IMPROVED INEQUALITIES FOR OPERATOR SPACE NUMERICAL RADIUS

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In this paper, we present some upper and lower bounds for operator space numerical radius of $k \times k$ block matrices, with entries in $\mathcal{M}_n(X)$, when X is a numerical radius operator space. We also derive formulas for operator space numerical radius of some special bidiagonal block matrices. One of our main results states that if $(X, (W_n))$ is a numerical radius operator space and $x_1, \dots, x_k \in \mathcal{M}_n(X)$, then

$$\begin{aligned} & W_{kn}\left(\operatorname{offdiag}(x_1, x_2, \dots, x_k)\right) \\ & \leq \frac{1}{k} \max\left\{ W_n\left(\sum_{r=1}^{k} \omega^{2r} x_r\right), W_n\left(\sum_{r=1}^{k} x_r\right)\right\} + \frac{1}{k}\sum_{s=3}^{2k-1} W_n\left(\sum_{r=1}^{k} \omega^{rs} x_r\right) \\ & \leq \frac{1}{k}\sum_{s=2}^{2k} W_n\left(\sum_{r=1}^{k} \omega^{rs} x_r\right), \end{aligned}$$

where $\omega = e^{\frac{2\pi i}{k}}$ is the k-th root of unity.

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1. INTRODUCTION

Let $\mathcal{B}(H)$ denote the C^*-algebra of all bounded linear operators acting on a complex Hilbert space H, and let $H^{(n)}$ be the direct sum of n copies of H. We identify the $n \times n$ matrix algebra $\mathcal{M}_n(\mathcal{B}(H))$ with $\mathcal{B}(H^{(n)})$. We denote by $\| a \|_n$ the operator norm for $a = [a_{ij}] \in \mathcal{M}_n(\mathcal{B}(H))$ and recall that the numerical radius norm of a is given by

$$w_n(a) = \sup\left\{ | \langle ax, x\rangle | : x \in H^{(n)}, \| x \| = 1\right\}.$$

Several aspects of the numerical radius have been investigated by some mathematicians; see, for example, [8, 11, 13, 14, 5]. They are useful in investigation of quantum error correction and perturbation theory (*e.g.*, see [2, 4, 10] and references therein.

In [12], Ruan introduced the striking concept of operator space. An (abstract) operator space or briefly an OS is a complex linear space X together with a sequence of norms $O_n(\cdot)$ on the $n \times n$ matrix space $\mathcal{M}_n(X)$, which satisfies the following Ruan's axioms (OI) and (OII):

$$\text{(OI)} \qquad O_{m+n}\left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right) = \max\left\{O_m(x), O_n(y)\right\},$$

$$\text{(OII)} \qquad O_n(\alpha x \beta) \leq \| \alpha \| \, O_m(x) \, \| \beta \|,$$

for all $x \in \mathcal{M}_m(X)$, $y \in \mathcal{M}_n(X)$, $\alpha \in \mathcal{M}_{n,m}(\mathbb{C})$, and $\beta \in \mathcal{M}_{m,n}(\mathbb{C})$, where $\| \cdot \|$ denotes the usual operator norm on space of matrices of appropriate size with complex entries.

Itoh and Nagisa [7] introduced the notion of (abstract) numerical radius operator space, see also [6]. Following [7], a complex linear space X is a (abstract) numerical radius operator space or briefly $NROS$, if it admits a sequence of norms $W_n(\cdot)$ on $\mathcal{M}_n(X)$ for each $n \in \mathbb{N}$, which satisfies the following pair of conditions (WI) and (WII):

$$\text{(WI)} \qquad W_{m+n}\left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right) = \max\left\{W_m(x), W_n(y)\right\},$$

$$\text{(WII)} \qquad W_n(\alpha x \alpha^*) \leq \| \alpha \|^2 \, W_m(x),$$

for all $x \in \mathcal{M}_m(X)$, $y \in \mathcal{M}_n(X)$, and $\alpha \in \mathcal{M}_{n,m}(\mathbb{C})$, where α^* is the adjoint (the conjugate transpose) of α. It is easily seen that Ruan's axioms (OI) and (WI) hold for $k \times k$ diagonal block matrices.

Given abstract numerical radius operator spaces (resp., operator spaces) X, Y and a linear map φ from X to Y, we define φ_n from $\mathcal{M}_n(X)$ to $\mathcal{M}_n(Y)$ by

$$\varphi_n\left([x_{ij}]\right) = \left[\varphi(x_{ij})\right]_{n \times n}, \qquad [x_{ij}] \in \mathcal{M}_n(X).$$

We denote the numerical radius norm (resp., the norm) of $x = [x_{ij}] \in \mathcal{M}_n(X)$ by $W_n(x)$ (resp., $O_n(x)$) and the norm of φ_n by

$$W_n(\varphi_n) = \sup\left\{W_n(\varphi_n(x)) : x \in \mathcal{M}_n(X), W_n(x) \leq 1\right\}$$

(resp., $O_n(\varphi_n) = \sup\left\{O_n(\varphi_n(x)) : x \in \mathcal{M}_n(X), O_n(x) \leq 1\right\}$).

The W-completely bounded norm (resp., completely bounded norm) of φ is defined by

$$W(\varphi)_{cb} = \sup\left\{W_n(\varphi_n) : n \in \mathbb{N}\right\} \quad (\text{resp.}, O(\varphi)_{cb} = \sup\left\{O_n(\varphi_n) : n \in \mathbb{N}\right\})$$

if it exists. In addition, we call φ a W-complete isometry (resp., a complete isometry) if $W_n(\varphi_n(x)) = W_n(x)$ (resp., $O_n(\varphi_n(x)) = O_n(x)$) for all $x \in \mathcal{M}_n(X)$ and all $n \in \mathbb{N}$.

Ruan [12] proved that if $(X, (O_n))$ is an operator space, then there is a complete isometry ψ from X into $\mathcal{B}(H)$ for some Hilbert space H; that is, $O_n(x) = \parallel \psi_n(x) \parallel_n$ for all $x \in \mathcal{M}_n(X)$ and all $n \in \mathbb{N}$.
It is also proved that if $(X, (W_n))$ is a numerical radius operator space, then there is a W-complete isometry φ from X into $\mathcal{B}(H)$ for some Hilbert space H; that is, $W_n(x) = w_n(\varphi_n(x))$ for all $x \in \mathcal{M}_n(X)$ and all $n \in \mathbb{N}$, where $w_n(\cdot)$ is the usual numerical radius norm on $\mathcal{B}(H^{(n)})$. Utilizing this result, one can use the terminology "numerical radius operator space" without confusion.
Having a look at the known equality

$$\frac{1}{2} \parallel x \parallel = w\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right), \quad x \in \mathcal{B}(H),$$

it is shown in [7] that for a numerical radius operator space $(X, (W_n))$ if one defines O_n $(n \in \mathbb{N})$ by

$$O_n(x) := 2W_{2n}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right), \quad x \in \mathcal{M}_n(X),$$

then X turns into an operator space.

In [9], the authors investigated some relationships between the operator space norm and the operator space numerical radius on the matrix space $\mathcal{M}_n(X)$, when X is a numerical radius operator space. It is easy to verify that the norm of operator space and operator space numerical radius are unitarily invariant and weakly unitary invariant, respectively. These mean that $O_n(UxV) = O_n(x)$ and $W_n(UxU^*) = W_n(x)$ for all $x \in \mathcal{M}_n(X)$ and all unitaries $U, V \in \mathcal{M}_n(\mathbb{C})$.

In this paper, we adopt the terminology and notation in [9] to establish several upper and lower bounds for operator space numerical radius of off-diagonal $k \times k$ block matrices. These generalize the result of [9] and refine some inequalities in [1]. Moreover, we get formulas for operator space numerical radius of special bidiagonal block matrices with entries in $\mathcal{M}_n(X)$.
Throughout this paper, we suppose that $(X, (W_n))$ is an $NROS$.

2. BOUNDS FOR OFF-DIAGONAL MATRICES

In this section, we obtain some upper and lower bounds for the operator space numerical radius of off-diagonal matrices with entries in $\mathcal{M}_n(X)$. Let us fix our notation for diagonal and off-diagonal matrices

$$\operatorname{diag}(x_1, x_2, \dots, x_k) := \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & x_k \end{bmatrix}$$

and

$$\operatorname{offdiag}(x_1, x_2, \dots, x_k) := \begin{bmatrix} 0 & \cdots & 0 & x_1 \\ \vdots & & x_2 & 0 \\ 0 & \iddots & & \vdots \\ x_k & 0 & \cdots & 0 \end{bmatrix}.$$

First of all, we present a key lemma which states that discarding a zero row together with a zero column of a block matrix, increases the operator space numerical radius.

Lemma 2.1. *Let $x_{ij} \in \mathcal{M}_n(X)(1 \le i, j \le k)$. For given $1 \le r, s \le k$, if $S = [x_{ij}]$ with $x_{rj} = x_{is} = 0$, for all $1 \le i, j \le k$, and T is the $(k-1) \times (k-1)$ matrix obtained from S by removing both its r-th row and its s-th column, then*

$$W_{kn}(S) \le W_{(k-1)n}(T).$$

In particular, if $x_1, x_2, \dots, x_k \in \mathcal{M}_n(X)$, then for given $1 \le r \le k$,

$$\begin{aligned} &W_{kn}\left(\operatorname{offdiag}(x_1, \dots, x_{r-1}, 0, x_{r+1}, \dots, x_k)\right) \\ &\qquad \le W_{(k-1)n}\left(\operatorname{offdiag}(x_1, \dots, x_{r-1}, x_{r+1}, \dots, x_k)\right). \end{aligned}$$

Proof. Since $(X, (W_n))$ is an $NROS$, Ruan's Theorem states that there exists a W-complete isometry Φ from X into $\mathcal{B}(H)$. As Φ is a W-complete isometry, we have $W_n(x) = w_n(\Phi_n(x))$ for all $x \in \mathcal{M}_n(X)$. We get

$$W_{kn}(S) = w_{kn}\left(\Phi_{kn}(S)\right) \le w_{(k-1)n}\left(\Phi_{(k-1)n}(T)\right) = W_{(k-1)n}(T).$$

□

Lemma 2.2. $O_n(x) = 2W_{kn}\left(\operatorname{offdiag}(x, 0, \dots, 0)\right)$, *where all entries are in $\mathcal{M}_n(X)$.*

Proof. Since $(X, (W_n))$ is an $NROS$, Ruan's Theorem says that there exists a complete and W-complete isometry Φ from X into $\mathcal{B}(H)$. As Φ is a complete isometry, we have
$O_n(x) = \|\Phi_n(x)\|$. In addition, since Φ is a W-complete isometry, we have $W_n(x) = w_n(\Phi_n(x))$. It follows from nilpotency of $\operatorname{offdiag}(\Phi_n(x), 0, \dots, 0) \in \mathcal{M}_{kn}(\mathcal{B}(H))$ that

$$O_n(x) = \|\Phi_n(x)\|$$

$$
\begin{aligned}
&= \|\text{offdiag}\,(\Phi_n(x), 0, \ldots, 0)\| \\
&= 2w_{kn}\,(\text{offdiag}\,(\Phi_n(x), 0, \ldots, 0)) \\
&= 2w_{kn}\,(\Phi_{kn}\,(\text{offdiag}\,(x, 0, \ldots, 0))) \\
&= 2W_{kn}\,(\text{offdiag}\,(x, 0, \ldots, 0))\,.
\end{aligned}
$$

□

The following theorem is obtained by a modification of [9, Lemma 2.3]. We state its proof for the sake of completeness. We observe that it refines [1, Theorem 2.2], when k is an odd positive integer.

THEOREM 2.3. *If* $x_1, x_2, \ldots, x_k \in \mathcal{M}_n(X)$, *then*

$$
\frac{1}{2}\max_{1\le i\le k}\{O_n(x_i)\} \le W_{kn}\,(\text{offdiag}\,(x_1, x_2, \ldots, x_k)) \le \frac{1}{2}\sum_{i=1}^{k} O_n(x_i). \tag{2.1}
$$

Proof. For each $1 \le i, j \le k$, let E_{ij} be the unit matrix whose ij entry is 1 and the other ones are 0. Putting $U = E_{ii}$ and $V = E_{k-i+1,i}$, by inequality (OII), we obtain

$$
\begin{aligned}
\frac{1}{2}O_n(x_i) &= \frac{1}{2}O_{kn}\,(\text{diag}\,(0, \ldots, 0, x_i, 0, \ldots, 0)) \\
&= \frac{1}{2}O_{kn}\,(U\text{offdiag}\,(x_1, \ldots, x_i, \ldots, x_k)\,V) \\
&\le \frac{1}{2}O_{kn}\,(\text{offdiag}\,(x_1, \ldots, x_i, \ldots, x_k)) \\
&\le W_{kn}\,(\text{offdiag}\,(x_1, \ldots, x_i, \ldots, x_k))\,.
\end{aligned}
$$

Now by taking maximum over $1 \le i \le k$, we get the left hand inequality of (2.1). In addition, using the triangle inequality, Lemmas 2.2 and 2.1 and the weakly unitary invariance of W_{2n}, we reach

$$
\begin{aligned}
&W_{kn}\,(\text{offdiag}(x_1, x_2, \ldots, x_k)) \\
&\le W_{kn}\,(\text{offdiag}\,(x_1, 0, \ldots, 0, 0)) + W_{kn}\,(\text{offdiag}\,(0, x_2, \ldots, x_{k-1}, x_k)) \\
&\le \frac{1}{2}O_n(x_1) + W_{(k-1)n}\,(\text{offdiag}(x_2, x_3, \ldots, x_{k-1}, x_k)) \\
&\le \cdots \\
&\le \frac{1}{2}\sum_{i=1}^{k-1} O_n(x_i) + W_{2n}\left(\begin{bmatrix} 0 & 0 \\ x_k & 0 \end{bmatrix}\right) \\
&= \frac{1}{2}\sum_{i=1}^{k-1} O_n(x_i) + W_{2n}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ x_k & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)
\end{aligned}
$$

$$= \frac{1}{2}\sum_{i=1}^{k-1} O_n(x_i) + W_{2n}\left(\begin{bmatrix} 0 & x_k \\ 0 & 0 \end{bmatrix}\right)$$
$$= \frac{1}{2}\sum_{i=1}^{k} O_n(x_i).$$

□

The following corollary is proved in a special case for operators in [1, Theorem 2.10]. We generalize it in a new fashion.

COROLLARY 2.4. *If $x_{ij} \in \mathcal{M}_n(X)$ for $1 \le i, j \le k$, then*

$$W_{kn}\left([x_{ij}]\right) \le \max_{1\le i\le k}\{W_n(x_{ii})\} + \frac{1}{2}\sum_{i\ne j} O_n(x_{ij}).$$

Proof. For each $1 \le r, s \le k$, let $T_{rs} := [t_{ij}]$ be such that

$$t_{ij} = \begin{cases} x_{ij}, & i = r, j = s, \\ 0, & \text{otherwise.} \end{cases}$$

Using Lemmas 2.1 and 2.2, for $r \ne s$ we have

$$W_{kn}(T_{rs}) \le W_{2n}\left(\begin{bmatrix} 0 & x_{rs} \\ 0 & 0 \end{bmatrix}\right) = \frac{1}{2}O_n(x_{rs}),$$

for $r = 1$ or $s = k$, and

$$W_{kn}(T_{rs}) \le W_{2n}\left(\begin{bmatrix} 0 & 0 \\ x_{rs} & 0 \end{bmatrix}\right) = \frac{1}{2}O_n(x_{rs})$$

for $r \ne 1$ and $s \ne k$.

Now, employing equality (WI), we get

$$W_{kn}\left([x_{ij}]\right) = W_{kn}\left(\operatorname{diag}(x_{11}, x_{22}, \ldots, x_{kk}) + \sum_{r\ne s} T_{rs}\right)$$
$$\le W_{kn}\left(\operatorname{diag}(x_{11}, x_{22}, \ldots, x_{kk})\right) + \sum_{r\ne s} W_{kn}(T_{rs})$$
$$\le \max_{1\le i\le k}\{W_n(x_{ii})\} + \frac{1}{2}\sum_{r\ne s} O_n(x_{rs}).$$

□

To achieve our next result we need the following lemma.

LEMMA 2.5. *Let $x_1, x_2, \dots, x_k \in \mathcal{M}_n(X)$. If $\theta_1, \theta_2, \dots, \theta_k \in \mathbb{R}$ are such that for each $1 \le r, s \le k, \theta_r + \theta_{k+1-r} \equiv \theta_s + \theta_{k+1-s} \pmod{4\pi}$, then*

$$W_{kn}\left(\operatorname{offdiag}\left(e^{i\theta_1}x_1, e^{i\theta_2}x_2, \dots, e^{i\theta_k}x_k\right)\right) = W_{kn}\left(\operatorname{offdiag}(x_1, x_2, \dots, x_k)\right).$$

Proof. If

$$U = \operatorname{diag}\left(e^{\frac{i\theta_1}{2}}1, e^{\frac{i\theta_2}{2}}1, \dots, e^{\frac{i\theta_k}{2}}1\right)$$

and

$$T = \operatorname{offdiag}\left(e^{i\theta_1}x_1, e^{i\theta_2}x_2, \dots, e^{i\theta_k}x_k\right),$$

then U is a unitary matrix in $\mathcal{M}_{kn}(\mathbb{C})$ and

$$\begin{aligned} U^*TU &= \operatorname{offdiag}\left(e^{i(\frac{\theta_1+\theta_k}{2})}x_1, e^{i(\frac{\theta_2+\theta_{k-1}}{2})}x_2, \dots, e^{i(\frac{\theta_k+\theta_1}{2})}x_k\right) \\ &= e^{i(\frac{\theta_1+\theta_k}{2})}\left(\operatorname{offdiag}(x_1, x_2, \dots, x_k)\right). \end{aligned}$$

Now, the assertion follows by the weakly unitary invariance of W_{kn}. □

COROLLARY 2.6. *Let $k = 2l$.*

(a) *If $\theta_1 = \theta_3 = \theta_5 = \cdots = \theta_{2l-1} = 0$ and $\theta_2 = \theta_4 = \theta_6 = \cdots = \theta_{2l} = \pi$, then*

$$W_{kn}\left(\operatorname{offdiag}(x_1, x_2, \dots, x_k)\right) = W_{kn}\left(\operatorname{offdiag}(x_1, -x_2, \dots, x_{k-1}, -x_k)\right).$$

(b) *If $\theta_1 = \theta_2 = \cdots = \theta_l = 0$ and $\theta_{l+1} = \theta_{l+2} = \cdots = \theta_{2l} = \pi$, then*

$$W_{kn}\left(\operatorname{offdiag}(x_1, x_2, \dots, x_k)\right) = W_{kn}\left(\operatorname{offdiag}(x_1, \dots, x_l, -x_{l+1}, \dots, -x_k)\right).$$

The next theorem generalizes [9, Theorem 2.9].

THEOREM 2.7. *If $x_1, x_2, \dots, x_k \in \mathcal{M}_n(X)$ and $\theta_1, \theta_2, \dots, \theta_k \in \mathbb{R}$, then*

$$\begin{aligned} &W_{kn}\left(\operatorname{offdiag}(x_1, x_2, \dots, x_k)\right) \\ &\ge \frac{1}{k}\sup\left\{W_n\left(\sum_{r=1}^{k} e^{i\theta_r}x_r\right) : \theta_r + \theta_{k+1-r} \equiv \theta_s + \theta_{k+1-s} \pmod{4\pi}, 1 \le r, s \le k\right\}. \end{aligned}$$

Proof. If for each $1 \le r, s \le k$, $\theta_r + \theta_{k+1-r} \equiv \theta_s + \theta_{k+1-s} \pmod{4\pi}$, then by Lemma 2.5 and inequality (WII),

$$\begin{aligned} &W_n\left(\sum_{r=1}^{k} e^{i\theta_r}x_r\right) \\ &= W_n\left(\begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix}\left(\operatorname{offdiag}\left(e^{i\theta_1}x_1, e^{i\theta_2}x_2, \dots, e^{i\theta_k}x_k\right)\right)\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}\right) \end{aligned}$$

$$\leq \left\| \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \right\| W_{kn} \left(\operatorname{offdiag} \left(e^{i\theta_1} x_1, e^{i\theta_2} x_2, \ldots, e^{i\theta_k} x_k \right) \right) \left\| \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \right\|$$

$$= k W_{kn} \left(\operatorname{offdiag}(x_1, x_2, \ldots, x_k) \right).$$

□

The first part of the next corollary can be obtained via another method in [1, Corollary 2.9].

COROLLARY 2.8. *Let* $x_1, x_2, \ldots, x_k \in \mathcal{M}_n(X)$.
(a) *If* $\theta_r = 0$ $(r = 1, \ldots, k)$, *then*

$$W_{kn} \left(\operatorname{offdiag}(x_1, x_2, \ldots, x_k) \geq \frac{1}{k} W_n \left(\sum_{r=1}^{k} x_r \right). \right.$$

(b) *If* $\theta_r = \frac{2\pi r}{k}$ $(r = 1, \ldots, k)$, *then*

$$W_{kn} \left(\operatorname{offdiag}(x_1, x_2, \ldots, x_k) \right) \geq \frac{1}{k} W_n \left(\sum_{r=1}^{k} \omega_r x_r \right),$$

where $\omega_r = e^{\frac{2\pi r i}{k}}$ $(r = 1, \ldots, k)$ *are the* k *roots of unity.*

The following theorem refines [1, Theorem 2.8].

THEOREM 2.9. *If* $x_i, y_i \in \mathcal{M}_n(X)$ *and* $\alpha_i, \beta_i \in \mathcal{M}_n(\mathbb{C})$ *for* $1 \leq i \leq k$, *then*

$$W_n \left(\sum_{i=1}^{k} \alpha_i x_i \beta_{k-i+1}^* + \beta_i y_i \alpha_{k-i+1}^* \right) \leq 2 \left(\sum_{i=1}^{k} \| \alpha_i \| \, \| \beta_{k-i+1} \| \right) W_{2kn}(T)$$

$$\leq 2 \sqrt{ \left(\sum_{i=1}^{k} \|\alpha_i\|^2 \right) \left(\sum_{i=1}^{k} \|\beta_i\|^2 \right) } W_{2kn}(T), \tag{2.2}$$

where

$$T = \operatorname{offdiag}\left(x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k \right) \in \mathcal{M}_{2kn}(X).$$

Proof. Put $C = \begin{bmatrix} \alpha_1 & \cdots & \alpha_k & \beta_1 & \cdots & \beta_k \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}$. So,

$$W_n \left(\sum_{i=1}^{k} \alpha_i x_i \beta_{k-i+1}^* + \beta_i y_i \alpha_{k-i+1}^* \right)$$

$$
\begin{aligned}
&= W_{2n}\left(\begin{bmatrix} \sum_{i=1}^{k}\left(\alpha_i x_i \beta_{k-i+1}^* + \beta_i y_i \alpha_{k-i+1}^*\right) & 0 \\ 0 & 0 \end{bmatrix}\right) \\
&= W_{2n}(CTC^*) \leq \| C \|^2 \, W_{2kn}(T) \\
&= \left\| \sum_{i=1}^{k} (\alpha_i \alpha_i^* + \beta_i \beta_i^*) \right\| W_{2kn}(T) \\
&\leq \left(\sum_{i=1}^{k} \| \alpha_i \|^2 + \| \beta_i \|^2 \right) W_{2kn}(T).
\end{aligned}
$$

Let $t_i > 0$ $(i = 1, \ldots, k)$. Replace α_i and β_i by $t_i\alpha_i$ and $t_{k-i+1}^{-1}\beta_i$ $(i = 1, \ldots, k)$, respectively, and use the following equality

$$\inf_{t_i > 0} \frac{t_i^2 u + t_i^{-2} v}{2} = \sqrt{uv}$$

to get

$$W_n\left(\sum_{i=1}^{k} \alpha_i x_i \beta_{k-i+1}^* + \beta_i y_i \alpha_{k-i+1}^*\right) \leq 2\left(\sum_{i=1}^{k} \| \alpha_i \| \, \| \beta_{k-i+1} \|\right) W_{2kn}(T).$$

The second inequality of (2.2) is obtained by a usage of the Cauchy–Schwarz inequality. □

COROLLARY 2.10 (see [3]). *If $x \in \mathcal{M}_n(X)$ and $\alpha \in \mathcal{M}_n(\mathbb{C})$, then*

$$W_n(\alpha x \pm x\alpha^*) \leq 2 \, \| \alpha \| \, W_n(x).$$

Proof. Put $k = 1$, $\alpha_1 = \alpha$, $\beta_1 = I$, and $x_1 = y_1 = x$, and consider

$$W_{2n}\left(\begin{bmatrix} 0 & x \\ x & 0 \end{bmatrix}\right) = W_n(x). \tag{2.3}$$

See also [9, Lemma 2.8 (c)]. □

COROLLARY 2.11. *If $x, y, z, w \in \mathcal{M}_n(X)$, then*

$$
\begin{aligned}
\max&\left\{\max\{W_n(x), W_n(w)\}, \frac{1}{2}W_n(y+z)\right\} \\
&\leq W_{2n}\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right) \\
&\leq \max\{W_n(x), W_n(w)\} + \frac{O_n(y) + O_n(z)}{2}.
\end{aligned}
$$

Proof. Use [9, Lemma 3.1] stating that

$$\max\left\{W_{2n}\left(\begin{bmatrix} x & 0 \\ 0 & w \end{bmatrix}\right), W_{2n}\left(\begin{bmatrix} 0 & y \\ z & 0 \end{bmatrix}\right)\right\} \leqslant W_{2n}\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right)$$

and Corollary 2.8 (a) for the left inequality, and the triangle inequality and Corollary 2.4 for the right one. See also [1, Corollary 2.11]. □

3. SPECIAL BLOCK MATRICES

In this section, we first provide upper bounds for some special types of matrices, with only either one nonzero row or one nonzero column. Then, we establish formulas for operator space numerical radius of some special bidiagonal matrices. At the end, we give a suitable upper bound for operator space numerical radius of general off-diagonal block matrices. Throughout this section, we use the following notation:

$$\mathrm{row}_r(x_1, x_2, \ldots, x_k) := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ x_1 & x_2 & \cdots & x_k \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

where $x_1, x_2, \ldots, x_k$ are located in the r-th row, and similarly

$$\mathrm{col}_s(x_1, x_2 \ldots, x_k) := \begin{bmatrix} 0 & \cdots & 0 & x_1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & x_2 & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & x_k & 0 & \cdots & 0 \end{bmatrix},$$

where $x_1, x_2, \ldots, x_k$ are located in the s-th column for $r, s = 1, \ldots, k$.

THEOREM 3.1. *If $x_1, x_2, \ldots, x_k \in \mathcal{M}_n(X)$, then*

$$O_{kn}\left(\mathrm{row}_r(x_1, x_2, \ldots, x_k)\right) \le \frac{1}{\sqrt{k}} \sum_{s=1}^{k} O_n\left(\sum_{l=1}^{k} \omega^{-(s-1)(l-1)} x_l\right) \tag{3.1}$$

and

$$O_{kn}\left(\mathrm{col}_s(x_1, x_2, \ldots, x_k)\right) \le \frac{1}{\sqrt{k}} \sum_{r=1}^{k} O_n\left(\sum_{l=1}^{k} \omega^{(r-1)(l-1)} x_l\right), \tag{3.2}$$

where $\omega = e^{\frac{2\pi i}{k}}$ is the k-th root of unity.

Proof. Let V be the unit matrix obtained by changing the first and r-th rows of the identity matrix in $\mathcal{M}_k(\mathcal{M}_n(\mathbb{C}))$. Since

$$\mathrm{row}_r(x_1, x_2, \ldots, x_k) = V\left(\mathrm{row}_1(x_1, x_2, \ldots, x_k)\right),$$

it suffices to prove inequality (3.1) only when $r = 1$. So, let $1, \omega, \omega^2, \ldots, \omega^{k-1}$ be the k roots of unity, where $\omega = e^{\frac{2\pi i}{k}}$ and

$$U = \frac{1}{\sqrt{k}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega 1 & \omega^2 1 & \cdots & \omega^{k-1} 1 \\ 1 & \omega^2 1 & \omega^4 1 & \cdots & \omega^{k-2} 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \omega^{k-1} 1 & \omega^{k-2} 1 & \cdots & \omega 1 \end{bmatrix}.$$

It is seen that U is a unitary matrix in $\mathcal{M}_{kn}(\mathbb{C})$ and

$$kU\left(\mathrm{row}_1(x_1, x_2, \ldots, x_k)\right) U^* = \left[\sum_{l=1}^{k} \omega^{-(s-1)(l-1)} x_l\right].$$

Utilizing unitary invariance of O_{kn}, we get

$$kO_{kn}\left(\mathrm{row}_1(x_1, x_2, \ldots, x_k)\right) = O_{kn}\left(\left[\sum_{l=1}^{k} \omega^{-(s-1)(l-1)} x_l\right]\right).$$

For each $x \in \mathcal{M}_n(X)$, using equality (OI) and inequality (OII), we get

$$\begin{aligned} O_{kn}\left(\mathrm{col}_s(x, x, \ldots, x)\right) &= O_{kn}\left(\mathrm{diag}(x, x, \ldots, x)\ \mathrm{col}_s(1, 1, \ldots, 1)\right) \\ &\leq O_{kn}\left(\mathrm{diag}(x, x, \ldots, x)\right)\ \|\mathrm{col}_s(1, 1, \ldots, 1)\| \\ &= \sqrt{k} O_n(x). \end{aligned}$$

Therefore,

$$\begin{aligned} &O_{kn}\left(\left[\sum_{l=1}^{k} \omega^{-(s-1)(l-1)} x_l\right]\right) \\ &= O_{kn}\left(\sum_{s=1}^{k}\left(\mathrm{col}_s\left(\sum_{l=1}^{k} \omega^{-(s-1)(l-1)} x_l, \ldots, \sum_{l=1}^{k} \omega^{-(s-1)(l-1)} x_l\right)\right)\right) \\ &\leq \sqrt{k} \sum_{s=1}^{k} O_n\left(\sum_{l=1}^{k} \omega^{-(s-1)(l-1)} x_l\right). \end{aligned}$$

The inequality (3.2) follows by a similar method, and so we omit its proof. □

Corollary 3.2. *If $x_{rs} \in \mathcal{M}_n(X)$ for $1 \leq r, s \leq k$, then*

$$O_{kn}\left([x_{rs}]\right) \leq \frac{1}{\sqrt{k}} \sum_{r,s=1}^{k} O_n\left(\sum_{l=1}^{k} \omega^{(r-1)(l-1)} x_{ls}\right)$$

and

$$O_{kn}\left([x_{rs}]\right) \leq \frac{1}{\sqrt{k}} \sum_{r,s=1}^{k} O_n\left(\sum_{l=1}^{k} \omega^{-(s-1)(l-1)} x_{rl}\right),$$

where $\omega = e^{\frac{2\pi i}{k}}$ *is the* k*-th root of unity.*

Proof.

$$[x_{rs}] = \sum_{s=1}^{k} \mathrm{col}_s(x_{1s}, x_{2s}, \ldots, x_{ks}) = \sum_{r=1}^{k} \mathrm{row}_r(x_{r1}, x_{r2}, \ldots, x_{rk}).$$

□

Now we obtain some interesting formulas for following bidiagonal block matrices. First assume that the size of our bidiagonal square matrix is even.

THEOREM 3.3. *If* $x, y \in \mathcal{M}_n(X)$ *and* $T \in \mathcal{M}_{2kn}(X)$ *such that*

(3.3)

$$T = \begin{bmatrix} x & 0 & \cdots & 0 & 0 & \cdots & 0 & y \\ 0 & x & \cdots & 0 & 0 & \cdots & y & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & x & y & \cdots & 0 & 0 \\ 0 & 0 & \cdots & y & x & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & y & \cdots & 0 & 0 & \cdots & x & 0 \\ y & 0 & \cdots & 0 & 0 & \cdots & 0 & x \end{bmatrix} = \begin{bmatrix} \mathrm{diag}(x, \ldots, x) & \mathrm{offdiag}(y, \ldots, y) \\ \mathrm{offdiag}(y, \ldots, y) & \mathrm{diag}(x, \ldots, x) \end{bmatrix},$$

then

$$W_{2kn}(T) = \max\Big\{W_n(x+y), W_n(x-y)\Big\}.$$

Proof. Let U be the unitary matrix in $\mathcal{M}_{2kn}(\mathbb{C})$, given by

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathrm{diag}(1, \ldots, 1) & \mathrm{offdiag}(1, \ldots, 1) \\ \mathrm{offdiag}(1, \ldots, 1) & -\mathrm{diag}(1, \ldots, 1) \end{bmatrix}.$$

It is easy to see that

$$UTU^* = \mathrm{diag}\left(x+y, \ldots, x+y, x-y, \ldots, x-y\right) \in \mathcal{M}_{2kn}(X).$$

Now, the assertion follows by the weakly unitary invariance of W_{2kn} and equality (WI). □

COROLLARY 3.4. *If* $x \in \mathcal{M}_n(X)$ *and* $\alpha, \beta \in \mathbb{C}$*, then*

$$W_{2kn}\left(\begin{bmatrix} \mathrm{diag}(\alpha x, \ldots, \alpha x) & \mathrm{offdiag}(\beta x, \ldots, \beta x) \\ \mathrm{offdiag}(\beta x, \ldots, \beta x) & \mathrm{diag}(\alpha x, \ldots, \alpha x) \end{bmatrix}\right)$$

$$= \max\{| \alpha+\beta |, | \alpha-\beta |\} W_n(x).$$

In particular,

$$W_{2kn}\left(\begin{bmatrix} \mathrm{diag}(x,\dots,x) & \mathrm{offdiag}(-x,\dots,-x) \\ \mathrm{offdiag}(-x,\dots,-x) & \mathrm{diag}(x,\dots,x) \end{bmatrix}\right) = 2W_n(x).$$

Next, suppose that the size of our bidiagonal square matrix is odd.

THEOREM 3.5. *If $x, y \in \mathcal{M}_n(X)$ and $S, T \in \mathcal{M}_{(2k+1)n}(X)$ such that*

$$S = \begin{bmatrix} \mathrm{diag}(x,\dots,x) & 0 & \mathrm{offdiag}(y,\dots,y) \\ 0 & x & 0 \\ \mathrm{offdiag}(y,\dots,y) & 0 & \mathrm{diag}(x,\dots,x) \end{bmatrix}$$

and

$$T = \begin{bmatrix} \mathrm{diag}(x,\dots,x) & 0 & \mathrm{offdiag}(y,\dots,y) \\ 0 & y & 0 \\ \mathrm{offdiag}(y,\dots,y) & 0 & \mathrm{diag}(x,\dots,x) \end{bmatrix}, \tag{3.4}$$

then

$$W_{(2k+1)n}(S) = \max\{W_n(x+y), W_n(x), W_n(x-y)\}$$

and

$$W_{(2k+1)n}(T) = \max\{W_n(x+y), W_n(y), W_n(x-y)\}.$$

Proof. Let U be the following unitary matrix in $\mathcal{M}_{(2k+1)n}(\mathbb{C})$:

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathrm{diag}(1,\dots,1) & 0 & \mathrm{offdiag}(1,\dots,1) \\ 0 & \sqrt{2} & 0 \\ \mathrm{offdiag}(1,\dots,1) & 0 & -\mathrm{diag}(1,\dots,1) \end{bmatrix}.$$

It is easy to verify that

$$USU^* = \mathrm{diag}(x+y,\dots,x+y,x,x-y,\dots,x-y) \in \mathcal{M}_{(2k+1)n}(X)$$

and

$$UTU^* = \mathrm{diag}(x+y,\dots,x+y,y,x-y,\dots,x-y) \in \mathcal{M}_{(2k+1)n}(X).$$

Now, the assertion follows by the weakly unitary invariance of $W_{(2k+1)n}$ and equality (WI). □

COROLLARY 3.6. *If $x \in \mathcal{M}_n(X)$ and $\alpha, \beta \in \mathbb{C}$, then*

$$W_{(2k+1)n}\left(\begin{bmatrix} \mathrm{diag}(\alpha x,\dots,\alpha x) & 0 & \mathrm{offdiag}(\beta x,\dots,\beta x) \\ 0 & \alpha x & 0 \\ \mathrm{offdiag}(\beta x,\dots,\beta x) & 0 & \mathrm{diag}(\alpha x,\dots,\alpha x) \end{bmatrix}\right)$$

$$= W_{(2k+1)n}\left(\begin{bmatrix} \mathrm{diag}(\alpha x,\dots,\alpha x) & 0 & \mathrm{offdiag}(\beta x,\dots,\beta x) \\ 0 & \beta x & 0 \\ \mathrm{offdiag}(\beta x,\dots,\beta x) & 0 & \mathrm{diag}(\alpha x,\dots,\alpha x) \end{bmatrix}\right)$$

$$= \max\{|\ \alpha+\beta\ |, |\ \alpha-\beta\ |\}\, W_n(x).$$

In particular,

$$W_{(2k+1)n}\left(\begin{bmatrix} \operatorname{diag}(x,\dots,x) & 0 & \operatorname{offdiag}(-x,\dots,-x) \\ 0 & x & 0 \\ \operatorname{offdiag}(-x,\dots,-x) & 0 & \operatorname{diag}(x,\dots,x) \end{bmatrix}\right)$$
$$= W_{(2k+1)n}\left(\begin{bmatrix} \operatorname{diag}(x,\dots,x) & 0 & \operatorname{offdiag}(-x,\dots,-x) \\ 0 & -x & 0 \\ \operatorname{offdiag}(-x,\dots,-x) & 0 & \operatorname{diag}(x,\dots,x) \end{bmatrix}\right)$$
$$= W_{(2k+1)n}\left(\begin{bmatrix} \operatorname{diag}(x,\dots,x) & 0 & \operatorname{offdiag}(x,\dots,x) \\ 0 & x & 0 \\ \operatorname{offdiag}(x,\dots,x) & 0 & \operatorname{diag}(x,\dots,x) \end{bmatrix}\right)$$
$$= 2W_n(X).$$

Proof. Consider

$$\max\{|\ \alpha+\beta\ |, |\ \alpha\ |, |\ \alpha-\beta\ |\} = \max\{|\ \alpha+\beta\ |, |\ \alpha-\beta\ |\}.$$

□

Remark 3.7. If $y \in \mathcal{M}_n(X)$, then taking $x = 0$ in (3.3) and (3.4), we conclude that for each $l = 1, 2, \dots$,

$$W_{ln}\left(\operatorname{offdiag}(y, y, \dots, y)\right) = W_n(y).$$

This is the straightforward generalization of equality (2.3).

The following theorem extends the second inequality in [9, Theorem 2.9] for $k \times k$ off-diagonal matrices with entries in $\mathcal{M}_n(X)$.

Theorem 3.8. *If $x_1, x_2, \dots, x_k \in \mathcal{M}_n(X)$, then*

$$W_{kn}\left(\operatorname{offdiag}(x_1, x_2, \dots, x_k)\right)$$
$$\leq \frac{1}{k}\max\left\{W_n\left(\sum_{r=1}^{k}\omega^{2r}x_r\right), W_n\left(\sum_{r=1}^{k}x_r\right)\right\} + \frac{1}{k}\sum_{s=3}^{2k-1}W_n\left(\sum_{r=1}^{k}\omega^{rs}x_r\right)$$
$$\leq \frac{1}{k}\sum_{s=2}^{2k}W_n\left(\sum_{r=1}^{k}\omega^{rs}x_r\right), \tag{3.5}$$

where $\omega = e^{\frac{2\pi i}{k}}$ is the k-th root of unity.

Proof. If $U = \frac{1}{\sqrt{k}} \begin{bmatrix} 1 & \omega 1 & \omega^2 1 & \cdots & \omega^{(k-1)} 1 \\ 1 & \omega^2 1 & \omega^4 1 & \cdots & \omega^{2(k-1)} 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \omega^{(k-1)} 1 & \omega^{2(k-1)} 1 & \cdots & \omega^{(k-1)(k-1)} 1 \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$,

where $\omega = e^{\frac{2\pi i}{k}}$, then U is a unitary matrix in $\mathcal{M}_{kn}(\mathbb{C})$. But,

$$U\left(\operatorname{offdiag}(x_1, x_2, \ldots, x_k)\right) U^* = \frac{1}{k}\left[\sum_{r=1}^{k} \omega^{-r(i+j)+j} x_{k-r+1}\right].$$

Using the weakly unitary invariance of W_{kn}, we have

$$kW_{kn}\left(\operatorname{offdiag}(x_1, x_2, \ldots, x_k)\right) = W_{kn}\left(\left[\sum_{r=1}^{k} \omega^{-r(i+j)+j} x_{k-r+1}\right]\right).$$

Employing the triangle inequality and sweeping along the off-diagonal, we get

$$\begin{aligned}
&kW_{kn}\left(\operatorname{offdiag}(x_1, x_2, \ldots, x_k)\right)\\
&\leq W_{kn}\left(\operatorname{diag}\left(\sum_{r=1}^{k} \omega^{-2r+1} x_{k-r+1}, 0, \ldots, 0, \sum_{r=1}^{k} \omega^{-2rk+k} x_{k-r+1}\right)\right)\\
&\quad + \sum_{s=3}^{k} W_{kn}\left(\begin{bmatrix} \operatorname{offdiag}\left(\sum_{r=1}^{k} \omega^{-rs+s-1} x_{k-r+1}, \ldots, \sum_{r=1}^{k} \omega^{-rs+1} x_{k-r+1}\right) & 0 \\ 0 & 0 \end{bmatrix}\right)\\
&\quad + W_{kn}\left(\operatorname{offdiag}\left(\sum_{r=1}^{k} \omega^{-r(k+1)+k} x_{k-r+1}, \ldots, \sum_{r=1}^{k} \omega^{-r(k+1)+1} x_{k-r+1}\right)\right)\\
&\quad + \sum_{s=k+2}^{2k-1} W_{kn}\left(\begin{bmatrix} 0 & 0 \\ 0 & \operatorname{offdiag}\left(\sum_{r=1}^{k} \omega^{-rs+k} x_{k-r+1}, \ldots, \sum_{r=1}^{k} \omega^{-rs+s-k} x_{k-r+1}\right) \end{bmatrix}\right).
\end{aligned}$$

Now utilizing equality (WI) and Lemma 2.1, we have

$$\begin{aligned}
&kW_{kn}\left(\operatorname{offdiag}(x_1, x_2, \ldots, x_k)\right)\\
&\leq \max\left\{W_{2n}\left(\sum_{r=1}^{k} \omega^{-2r+1} x_{k-r+1}\right), W_{2n}\left(\sum_{r=1}^{k} \omega^{-2rk+k} x_{k-r+1}\right)\right\}\\
&\quad + \sum_{s=3}^{k+1} W_{(s-1)n}\left(\operatorname{offdiag}\left(\sum_{r=1}^{k} \omega^{-rs+s-1} x_{k-r+1}, \ldots, \sum_{r=1}^{k} \omega^{-rs+1} x_{k-r+1}\right)\right)\\
&\quad + \sum_{s=k+2}^{2k-1} W_{(2k+1-s)n}\left(\operatorname{offdiag}\left(\sum_{r=1}^{k} \omega^{-rs+k} x_{k-r+1}, \ldots, \sum_{r=1}^{k} \omega^{-rs+s-k} x_{k-r+1}\right)\right). \qquad (3.6)
\end{aligned}$$

By Lemma 2.5 and remark 3.7, we obtain

$$\sum_{s=3}^{k+1} W_{(s-1)n}\left(\operatorname{offdiag}\left(\sum_{r=1}^{k} \omega^{-rs+s-1} x_{k-r+1}, \ldots, \sum_{r=1}^{k} \omega^{-rs+1} x_{k-r+1}\right)\right)$$

$$
\begin{aligned}
&+\sum_{s=k+2}^{2k-1} W_{(2k+1-s)n}\left(\operatorname{offdiag}\left(\sum_{r=1}^{k}\omega^{-rs+k}x_{k-r+1},\dots,\sum_{r=1}^{k}\omega^{-rs+s-k}x_{k-r+1}\right)\right)\\
&=\sum_{s=3}^{k+1} W_{(s-1)n}\left(\operatorname{offdiag}\left(\omega\sum_{r=1}^{k}\omega^{-rs+s-1}x_{k-r+1},\dots,\omega^{s-1}\sum_{r=1}^{k}\omega^{-rs+1}x_{k-r+1}\right)\right)\\
&+\sum_{s=k+2}^{2k-1} W_{(2k+1-s)n}\left(\operatorname{offdiag}\left(\omega^{s}\sum_{r=1}^{k}\omega^{-rs+k}x_{k-r+1},\dots,\omega^{2k}\sum_{r=1}^{k}\omega^{-rs+s-k}x_{k-r+1}\right)\right)\\
&=\sum_{s=3}^{k+1} W_{n}\left(\sum_{r=1}^{k}\omega^{-rs+k+s}x_{k-r+1}\right)+\sum_{s=k+2}^{2k-1} W_{n}\left(\sum_{r=1}^{k}\omega^{-rs+k+s}x_{k-r+1}\right)\\
&=\sum_{s=3}^{2k-1} W_{n}\left(\sum_{r=1}^{k}\omega^{rs}x_{r}\right). \qquad (3.7)
\end{aligned}
$$

Now, inequality (3.6) and equality (3.7) yeild the first inequality of (3.5). The second inequality of (3.5) is obvious. □

COROLLARY 3.9. *If $k = 3$, then*

$$
\begin{aligned}
&W_{3n}\left(\operatorname{offdiag}(x_1x_2,x_3)\right)\\
&\quad\le \frac{1}{3}\left\{W_n(x_1+x_2+x_3)+W_n(x_1+\omega x_2+\omega^2x_3)+W_n(x_1+\omega^2x_2+\omega x_3)\right\}\\
&\qquad+\frac{1}{3}\max\left\{W_n(\omega x_1+x_2+\omega^2x_3),W_n(x_1+x_2+x_3)\right\}.
\end{aligned}
$$

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