ *in* R*-Mod is flat-pure.*

Proof. (1) ⇒ (2) For any flat left R-module F, the exact sequence $0 \to M \to N \to L \to 0$ induces the exact sequence

$$0 \to \mathrm{Hom}_R(F, M) \to \mathrm{Hom}_R(F, N) \to \mathrm{Hom}_R(F, L) \to \mathrm{Ext}^1_R(F, M) = 0.$$

So the exact sequence $0 \to M \to N \to L \to 0$ is flat-pure.

(2) ⇒ (3) and (2) ⇒ (4) are trivial.

(3) ⇒ (1) For any flat left R-module F, the exact sequence $0 \to M \stackrel{\iota}{\to} E(M) \to H \to 0$ gives rise to the exactness of the sequence

$$\mathrm{Hom}_R(F, E(M)) \to \mathrm{Hom}_R(F, H) \to \mathrm{Ext}^1_R(F, M) \to \mathrm{Ext}^1_R(F, E(M)) = 0.$$

Since $\mathrm{Hom}_R(F, E(M)) \to \mathrm{Hom}_R(F, H)$ is an epimorphism, $\mathrm{Ext}^1_R(F, M) = 0$, *i.e.*, M is a cotorsion left R-module.

(4) ⇒ (1) Note that G is flat by Wakamatsu's Lemma (see [16, Section 2.1]). So $0 \to M \stackrel{\lambda}{\to} C(M) \to G \to 0$ is split by (4). Thus M is a cotorsion left R-module. □

Definition 2.2. Let R be a ring. A morphism $f : M \to N$ of left R-modules is said to be a *twist morphism* if the induced morphism $\mathrm{Ext}^1_R(F, f) : \mathrm{Ext}^1_R(F, M) \to \mathrm{Ext}^1_R(F, N)$ is 0 for every flat left R-module F.

Recall that R is a *left phantomless ring* [7] if every phantom morphism in R-Mod factors through a flat left R-module. The following result shows that any twist morphism in R-Mod factors through a cotorsion left R-module.

THEOREM 2.3. *The following are equivalent for a morphism* $f : M \to N$ *in* R*-Mod:*

1. f *is a twist morphism.*

2. *For any left* R*-module* L, $\mathrm{Ext}^1_R(L, f) : \mathrm{Ext}^1_R(L, M) \to \mathrm{Ext}^1_R(L, N)$ *takes values in the subgroup consisting of flat-pure exact sequences.*

3. f *factors through a cotorsion left* R*-module.*

4. *There exists a flat-pure monomorphism* g *such that* gf *is a twist morphism.*

5. $\mathrm{Ext}^n_R(F, f) : \mathrm{Ext}^n_R(F, M) \to \mathrm{Ext}^n_R(F, N)$ *is 0 for any flat left* R*-module* F *and any* $n \geq 1$.

Proof. (1) ⇒ (2) Let $\eta : 0 \to M \to X \to L \to 0$ be any exact sequence in R-Mod. Then we get the pushout η' of η along f:

$$\begin{array}{lccccccccc}
\eta: & 0 & \longrightarrow & M & \longrightarrow & X & \longrightarrow & L & \longrightarrow & 0 \\
 & & & {\scriptstyle f}\downarrow & & \downarrow & & \| & & \\
\eta': & 0 & \longrightarrow & N & \longrightarrow & Q & \xrightarrow{h} & L & \longrightarrow & 0.
\end{array}$$

For any flat left R-module F, we obtain the commutative diagram with exact rows:

$$\begin{array}{ccccc}
\mathrm{Hom}_R(F,X) & \longrightarrow & \mathrm{Hom}_R(F,L) & \xrightarrow{\varphi} & \mathrm{Ext}^1_R(F,M) \\
\downarrow & & \| & & \downarrow{\scriptstyle \mathrm{Ext}^1_R(F,f)} \\
\mathrm{Hom}_R(F,Q) & \xrightarrow{h_*} & \mathrm{Hom}_R(F,L) & \xrightarrow{\theta} & \mathrm{Ext}^1_R(F,N).
\end{array}$$

Since $\mathrm{Ext}^1_R(F,f) = 0$, we have $\theta = \mathrm{Ext}^1_R(F,f)\varphi = 0$. So h_* is an epimorphism. Thus η' is a flat-pure exact sequence.

(2) ⇒ (3) There exists an exact sequence in R-Mod:

$$\zeta : 0 \to M \overset{\lambda}{\to} C(M) \to W \to 0.$$

Then we get the pushout ζ' of ζ along f:

$$\begin{array}{lccccccccc}
\zeta: & 0 & \longrightarrow & M & \xrightarrow{\lambda} & C(M) & \longrightarrow & W & \longrightarrow & 0 \\
 & & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \| & & \\
\zeta': & 0 & \longrightarrow & N & \xrightarrow{\iota} & T & \longrightarrow & W & \longrightarrow & 0.
\end{array}$$

Note that W is flat by Wakamatsu's Lemma and $\zeta' : 0 \to N \overset{\iota}{\to} T \to W \to 0$ is a flat-pure exact sequence by (2). So ζ' is a split exact sequence. Thus there exists $\rho : T \to N$ such that $\rho\iota = 1$. Hence we have

$$f = (\rho\iota)f = (\rho g)\lambda,$$

i.e., f factors through the cotorsion left R-module $C(M)$.

(3) ⇒ (1) There exist a cotorsion left R-module U, $\alpha : M \to U$ and $\beta : U \to N$ such that $f = \beta\alpha$. For any flat left R-module F, we have

$$\mathrm{Ext}^1_R(F,f) = \mathrm{Ext}^1_R(F,\beta)\mathrm{Ext}^1_R(F,\alpha) = 0.$$

So f is a twist morphism.

(1) ⇒ (4) is clear by choosing $g = 1_N$.

(4) ⇒ (1) Let $g : N \to H$ be a flat-pure monomorphism such that gf is a twist morphism. For any flat left R-module F, we have

$$\mathrm{Ext}^1_R(F,g)\mathrm{Ext}^1_R(F,f) = \mathrm{Ext}^1_R(F,gf) = 0.$$

There exists an exact sequence $0 \to N \xrightarrow{g} H \to G \to 0$. So we get the induced exact sequence

$$\mathrm{Hom}_R(F,H) \to \mathrm{Hom}_R(F,G) \to \mathrm{Ext}^1_R(F,N) \xrightarrow{\mathrm{Ext}^1_R(F,g)} \mathrm{Ext}^1_R(F,H).$$

Since $\mathrm{Hom}_R(F,H) \to \mathrm{Hom}_R(F,G)$ is an epimorphism, $\mathrm{Ext}^1_R(F,g)$ is a monomorphism. Thus $\mathrm{Ext}^1_R(F,f) = 0$, *i.e.*, f is a twist morphism.

(1) $\Rightarrow$ (5) For any flat left R-module F, there is an exact sequence

$$0 \to K \to P \to F \to 0$$

with P projective and K flat.

Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccc}
\mathrm{Ext}^1_R(K,M) & \xrightarrow{\gamma} & \mathrm{Ext}^2_R(F,M) & \longrightarrow & \mathrm{Ext}^2_R(P,M) = 0 \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \mathrm{Ext}^2_R(F,f)} & & \downarrow \\
\mathrm{Ext}^1_R(K,N) & \longrightarrow & \mathrm{Ext}^2_R(F,N) & \longrightarrow & \mathrm{Ext}^2_R(P,N) = 0.
\end{array}$$

Then $\mathrm{Ext}^2_R(F,f)\gamma = 0$. Since γ is an epimorphism, $\mathrm{Ext}^2_R(F,f) = 0$. By induction, $\mathrm{Ext}^n_R(F,f) = 0$ for any $n \geq 1$.

(5) $\Rightarrow$ (1) is trivial. $\square$

Remark 2.4. (1) Obviously, M is a cotorsion left R-module if and only if the identity map 1_M is a twist morphism. So, in the context of modules, just as we know that a phantom morphism may be viewed as the morphism version of a flat module, a twist morphism may be viewed as the morphism version of a cotorsion module.

(2) Recall that a morphism $g : X \to Y$ of left R-modules is a *cotorsion representation* of the quiver $\bullet \to \bullet$ by left R-modules [5] if $\mathrm{Ext}^1_R(G,g) = 0$ for any flat morphism G. Here we simply call the morphism $g : X \to Y$ a *cotorsion morphism*. By [5, Theorem 5.3.5], $g : X \to Y$ is a cotorsion morphism if and only if X and Y are cotorsion modules. So the concept of twist morphisms is a proper generalization of cotorsion morphisms.

(3) It is well known that any pure-injective module is cotorsion. So, if a left R-morphism f factors through a pure-injective left R-module, then f is a twist morphism by Theorem 2.3. In particular, either morphism $M^+ \to N$ or $M \to N^+$ in R-Mod is a twist morphism.

PROPOSITION 2.5. *Let $0 \to X \xrightarrow{\varphi} Y \xrightarrow{\pi} Z \to 0$ be a flat-pure exact sequence in R-Mod and $0 \to A \xrightarrow{\lambda} B \xrightarrow{f} Z \to 0$ be an exact sequence in R-Mod.*

Consider the following pullback:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & X & = & X & & \\ & & & & \downarrow & & \downarrow \varphi & & \\ 0 & \longrightarrow & A & \xrightarrow{\iota} & M & \xrightarrow{h} & Y & \longrightarrow & 0 \\ & & \| & & \downarrow \varphi & & \downarrow \pi & & \\ 0 & \longrightarrow & A & \xrightarrow{\lambda} & B & \xrightarrow{f} & Z & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

1. $0 \to A \overset{\lambda}{\to} B \overset{f}{\to} Z \to 0$ *is flat-pure if and only if* $0 \to A \overset{\iota}{\to} M \overset{h}{\to} Y \to 0$ *is flat-pure.*

2. λ *is a twist morphism if and only if* ι *is a twist morphism.*

Proof. For any flat left R-module F, we get the following commutative diagram with exact rows and column:

$$\begin{array}{ccccccc} \mathrm{Hom}_R(F,M) & \xrightarrow{h_*} & \mathrm{Hom}_R(F,Y) & \xrightarrow{\theta} & \mathrm{Ext}^1_R(F,A) & \xrightarrow{\mathrm{Ext}^1_R(F,\iota)} & \mathrm{Ext}^1_R(F,M) \\ \downarrow \varphi_* & & \pi_* \downarrow & & \| & & \downarrow \mathrm{Ext}^1_R(F,\varphi) \\ \mathrm{Hom}_R(F,B) & \xrightarrow{f_*} & \mathrm{Hom}_R(F,Z) & \xrightarrow{\gamma} & \mathrm{Ext}^1_R(F,A) & \xrightarrow{\mathrm{Ext}^1_R(F,\lambda)} & \mathrm{Ext}^1_R(F,B) \\ & & \downarrow & & & & \\ & & 0 & & & & \end{array}$$

(1) $0 \to A \overset{\lambda}{\to} B \overset{f}{\to} Z \to 0$ is flat-pure if and only if $\gamma = 0$ if and only if $\theta = 0$ if and only if $0 \to A \overset{\iota}{\to} M \overset{h}{\to} Y \to 0$ is flat-pure.

(2) λ is a twist morphism if and only if γ is an epimorphism if and only if θ is an epimorphism if and only if ι is a twist morphism. □

Following [7], an additive subbifunctor of the bifunctor $\mathrm{Hom}_R(-,-) : R\text{-}\mathrm{Mod}^{op} \times R\text{-Mod} \to \mathrm{Ab}$ is called an *ideal* $\mathcal{I}$ of R-Mod. This means that, for every pair of left R-modules M and N, the morphisms $M \to N$ in $\mathcal{I}$ form a subgroup of the abelian group $\mathrm{Hom}_R(M,N)$, and given any three left R-module morphisms f, g, h for which fgh is defined and $g \in \mathcal{I}$, we have $fgh \in \mathcal{I}$.

Let $\mathcal{I}$ be an ideal of R-Mod. Recall that a morphism $\phi : M \to N$ in $\mathcal{I}$ is an *$\mathcal{I}$-preenvelope* of a left R-module M [7] if for any morphism $\psi : M \to L$

in $\mathcal{I}$, there is a morphism $\theta : N \to L$ such that $\theta\phi = \psi$. An $\mathcal{I}$-preenvelope $\phi : M \to N$ is called an $\mathcal{I}$-*envelope* if every endomorphism h of N such that $h\phi = \phi$ is an isomorphism. An $\mathcal{I}$-precover and an $\mathcal{I}$-cover of a left R-module are defined dually.

Clearly, the collection of twist morphisms in R-Mod is an ideal of R-Mod.

PROPOSITION 2.6. *Let R be a ring.*

1. *Every twist monomorphism with cokernel flat is a twist preenvelope.*
2. *Every left R-module has a twist envelope.*

Proof. (1) Let $f : A \to B$ be a twist monomorphism in R-Mod with $L = \mathrm{coker}(f)$ flat. For any twist morphism $g : A \to D$, we get the following pushout:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \| & & \\ 0 & \longrightarrow & D & \xrightarrow{\iota} & H & \longrightarrow & L & \longrightarrow & 0. \end{array}$$

By Theorem 2.3, $0 \to D \to H \to L \to 0$ is a flat-pure exact sequence. Since L is flat, the exact sequence $0 \to D \to H \to L \to 0$ is split. Thus there exists $\pi : H \to D$ such that $\pi\iota = 1$. So $g = \pi\iota g = (\pi h)f$. Thus f is a twist preenvelope.

(2) Every left R-module M has a cotorsion envelope $\lambda : M \to C(M)$, which is certainly a twist morphism. For any twist morphism $\beta : M \to G$, β factors through a cotorsion left R-module N by Theorem 2.3, *i.e.*, there exist $\varphi : M \to N$ and $\gamma : N \to G$ such that $\beta = \gamma\varphi$. Then there exists $\theta : C(M) \to N$ such that $\theta\lambda = \varphi$. So $(\gamma\theta)\lambda = \gamma\varphi = \beta$. Hence $\lambda : M \to C(M)$ is a twist preenvelope. Since λ is a cotorsion envelope, it is also a twist envelope. □

Let R be a ring and R-Mor denote the category whose objects are left R-module morphisms and the morphism from a left R-module morphism $M_1 \stackrel{f}{\to} M_2$ to a left R-module morphism $N_1 \stackrel{g}{\to} N_2$ is a pair of left R-module morphisms $(M_1 \stackrel{d}{\to} N_1, M_2 \stackrel{s}{\to} N_2)$ such that $sf = gd$. The category R-Mor is also denoted by $\mathcal{A}_2$ in [6], which means the category of all representations of the quiver $\bullet \to \bullet$ by left R-modules.

Since the class of twist morphisms in R-Mor is an ideal, it is not closed under extensions in R-Mor by [6, Remark 3.4]. However we have

PROPOSITION 2.7. *The class of twist morphisms in R-Mor is closed under direct products, direct summands and cosyzygies.*

Proof. Let $(f_i : M_i \to N_i)_{i\in I}$ be a family of twist morphisms in R-Mor and $\prod_{i\in I} f_i : \prod_{i\in I} M_i \to \prod_{i\in I} N_i$ be the induced morphism. For any flat left R-module F, we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Ext}^1_R(F,\prod_{i\in I} M_i) & \xrightarrow{\mathrm{Ext}^1_R(F,\prod_{i\in I} f_i)} & \mathrm{Ext}^1_R(F,\prod_{i\in I} N_i) \\ \cong\downarrow & & \cong\downarrow \\ \prod_{i\in I}\mathrm{Ext}^1_R(F,M_i) & \xrightarrow{\prod_{i\in I}\mathrm{Ext}^1_R(F,f_i)} & \prod_{i\in I}\mathrm{Ext}^1_R(F,N_i). \end{array}$$

Since $\prod_{i\in I}\mathrm{Ext}^1_R(F,f_i) = 0, \mathrm{Ext}^1_R(F,\prod_{i\in I} f_i) = 0$. So $\prod_{i\in I} f_i : \prod_{i\in I} M_i \to \prod_{i\in I} N_i$ is a twist morphism.

It is easy to see that the class of twist morphisms in R-Mor is closed under direct summands.

Now let $f : A_1 \to A_2$ be a twist morphism of left R-modules. Consider the following exact sequence in R-Mor:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & E_1 & \longrightarrow & D_1 & \longrightarrow & 0 \\ & & f\downarrow & & g\downarrow & & h\downarrow & & \\ 0 & \longrightarrow & A_2 & \longrightarrow & E_2 & \longrightarrow & D_2 & \longrightarrow & 0, \end{array}$$

where $E_1 \to E_2$ is an injective morphism in R-Mor, *i.e.*, E_1 and E_2 are injective left R-modules and g is a split epimorphism. For any flat left R-module F, we get the following commutative diagram:

$$\begin{array}{ccccc} 0 = \mathrm{Ext}^1_R(F,E_1) & \longrightarrow & \mathrm{Ext}^1_R(F,D_1) & \longrightarrow & \mathrm{Ext}^2_R(F,A_1) \\ \downarrow & & \downarrow \mathrm{Ext}^1_R(F,h) & & \downarrow \mathrm{Ext}^2_R(F,f) \\ 0 = \mathrm{Ext}^1_R(F,E_2) & \longrightarrow & \mathrm{Ext}^1_R(F,D_2) & \xrightarrow{\gamma} & \mathrm{Ext}^2_R(F,A_2). \end{array}$$

Note that $\mathrm{Ext}^2_R(F,f) = 0$ by Theorem 2.3. So $\gamma\mathrm{Ext}^1_R(F,h) = 0$. Since γ is a monomorphism, $\mathrm{Ext}^1_R(F,h) = 0$. It follows that the class of twist morphisms in R-Mor is closed under cosyzygies. □

Next we give some characterizations of left perfect rings and von Neumann regular rings in terms of twist morphisms.

THEOREM 2.8. *The following are equivalent for a ring R:*

1. *R is a left perfect ring.*
2. *The class of twist morphisms in R-Mor is closed under direct limits.*
3. *The class of twist morphisms in R-Mor is closed under direct sums.*
4. *The class of twist morphisms in R-Mor is closed under subobjects.*

5. *Every left R-module has a twist (pre)cover.*
6. *Every pure exact sequence in R-Mod is flat-pure.*
7. *Every phantom morphism in R-Mod is twist.*
8. *Every twist morphism in R-Mod is cotorsion.*

Proof. By [16, Proposition 3.3.1], R is a left perfect ring if and only if every left R-module is cotorsion. Thus (1) $\Rightarrow$ (2)-(8) are trivial.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1) Let $(M_i)_{i\in I}$ be a family of cotorsion left R-modules. Then every morphism $M_i \xrightarrow{1} M_i$ is a twist morphism and so the induced morphism $\oplus_{i\in I}M_i \xrightarrow{1} \oplus_{i\in I}M_i$ is a twist morphism by (3). Hence $\mathrm{Ext}^1_R(F, \oplus_{i\in I}M_i) = 0$ for every flat left R-module F. Therefore $\oplus_{i\in I}M_i$ is a cotorsion left R-module and so R is a left perfect ring by [9, Theorem 19].

(4) $\Rightarrow$ (1) Let M be any left R-module. Then there is a monomorphism $M \to M^{++}$. So $M \xrightarrow{1} M$ is a subobject of $M^{++} \xrightarrow{1} M^{++}$ in R-Mor. Since $M^{++} \xrightarrow{1} M^{++}$ is a twist morphism, $M \xrightarrow{1} M$ is a twist morphism. Thus M is a cotorsion left R-module. So R is a left perfect ring.

(5) $\Rightarrow$ (1) Let $(M_i)_{i\in I}$ be a family of cotorsion left R-modules. Then $\oplus_{i\in I}M_i$ has a twist precover $\alpha : N \to \oplus_{i\in I}M_i$ by (5). So there exist $\beta : N \to C$ and $\gamma : C \to \oplus_{i\in I}M_i$ with C cotorsion such that $\alpha = \gamma\beta$ by Theorem 2.3. Let $\lambda_i : M_i \to \oplus_{i\in I}M_i$ be the injection. Then every λ_i is twist. So there exists $\theta_i : M_i \to N$ such that $\alpha\theta_i = \lambda_i$. Therefore there is $\xi : \oplus_{i\in I}M_i \to C$ such that $\xi\lambda_i = \beta\theta_i$. So for any $i \in I$, we have

$$\gamma\xi\lambda_i = \gamma\beta\theta_i = \alpha\theta_i = \lambda_i.$$

Thus $\gamma\xi = 1$. Hence $\oplus_{i\in I}M_i$ is isomorphic to a direct summand of C and so is a cotorsion left R-module. It follows that R is a left perfect ring by [9, Theorem 19].

(6) $\Rightarrow$ (1) Let M be any left R-module. Then there is an exact sequence $0 \to M \xrightarrow{\lambda} C(M) \to F \to 0$ with F flat. Since the exact sequence is pure, it is flat-pure by (6) and so is split. Thus M is a cotorsion left R-module. Whence R is a left perfect ring.

(7) $\Rightarrow$ (1) Let F be any flat left R-module. Then $F \xrightarrow{1} F$ is a phantom morphism and so is twist by (7). Hence $F \xrightarrow{1} F$ factors through a cotorsion left R-module by Theorem 2.3. Thus F is a cotorsion left R-module. So R is a left perfect ring by [16, Proposition 3.3.1].

(8) $\Rightarrow$ (1) Let M be any left R-module. Then $\lambda : M \to C(M)$ is a twist morphism and so is a cotorsion morphism by (8). Hence M is a cotorsion left R-module by [5, Theorem 5.3.5]. So R is a left perfect ring. $\square$

Recall that a left R-module M is *FP-injective* [14] if $\mathrm{Ext}^1_R(N, M) = 0$ for any finitely presented left R-module N.

PROPOSITION 2.9. *The following are equivalent for a ring R:*

1. *R is a von Neumann regular ring.*
2. *Every flat-pure exact sequence in R-Mod is pure exact.*
3. *Every twist morphism in R-Mod is an* Ext*-phantom morphism.*
4. *Every twist morphism in R-Mod is a phantom morphism.*

Proof. (1) $\Rightarrow$ (2)-(4) are trivial.

(2) $\Rightarrow$ (1) For any left R-module M, there is an exact sequence $0 \to M \stackrel{\lambda}{\to} C(M) \to F \to 0$ with F flat. Also there is an exact sequence $0 \to C(M) \to E \to L \to 0$ with E injective. By Proposition 2.1, the exact sequence $0 \to C(M) \to E \to L \to 0$ is flat-pure and so is pure exact by (2). Thus $C(M)$ is FP-injective. Hence M is FP-injective. So R is a von Neumann regular ring by [15, 37.6].

(3) $\Rightarrow$ (1) For any left R-module M, there is an exact sequence $0 \to M \to C(M) \to F \to 0$ with F flat. Since the twist morphism $C(M) \stackrel{1}{\to} C(M)$ is an Ext-phantom morphism by (3), $C(M)$ is FP-injective. Hence M is FP-injective. So R is a von Neumann regular ring.

(4) $\Rightarrow$ (1) For any cotorsion left R-module M, $M \stackrel{1}{\to} M$ is a twist morphism and so is a phantom morphism by (4). Thus M is flat. Hence R is a von Neumann regular ring by [16, Theorem 3.3.2]. □

3. TWIST MORPHISMS UNDER CHANGE OF RINGS

Let $R \to S$ be a ring homomorphism. Then S is an R-R-bimodule in a canonical way. Moreover any left (resp. right) S-module can be regarded as a left (resp. right) R-module and any left (resp. right) S-module morphism can be regarded as a left (resp. right) R-module morphism.

Let ${}_RM$ be a left R-module and ${}_SN$ be a left S-module. Then there are a natural R-module morphism $\varepsilon_M : \mathrm{Hom}_R(S, M) \to {}_RM$ defined by $\varepsilon_M(f) = f(1)$ for any $f \in \mathrm{Hom}_R(S, M)$ and a natural S-module morphism $\eta_N : {}_SN \to \mathrm{Hom}_R(S, N)$ defined by $\eta_N(y)(t) = ty$ for any $y \in N$ and $t \in S$.

It is not hard to verify that the composition of R-module morphisms ${}_SN \stackrel{\eta_N}{\to} \mathrm{Hom}_R(S, N) \stackrel{\varepsilon_N}{\to} {}_RN$ is the identity and the composition of S-module morphisms $\mathrm{Hom}_R(S, M) \stackrel{\eta_{\mathrm{Hom}_R(S,M)}}{\to} \mathrm{Hom}_R(S, \mathrm{Hom}_R(S, M)) \stackrel{(\varepsilon_M)_*}{\to} \mathrm{Hom}_R(S, M)$ is also the identity.

LEMMA 3.1. *Let $R \to S$ be a ring homomorphism.*

1. *If S_R is flat and $\varphi : {}_SM \to {}_SN$ is a twist morphism in S-Mod, then $\varphi : {}_RM \to {}_RN$ is a twist morphism in R-Mod.*

2. *If ${}_RS$ is flat and $\psi : {}_RU \to {}_RV$ is a twist morphism in R-Mod, then $\psi_* : \mathrm{Hom}_R(S,U) \to \mathrm{Hom}_R(S,V)$ is a twist morphism in S-Mod.*

Proof. (1) For any flat left R-module ${}_RL$, $S \otimes_R L$ is a flat left S-module. By [13, Theorem 11.65], we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Ext}^1_S(S\otimes_R L, M) & \xrightarrow{\cong} & \mathrm{Ext}^1_R(L,M) \\ {\scriptstyle \mathrm{Ext}^1_S(S\otimes_R L,\varphi)}\downarrow & & \downarrow{\scriptstyle \mathrm{Ext}^1_R(L,\varphi)} \\ \mathrm{Ext}^1_S(S\otimes_R L, N) & \xrightarrow{\cong} & \mathrm{Ext}^1_R(L,N). \end{array}$$

Since $\mathrm{Ext}^1_S(S\otimes_R L,\varphi) = 0$, we have $\mathrm{Ext}^1_R(L,\varphi) = 0$. So $\varphi : {}_RM \to {}_RN$ is a twist morphism in R-Mod.

(2) By Theorem 2.3, $\psi : {}_RU \to {}_RV$ factors through a cotorsion left R-module ${}_RW$. By [12, Lemma 2.16]. $\mathrm{Hom}_R(S,W)$ is a cotorsion left S-module. Thus $\psi_* : \mathrm{Hom}_R(S,U) \to \mathrm{Hom}_R(S,V)$ factors through the cotorsion left S-module $\mathrm{Hom}_R(S,W)$. It follows that $\psi_* : \mathrm{Hom}_R(S,U) \to \mathrm{Hom}_R(S,V)$ is a twist morphism in S-Mod by Theorem 2.3. □

THEOREM 3.2. *Let $R \to S$ be a ring homomorphism with ${}_RS$ and S_R flat.*

1. *If a left S-module morphism $\varphi : {}_SM \to {}_SN$ is a twist preenvelope in S-Mod, then $\varphi : {}_RM \to {}_RN$ is a twist preenvelope in R-Mod.*

2. *If a left R-module morphism $\psi : {}_RU \to {}_RV$ is a twist precover in R-Mod, then $\psi_* : \mathrm{Hom}_R(S,U) \to \mathrm{Hom}_R(S,V)$ is a twist precover in S-Mod.*

Proof. (1) By Lemma 3.1(1), $\varphi : {}_RM \to {}_RN$ is a twist morphism in R-Mod. Let $f : {}_RM \to {}_RA$ be a twist morphism in R-Mod. Then $f_* : \mathrm{Hom}_R(S,M) \to \mathrm{Hom}_R(S,A)$ is a twist morphism in S-Mod by Lemma 3.1(2). So $f_*\eta_M : {}_SM \overset{\eta_M}{\to} \mathrm{Hom}_R(S,M) \overset{f_*}{\to} \mathrm{Hom}_R(S,A)$ is also a twist morphism in S-Mod. Thus there exists $g : {}_SN \to \mathrm{Hom}_R(S,A)$ such that $g\varphi = f_*\eta_M$.

From the following commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_R(S,M) & \xrightarrow{f_*} & \mathrm{Hom}_R(S,A) \\ {\scriptstyle \varepsilon_M}\downarrow & & \downarrow{\scriptstyle \varepsilon_A} \\ {}_RM & \xrightarrow{f} & {}_RA, \end{array}$$

we have

$$(\varepsilon_A g)\varphi = \varepsilon_A f_* \eta_M = f \varepsilon_M \eta_M = f.$$

Hence $\varphi : {}_RM \to {}_RN$ is a twist preenvelope in R-Mod.

(2) By Lemma 3.1(2), $\psi_* : \mathrm{Hom}_R(S,U) \to \mathrm{Hom}_R(S,V)$ is a twist morphism in S-Mod. Let $h : {}_SB \to \mathrm{Hom}_R(S,V)$ be any twist morphism in S-Mod. Then $h : {}_RB \to \mathrm{Hom}_R(S,V)$ is a twist morphism in R-Mod by Lemma 3.1(1). Thus $\varepsilon_V h : {}_RB \xrightarrow{h} \mathrm{Hom}_R(S,V) \xrightarrow{\varepsilon_V} {}_RV$ is also a twist morphism in R-Mod. Hence there exists $\alpha : {}_RB \to {}_RU$ such that $\psi\alpha = \varepsilon_V h$. From the following commutative diagram

$$\begin{array}{ccc} {}_SB & \xrightarrow{\quad h \quad} & \mathrm{Hom}_R(S,V) \\ \downarrow{\scriptstyle \eta_B} & & \downarrow{\scriptstyle \eta_{\mathrm{Hom}_R(S,V)}} \\ \mathrm{Hom}_R(S,B) & \xrightarrow{\quad h_* \quad} & \mathrm{Hom}_R(S,\mathrm{Hom}_R(S,V)), \end{array}$$

we obtain

$$\psi_*(\alpha_*\eta_B) = (\psi\alpha)_*\eta_B = (\varepsilon_V h)_*\eta_B = (\varepsilon_V)_*(h_*\eta_B) = (\varepsilon_V)_*\eta_{\mathrm{Hom}_R(S,V)}h = h.$$

Therefore $\psi_* : \mathrm{Hom}_R(S,U) \to \mathrm{Hom}_R(S,V)$ is a twist precover in S-Mod. □

Let S be a multiplicative subset of a commutative ring R. We can form the ring of fractions $S^{-1}R$. There is a canonical ring homomorphism $R \to S^{-1}R$. For an R-module M, we also can construct the localization of M with respect to S, denoted by $S^{-1}M$, which is an $S^{-1}R$-module and hence an R-module.

It is well known that $S^{-1}R$ is a flat R-module. So the following result is an immediate consequence of Theorem 3.2.

Corollary 3.3. *Let S be a multiplicative subset of a commutative ring R.*

1. *If an $S^{-1}R$-module morphism $\varphi : {}_{S^{-1}R}M \to {}_{S^{-1}R}N$ is a twist preenvelope in $S^{-1}R$-Mod, then $\varphi : {}_RM \to {}_RN$ is a twist preenvelope in R-Mod.*

2. *If an R-module morphism $\psi : {}_RU \to {}_RV$ is a twist precover in R-Mod, then $\psi_* : \mathrm{Hom}_R(S^{-1}R,U) \to \mathrm{Hom}_R(S^{-1}R,V)$ is a twist precover in $S^{-1}R$-Mod.*

Lemma 3.4. *Let $R \to S$ be a surjective ring homomorphism and ${}_SM$ a left S-module. Then $\eta_M : {}_SM \to \mathrm{Hom}_R(S,M)$ is an isomorphism.*

Proof. It is routine. □

Lemma 3.5. *Let $R \to S$ be a surjective ring homomorphism and φ: ${}_SM \to {}_SN$ be a morphism in S-Mod. If φ: ${}_SM \to {}_SN$ is a twist morphism in S-Mod, then $\varphi : {}_RM \to {}_RN$ is a twist morphism in R-Mod. The converse holds if ${}_RS$ is a flat left R-module.*

Proof. By Theorem 2.3, $\varphi : {}_SM \to {}_SN$ factors through a cotorsion left S-module ${}_SL$. By [16, Proposition 3.3.3], ${}_RL$ is a cotorsion left R-module. So $\varphi : {}_RM \to {}_RN$ factors through the cotorsion left R-module ${}_RL$. Hence φ is a twist morphism in R-Mod.

Conversely, if $\varphi : {}_RM \to {}_RN$ is twist as a morphism of left R-modules and ${}_RS$ is a flat left R-module, then $\psi_* : \mathrm{Hom}_R(S, M) \to \mathrm{Hom}_R(S, N)$ is a twist morphism in S-Mod by Lemma 3.1(2). So $\varphi : {}_SM \to {}_SN$ is a twist morphism in S-Mod by Lemma 3.4. □

THEOREM 3.6. *Let $R \to S$ be a surjective ring homomorphism with ${}_RS$ flat.*

1. *A left S-module morphism $\varphi : {}_SM \to {}_SN$ is a twist preenvelope (resp. twist envelope) in S-Mod if and only if the induced morphism $\varphi : {}_RM \to {}_RN$ is a twist preenvelope (resp. twist envelope) in R-Mod.*

2. *If a left R-module morphism $\psi : {}_RU \to {}_RV$ is a twist precover in R-Mod, then $\psi_* : \mathrm{Hom}_R(S, U) \to \mathrm{Hom}_R(S, V)$ is a twist precover in S-Mod.*

Proof. (1) "$\Rightarrow$" If $\varphi : {}_SM \to {}_SN$ is a twist preenvelope in S-Mod, then $\varphi : {}_RM \to {}_RN$ is a twist morphism in R-Mod by Lemma 3.5. Let $f : {}_RM \to {}_RA$ be a twist morphism in R-Mod. Then $f_* : \mathrm{Hom}_R(S, M) \to \mathrm{Hom}_R(S, A)$ is a twist morphism in S-Mod by Lemma 3.1(2). So $f_*\eta_M : {}_SM \xrightarrow{\eta_M} \mathrm{Hom}_R(S, M) \xrightarrow{f_*} \mathrm{Hom}_R(S, A)$ is also a twist morphism in S-Mod. Thus there exists $g : {}_SN \to \mathrm{Hom}_R(S, A)$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Hom}_R(S, M) & \xrightarrow{f_*} & \mathrm{Hom}_R(S, A) \\ \uparrow \eta_M & & \uparrow g \\ {}_SM & \xrightarrow{\varphi} & {}_SN. \end{array}$$

So we have

$$(\varepsilon_A g)\varphi = \varepsilon_A f_* \eta_M = f \varepsilon_M \eta_M = f.$$

Hence $\varphi : {}_RM \to {}_RN$ is a twist preenvelope in R-Mod.

Furthermore suppose that $\varphi : {}_SM \to {}_SN$ is a twist envelope in S-Mod. Let $\theta : {}_RN \to {}_RN$ be a left R-module morphism such that $\theta\varphi = \varphi$. Then $\theta_*\varphi_* = \varphi_*$.

By Lemma 3.4, we have

$$(\eta_N^{-1}\theta_*\eta_N)\varphi = \eta_N^{-1}\theta_*\varphi_*\eta_M = \eta_N^{-1}\varphi_*\eta_M = \eta_N^{-1}\eta_N\varphi = \varphi.$$

Hence $\eta_N^{-1}\theta_*\eta_N$ is an isomorphism. So $\theta = \varepsilon_N\theta_*\varepsilon_N^{-1}$ is an isomorphism. It follows that $\varphi : {}_RM \to {}_RN$ is a twist envelope in R-Mod.

“ $\Leftarrow$ ” If the induced morphism $\varphi : {}_RM \to {}_RN$ is a twist preenvelope in R-Mod, then $\varphi : {}_SM \to {}_SN$ is a twist morphism in S-Mod by Lemma 3.5. Let $\alpha : {}_SM \to {}_SB$ be a twist morphism in S-Mod. Then $\alpha : {}_RM \to {}_RB$ is a twist morphism in R-Mod by Lemma 3.5. Thus there exists $\beta : {}_RN \to {}_RB$ such that $\beta\varphi = \alpha$. By Lemma 3.4, $\beta = \varepsilon_B\beta_*\varepsilon_N^{-1} = \eta_B^{-1}\beta_*\eta_N$ is an S-module morphism. Hence $\varphi : {}_SM \to {}_SN$ is a twist preenvelope in S-Mod.

Furthermore, if the induced morphism $\varphi : {}_RM \to {}_RN$ is a twist envelope in R-Mod, then it is easy to verify that $\varphi : {}_SM \to {}_SN$ is a twist envelope in S-Mod.

(2) By Lemma 3.1(2), $\psi_* : \mathrm{Hom}_R(S,U) \to \mathrm{Hom}_R(S,V)$ is a twist morphism in S-Mod. Let $\gamma : {}_SC \to \mathrm{Hom}_R(S,V)$ be any twist morphism in S-Mod. Then $\gamma : {}_RC \to \mathrm{Hom}_R(S,V)$ is a twist morphism in R-Mod by Lemma 3.5. Thus $\varepsilon_V\gamma : {}_RC \xrightarrow{\gamma} \mathrm{Hom}_R(S,V) \xrightarrow{\varepsilon_V} {}_RV$ is also a twist morphism in R-Mod. Hence there exists $\delta : {}_RC \to {}_RU$ such that the following diagram is commutative:

$$\begin{array}{ccc} {}_RC & \xrightarrow{\quad\gamma\quad} & \mathrm{Hom}_R(S,V) \\ \Big\downarrow{\scriptstyle\delta} & & \Big\downarrow{\scriptstyle\varepsilon_V} \\ {}_RU & \xrightarrow{\quad\psi\quad} & {}_RV. \end{array}$$

So we have

$$\psi_*(\delta_*\eta_C) = (\psi\delta)_*\eta_C = (\varepsilon_V\gamma)_*\eta_C = (\varepsilon_V)_*(\gamma_*\eta_C) = (\varepsilon_V)_*\eta_{\mathrm{Hom}_R(S,V)}\gamma = \gamma.$$

Therefore $\psi_* : \mathrm{Hom}_R(S,U) \to \mathrm{Hom}_R(S,V)$ is a twist precover in S-Mod. $\square$

Acknowledgments. This research was supported by NSFC (12171230, 12271249) and NSF of Jiangsu Province of China (BK20211358). The author wants to express his gratitude to the referee for the very helpful comments and suggestions.

REFERENCES

[1] M. Auslander and S. Smalø, *Preprojective modules over artin algrbras.* J. Algebra **66** (1980), 61–122.

[2] E. E. Enochs, *Injective and flat covers, envelopes and resolvents.* Israel J. Math. **39** (1981), 189–209.

[3] E. E. Enochs, *Flat covers and flat cotorsion modules.* Proc. Amer. Math. Soc. **92** (1984), 179–184.

[4] E. E. Enochs and O. M. G. Jenda, *Relative Homological Algebra.* Walter de Gruyter, Berlin-New York, 2000.

[5] E. E. Enochs and L. Oyonarte, *Covers, Envelopes and Cotorsion Theories.* Nova Science Publishere, Inc., New York, 2002.

[6] S. Estrada, P. A. Guil Asensio and F. Ozbek, *Covering ideals of morphisms and module representations of the quiver* $\mathbb{A}_2$. J. Pure Appl. Algebra **218** (2014), 1953–1963.

[7] X. H. Fu, P. A. Guil Asensio, I. Herzog and B. Torrecillas, *Ideal approximation theory.* Adv. Math. **244** (2013), 750–790.

[8] R. Göbel and J. Trlifaj, *Approximations and Endomorphism Algebras of Modules.* GEM 41, Walter de Gruyter, Berlin-New York, 2006.

[9] P. A. Guil Asensio and I. Herzog, *Sigma-cotorsion rings.* Adv. Math. **191** (2005), 11–28.

[10] I. Herzog, *The phantom cover of a module*, Adv. Math. **215** (2007), 220–249.

[11] I. Herzog, *Contravariant functors on the category of finitely presented modules.* Israel J. Math. **167** (2008), 347–410.

[12] L. X. Mao and N. Q. Ding, *Notes on cotorsion modules.* Comm. Algebra **33** (2015), 349–360.

[13] J. J. Rotman, *An Introduction to Homological Algebra.* Academic Press, New York, 1979.

[14] B. Stenström, *Coherent rings and FP-injective modules.* J. London Math. Soc. **2** (1970), 323–329.

[15] R. Wisbauer, *Foundations of Module and Ring Theory.* Gordon and Breach, Philadelphia, 1991.

[16] J. Z. Xu, *Flat Covers of Modules.* Lecture Notes in Math. 1634, Springer, Berlin, 1996.

[17] H. Y. Zhu and N. Q. Ding, $\mathbb{S}$-*superfluous and* $\mathbb{S}$-*essential homomorphisms.* Acta Math. Sci. **29B** (2009), 391–401.

Received August 18, 2018

Nanjing Institute of Technology
School of Mathematics and Physics
Nanjing 211167, China
maolx2@hotmail.com