DYNAMICAL SAMPLING: MIXED FRAME OPERATORS, REPRESENTATIONS AND PERTURBATIONS

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Motivated by recent progress in operator representation of frames, we investigate the frames of the form $\{T^n\varphi\}_{n\in I}$ for $I = \mathbb{N}, \mathbb{Z}$, and answer questions about representations, perturbations and frames induced by the action of powers of bounded linear operators. As a particular case, we discuss problems concerning representation of frames in terms of iterations of the mixed frame operators. As our another contribution, we consider frames of the form $\{a_nT^n\varphi\}_{n=0}^{\infty}$ for some non-zero scalars $\{a_n\}_{n=0}^{\infty}$, and we obtain some new results in dynamical sampling. Finally, we will present some auxiliary results related to the perturbation of sequences of the form $\{T^n\varphi\}_{n=0}^{\infty}$.

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1. INTRODUCTION

A frame in a separable Hilbert space $\mathcal{H}$ is a countable collection of elements in $\mathcal{H}$ that allows each $f \in \mathcal{H}$ to be written as an (infinite) linear combination of the frame elements, but linear independence between the frame elements is not required. Duffin and Schaeffer [12] introduced frames, and they used frames as a tool in the study sequences of the form $\{e^{i\lambda_n x}\}_{n\in\mathbb{Z}}$, where $\{\lambda_n\}_{n\in\mathbb{Z}}$ is a family of real or complex numbers. Dynamical sampling has already introduced in [1] by Aldroubi *et al.*, and it deals with frame properties of sequences of the form $\{T^n\varphi\}_{n=0}^{\infty}$, where $\varphi \in \mathcal{H}$ and $T : \mathcal{H} \to \mathcal{H}$ belongs to certain classes of linear operators.

Throughout this paper, let $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$. We let $\mathcal{H}$ denote a complex separable infinite-dimensional Hilbert space. Given a Hilbert space $\mathcal{H}$, we let $B(\mathcal{H})$ denote the set of all bounded linear operators $T : \mathcal{H} \to \mathcal{H}$. Moreover, $GL(\mathcal{H})$ will denote the set of all bijective operators in $B(\mathcal{H})$.

Definition 1.1. Let I denote a countable set and let $\{f_k\}_{k\in I}$ be a sequence in $\mathcal{H}$.

- $\{f_k\}_{k\in I}$ is called a frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that $A\|f\|^2 \leq \sum_{k\in I} |\langle f, f_k\rangle|^2 \leq B\|f\|^2$ for all $f \in \mathcal{H}$; it is a frame sequence if the stated inequalities hold for all $f \in \overline{\text{span}}\{f_k\}_{k\in I}$.
- $\{f_k\}_{k\in I}$ is called a Bessel sequence with Bessel bound B, if $\sum_{k\in I} |\langle f, f_k\rangle|^2 \leq B\|f\|^2$ for all $f \in \mathcal{H}$;
- $\{f_k\}_{k\in I}$ is called a Riesz sequence if there exist constants $A, B > 0$ such that $A\sum_{k\in I} |c_k|^2 \leq \|\sum_{k\in I} c_k f_k\|^2 \leq B\sum_{k\in I} |c_k|^2$ for all finite scalar sequences $\{c_k\}_{k\in I}$.
- $\{f_k\}_{k\in I}$ is called a Riesz basis for $\mathcal{H}$, if it is a Riesz sequence for which $\overline{\text{span}}\{f_k\}_{k\in I} = \mathcal{H}$.

The following theorem was proved in [4] which is about frames and operators:

THEOREM 1.2. *Consider a sequence $\{f_k\}_{k=1}^{\infty}$ in a separable Hilbert space $\mathcal{H}$. Then the following hold:*

- *$\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence if and only if $U : \{c_k\}_{k=1}^{\infty} \mapsto \sum_{k=1}^{\infty} c_k f_k$ is a well-defined mapping from $\ell^2(\mathbb{N})$ to $\mathcal{H}$, i.e, the infinite series is convergent for all $\{c_k\}_{k=1}^{\infty} \in \ell^2(\mathbb{N})$; in the affirmative case the operator U is linear and bounded.*
- *$\{f_k\}_{k=1}^{\infty}$ is a frame if and only if the mapping $\{c_k\}_{k=1}^{\infty} \mapsto \sum_{k=1}^{\infty} c_k f_k$ is well-defined from $\ell^2(\mathbb{N})$ to $\mathcal{H}$ and surjective.*
- *$\{f_k\}_{k=1}^{\infty}$ is a Riesz basis if and only if the mapping $\{c_k\}_{k=1}^{\infty} \mapsto \sum_{k=1}^{\infty} c_k f_k$ is well-defined from $\ell^2(\mathbb{N})$ to $\mathcal{H}$ and bijective.*

For $I = \mathbb{N}$ or $\mathbb{Z}$, Theorem 1.2 tells us that if $\{f_k\}_{k\in I}$ is a Bessel sequence, the synthesis operator

$$U : \ell^2(I) \to \mathcal{H}, \quad U\{c_k\}_{k\in I} := \sum_{k\in I} c_k f_k,$$

is well-defined and bounded. A central role will be played by the kernel of the operator U, *i.e.*, the subset of $\ell^2(I)$ given by

$$\mathcal{N}_U = \big\{\{c_k\}_{k\in I} \in \ell^2(I) : \sum_{k\in I} c_k f_k = 0\big\}.$$

The *excess* of a frame is the number of elements that can be removed in order for the remaining set to form a basis. Given a Bessel sequence $\{f_k\}_{k=1}^{\infty}$, the *frame operator* $S : \mathcal{H} \to \mathcal{H}$ is defined by

$$S := UU^*, \quad Sf := UU^*f = \sum_{k=1}^{\infty} \langle f, f_k\rangle f_k.$$

1.1. **Motivation and idea of dynamical sampling**

Dynamical sampling is a recent research was introduced earlier in [1] deals with frame properties of the sequence $\{T^n\varphi\}_{n=0}^{\infty}$ for some $T \in (\mathcal{H})$ and some $\varphi \in \mathcal{H}$. We will consider frames $\{f_k\}_{k\in I}$ with indexing over $I = \mathbb{N}$ or $I = \mathbb{Z}$. It is natural to ask whether we can find a linear operator T such that $f_{k+1} = Tf_k$ for all $k \in I$. Various characterizations of frames having the form $\{f_k\}_{k\in I} = \{T^k\varphi\}_{k\in I}$, where T is a linear (not necessarily bounded) operator can be found in [7, 8, 5]. We are interested in the structure of the set of iterations of the operator $T \in B(\mathcal{H})$ when acting on the vector $\varphi \in \mathcal{H}$. Indeed, we are interested in the following two questions:

- Under what conditions on T and I is the the iterated system of vectors $\{T^n\varphi\}_{n\in I}$ a frame or a Riesz basis for $\mathcal{H}$?

- If $\{T^n\varphi\}_{n\in I}$ is a frame or a Riesz basis for $\mathcal{H}$, what can be deduced about the operator T?

Example 1.3. Let $\{e_k\}_{k=1}^{\infty}$ denote an orthonormal basis for $\mathcal{H}$. Define the operator $T : \mathcal{H} \to \mathcal{H}$ by $T(f) = \sum_{k=1}^{\infty}\langle f, e_k\rangle e_{k+1}$. It is clear that $\{e_k\}_{k=1}^{\infty} = \{T^k e_1\}_{k=0}^{\infty}$.

Example 1.4. Assume that $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$, and define the bounded operator $T : \mathcal{H} \to \mathcal{H}$ by $T(f) = \sum_{k=1}^{\infty}\langle f, e_k\rangle 2^{-k}e_{k+1}$. In particular, T is compact, being the norm-limit of the finite-rank operators

$$T_N : \mathcal{H} \to \mathcal{H}, \quad T_N(f) = \sum_{k=1}^{N}\langle f, e_k\rangle 2^{-k}e_{k+1}.$$

On the other hand, by construction the sequence $\left\{\frac{T^k e_1}{\|T^k e_1\|}\right\}_{k=0}^{\infty}$ is $\{e_k\}_{k=1}^{\infty}$.

Definition 1.5. Suppose that $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are two frames (or Bessel sequences) for $\mathcal{H}$. The operator $T : \mathcal{H} \to \mathcal{H}$ defined by

$$Tf = \sum_{k=1}^{\infty}\langle f, g_k\rangle f_k$$

is called *the mixed frame operator* associated with $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$.

Obviously, any bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ is indeed a mixed frame operator. Because, if $T \in B(\mathcal{H})$ and $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$, then by applying T on the decomposition $f = \sum_{k=1}^{\infty}\langle f, e_k\rangle e_k$, we have that $Tf = \sum_{k=1}^{\infty}\langle f, e_k\rangle Te_k$ for all $f \in \mathcal{H}$. Hence, T is the mixed frame operator for the Bessel sequences $\{e_k\}_{k=1}^{\infty}$ and $\{Te_k\}_{k=1}^{\infty}$.

The following example of a mixed frame operator was already in [5]:

Example 1.6. Suppose that $\{f_k\}_{k=1}^{\infty} = \{T^n f_1\}_{n=0}^{\infty}$ is a frame for $\mathcal{H}$ for some $T \in B(\mathcal{H})$. Let $\{g_k\}_{k=1}^{\infty}$ be a dual frame of $\{f_k\}_{k=1}^{\infty}$. Then $Tf = \sum_{k=1}^{\infty} \langle f, g_k\rangle T f_k = \sum_{k=1}^{\infty} \langle f, g_k\rangle f_{k+1}$, for every $f \in \mathcal{H}$. Therefore, T is a mixed frame operator.

Let $\{f_k\}_{k=1}^{\infty}$ be a Bessel sequence and $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. Define the operator $T : \mathcal{H} \to \mathcal{H}$ by $Tf = \sum_{k=1}^{\infty} \langle f, e_k\rangle f_k$. It is clear that T is bounded and $Te_k = f_k$ for all k. Therefore we have the following:

PROPOSITION 1.7. *The Bessel sequences in $\mathcal{H}$ are precisely the sequences $\{Te_k\}_{k=1}^{\infty}$, where $T \in B(\mathcal{H})$ and $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$.*

1.2. **Recent results on dynamical sampling and frames**

Various aspect of the dynamical sampling problem and related frame theory have been studied by Aldroubi *et al.* and Christensen *et al.* in [1, 2, 3, 5, 6, 7, 8, 9, 10]. They deal with frame properties of sequences in a Hilbert space $\mathcal{H}$ of the form $\{T^n \varphi\}_{n=0}^{\infty}$, where $\varphi \in \mathcal{H}$ and $T \in B(\mathcal{H})$. However, some no-go results in dynamical sampling have been proved; for example, if T is a normal operator, then $\{T^n \varphi\}_{n=0}^{\infty}$ cannot be a basis [2]. Moreover, if T is a unitary operator or a compact operator, then $\{T^n \varphi\}_{n=0}^{\infty}$ cannot be a frame [3, 5]. The following recent results in dynamical sampling and frame representations with bounded operators can be found in [5, 7, 8, 10]. Suppose that $\{f_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$:

(i) $\{f_k\}_{k=1}^{\infty}$ has a representation$\{f_k\}_{k=1}^{\infty} = \{T^k f_1\}_{k=0}^{\infty}$ for some bounded operator $T : \mathcal{H} \to \mathcal{H}$ if and only if $\{f_k\}_{k=1}^{\infty}$ is linearly independent.

(ii) Let $T : \text{span}\{f_k\}_{k=0}^{\infty} \to \text{span}\{f_k\}_{k=0}^{\infty}$ be a linear operator and $\{f_k\}_{k=1}^{\infty} = \{T^k f_1\}_{k=0}^{\infty}$. Then T is bounded if and only if the kernel $\mathcal{N}_U$ of the synthesis operator is invariant under right-shifts; in particular T is bounded if $\{f_k\}_{k=1}^{\infty} = \{T^k f_1\}_{k=0}^{\infty}$ is a Riesz basis.

(iii) Assume that $\{f_k\}_{k=1}^{\infty}$ is linearly independent and overcomplete. Then $\{f_k\}_{k=1}^{\infty}$ has infinite excess.

For countable subsets $\mathcal{G} \subset \mathcal{H}$ and a normal operator T, Aldroubi *et al.* [2] proved that the iterative system $\{T^n \varphi\}_{\varphi \in \mathcal{G}, n \geq 0}$ can be a frame for $\mathcal{H}$, but cannot be a basis. However, it is difficult for a system of vectors of the form $\{T^n \varphi\}_{\varphi \in \mathcal{G}, n \geq 0}$ to be a frame. The difficulty is that the the spectrum of T must be very special. Such frames however do exist, as shown by the constructions in [1].

The paper is organized as follows. In section 2, we provide an alternative proof to show that $\bigcup_{j=1}^{k} \{T^n \varphi_j\}_{n=0}^{\infty}$ cannot form a frame for $\mathcal{H}$, whenever T is compact. Moreover, we provide necessary and sufficient conditions for T being surjective. The main purpose of this section is to characterize and compare the Bessel and frame properties of orbits $\{T^n\varphi\}_{n=0}^{\infty}$ with a bounded operator T in connection with frame operators and mixed frame operators. We also show that the iterative actions of the mixed frame operator associated with two orthonormal basis cannot form a frame. Section 3 discusses representations of frames which can be represented of the form $\{a_nT^n\varphi\}_{n=0}^{\infty}$ for some non-zero scalars $\{a_n\}_{n=0}^{\infty}$ with $\sup_n|\dfrac{a_n}{a_{n+1}}| < \infty$. Finally, in section 4 we illustrate some auxiliary results related to the perturbation of an operator to construct frame orbits in terms of the operator representations.

2. ITERATIVE ACTIONS OF FRAME OPERATOR AND MIXED FRAME OPERATOR

The representation of frames in the form $\{T^n\varphi\}_{n=0}^{\infty}$ and $\{T^n\varphi\}_{n\in\mathbb{Z}}$ for some $\varphi \in \mathcal{H}$ and some $T \in B(\mathcal{H})$ was already studied in [5, 7]. Aldroubi *et al.* [1] showed that iterative actions of compact self-adjoint operators cannot form a frame. However, for a normal operator, Philipp [13] proved that $\{T^n\varphi\}_{n\in\mathbb{N}}$ can be a Bessel sequence. It is clear that the iterative system $\{T^n\varphi\}_{n=0}^{\infty}$is a Bessel sequence if $\|T\| < 1$. Indeed, for any $f \in \mathcal{H}$, we have

$$\sum_{n=0}^{\infty}|\langle f, T^n\varphi\rangle|^2 \leq \sum_{n=0}^{\infty}\|f\|^2\|T^n\varphi\|^2 \leq \|f\|^2\|\varphi\|^2\sum_{n=0}^{\infty}\|T\|^{2n} = \frac{\|\varphi\|^2}{1-\|T\|^2}\|f\|^2.$$

It has already proved that if T is a compact operator on an infinite-dimensional Hilbert space $\mathcal{H}$ and $\varphi_1, ..., \varphi_k \in \mathcal{H}$, then $\bigcup_{j=1}^{k}\{T^n\varphi_j\}_{n=0}^{\infty}$ cannot be a frame for $\mathcal{H}$ [5]. Here we provide an alternative simple proof. We first prove a lemma.

LEMMA 2.1. *Let $T \in B(\mathcal{H})$ and $\varphi_1, ..., \varphi_k \in \mathcal{H}$. If $\bigcup_{j=1}^{k}\{T^n\varphi_j\}_{n=0}^{\infty}$ is a frame for $\mathcal{H}$, then T has closed rang and the range of T is $\mathcal{R}_T = \overline{\operatorname{span}}\{T^n\varphi_j : j = 1, 2, \cdots, k\}_{n=1}^{\infty}$.*

Proof. For each $x \in \mathcal{H}$ there exists a sequence $\{c_{n,j} : j = 1, 2, \cdots, k\}_{n=0}^{\infty}$ of scalars such that $x = \sum_{j=1}^{k}\sum_{n=0}^{\infty}c_{n,j}T^n\varphi_j$. Therefore

$$Tx = \sum_{j=1}^{k}\sum_{n=0}^{\infty}c_{n,j}T^{n+1}\varphi_j \in \overline{\operatorname{span}}\{T^n\varphi_j : j = 1, 2, \cdots, k\}_{n=1}^{\infty}.$$

Therefore $\mathcal{R}_T \subseteq \mathcal{K} := \overline{\operatorname{span}}\{T^n\varphi_j : j = 1, 2, \cdots, k\}_{n=1}^{\infty}$. On the other hand, since $\bigcup_{j=1}^{k}\{T^n\varphi_j\}_{n=1}^{\infty}$ is a frame for $\mathcal{K}$, for each $x \in K$ there is a sequence

$\{c_{n,j} : j = 1, 2, \cdots, k\}_{n=1}^{\infty}$ of scalars such that $x = \sum_{j=1}^{k}\sum_{n=1}^{\infty} c_{n,j}T^n\varphi_j = T\big(\sum_{j=1}^{k}\sum_{n=0}^{\infty} c_{n,j}T^n\varphi_j\big) \in \mathcal{R}_T$. Therefore

$$\mathcal{R}_T = \overline{\text{span}}\{T^n\varphi_j : j = 1, 2, \cdots, k\}_{n=1}^{\infty},$$

i.e., T has closed range. □

Proposition 2.2. *Suppose that* $\dim \mathcal{H} = \infty$, $\varphi_1, \cdots, \varphi_k \in \mathcal{H}$ *and* $T : \mathcal{H} \to \mathcal{H}$ *is a compact operator. Then* $\bigcup_{j=1}^{k}\{T^n\varphi_j\}_{n=0}^{\infty}$ *cannot form a frame for* $\mathcal{H}$.

Proof. Let $\bigcup_{j=1}^{k}\{T^n\varphi_j\}_{n=0}^{\infty}$ be a frame for $\mathcal{H}$. Then T has closed rang and $\mathcal{R}_T = \overline{\text{span}}\{T^n\varphi_j : j = 1, 2, \cdots, k\}_{n=1}^{\infty}$ by Lemma 2.1. We denote by $T^\dagger \in B(\mathcal{H})$ the pseudo-inverse of T, *i.e.*,

$$T^\dagger : \mathcal{H} \to \mathcal{H}, \quad TT^\dagger x = x, \quad x \in \mathcal{R}_T.$$

Since T is compact, $TT^\dagger = I_{\mathcal{R}_T}$ is compact. This implies that $\mathcal{R}_T$ is finite-dimensional, and it leads to conclude $\dim \mathcal{H} < \infty$, which is a contradiction. Therefore $\bigcup_{j=1}^{k}\{T^n\varphi_j\}_{n=0}^{\infty}$ cannot be a frame for $\mathcal{H}$. □

As we saw in Lemma 2.1, $\mathcal{R}_T$ is closed if $\{T^n\varphi\}_{n=0}^{\infty}$ is a frame. The following proposition provides necessary and sufficient conditions for T being surjective.

Proposition 2.3. *Let* $T \in B(\mathcal{H})$ *and* $\varphi \in \mathcal{H}$. *Assume that* $\{T^n\varphi\}_{n=0}^{\infty}$ *is a frame for* $\mathcal{H}$ *with frame operator* S. *Then the following hold:*

(i) T *is surjective if and only if there exists* $n \geq 1$ *such that* $\langle T^n\varphi, S^{-1}\varphi\rangle \neq 0$.

(ii) T *is surjective if and only if* $\varphi \in \mathcal{R}_T$.

(iii) T *is surjective if and only if* $S^{-1}\varphi \notin \ker T^*$.

(iv) T *is surjective if and only if* $\|S^{-1/2}\varphi\| \neq 1$.

Proof. (i) First assume that T is surjective. Then $\mathcal{H} = \overline{\text{span}}\{T^n\varphi\}_{n=1}^{\infty}$ by Lemma 2.1. If $\langle T^n\varphi, S^{-1}\varphi\rangle = 0$ for all $n \geq 1$, then $S^{-1}\varphi \perp \mathcal{H}$. This implies that $\varphi = 0$, which is a contradiction. Conversely, assume that $\langle T^n\varphi, S^{-1}\varphi\rangle \neq 0$ for some $n \geq 1$. Then

$$T^n\varphi = \sum_{i=0}^{\infty}\langle S^{-1}T^n\varphi, T^i\varphi\rangle T^i\varphi = \langle T^n\varphi, S^{-1}\varphi\rangle\varphi + \sum_{i=1}^{\infty}\langle S^{-1}T^n\varphi, T^i\varphi\rangle T^i\varphi.$$

Therefore $\varphi \in \mathcal{R}_T$. On the other hand, $\{T^n\varphi\}_{n=1}^\infty$ is a frame sequence, and $\mathcal{R}_T = \overline{\text{span}}\{T^n\varphi\}_{n=1}^\infty$ by Lemma 2.1. Hence $\varphi \in \mathcal{R}_T$ implies that $\mathcal{R}_T = \overline{\text{span}}\{T^n\varphi\}_{n=1}^\infty = \overline{\text{span}}\{T^n\varphi\}_{n=0}^\infty = \mathcal{H}$, as desired.

The result in (ii) follows from the proof of (i).

To prove (iii), it follows from (i) that T is surjective if and only if $S^{-1}\varphi \notin [\mathcal{R}_T]^\perp = \ker T^*$.

For the proof of (iv), assume that T is surjective and $\|S^{-1/2}\varphi\| = 1$. Since

$$\varphi = \langle S^{-1}\varphi, \varphi\rangle\varphi + \sum_{n=1}^{\infty}\langle S^{-1}\varphi, T^n\varphi\rangle T^n\varphi, \tag{2.1}$$

we get $\sum_{n=1}^\infty \langle S^{-1}\varphi, T^n\varphi\rangle T^n\varphi = 0$. Then $\sum_{n=1}^\infty \left|\langle S^{-1}\varphi, T^n\varphi\rangle\right|^2 = 0$. Applying (i), we conclude that T is not surjective, which is a contradiction. Conversely, if $\|S^{-1/2}\varphi\| \neq 1$, then (2.1) implies that there exists $n \geq 1$ such that $\langle T^n\varphi, S^{-1}\varphi\rangle \neq 0$. Hence T is surjective by (i). □

Since a Riesz base and its canonical dual are bi-orthogonal, we have

COROLLARY 2.4. *Let $T \in B(\mathcal{H})$ and $\varphi \in \mathcal{H}$. Assume that $\{T^n\varphi\}_{n=0}^\infty$ is a Riesz basis for $\mathcal{H}$. Then T is not surjective. In particular, $\varphi \notin \mathcal{R}_T$ and $S^{-1}\varphi \in \ker T^*$.*

Let $\{f_k\}_{k=1}^\infty$ be a frame for $\mathcal{H}$ with frame operator S. We investigate the question: Does there exist some $\varphi \in \mathcal{H}$ such that $\{S^n\varphi\}_{n=0}^\infty$ is a frame? There are many frames for which this cannot happen. For example, if $\{f_k\}_{k=1}^\infty$ is a tight frame for $\mathcal{H}$ with bound A, then for $\varphi(\neq 0) \in \mathcal{H}$, we have

$$\sum_{n=0}^{\infty}|\langle f, S^n\varphi\rangle|^2 = \sum_{n=0}^{\infty}|\langle f, A^n\varphi\rangle|^2 = |\langle f, \varphi\rangle|^2\sum_{n=0}^{\infty}A^{2n}, \quad f \in \mathcal{H}.$$

Therefore, $\{S^n\varphi\}_{n=0}^\infty$ is a frame for $\mathcal{H}$ if and only if $\dim\mathcal{H} = 1$ and $A < 1$.

The following exhibits a concrete example of a frame $\{f_k\}_{k=1}^\infty = \{T^n f_1\}_{n=0}^\infty$ for which T is a frame operator:

Example 2.5. Consider the operator $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ defined by

$$T\{c_k\}_{k=1}^\infty = \{(1-2^{-k})c_k\}_{k=1}^\infty, \quad \{c_k\}_{k=1}^\infty \in \ell^2(\mathbb{N}). \tag{2.2}$$

Letting $\lambda_k = 1 - 2^{-k}$ for $k \in \mathbb{N}$, Aldroubi *et al.* [1] proved that the sequence $\{T^n b\}_{n=0}^\infty$ is a frame for $\ell^2(\mathbb{N})$ whenever $b = \{\sqrt{1-\lambda_k^2}\}_{k=1}^\infty$. Defining the bounded operator $U : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by $U\{c_k\}_{k=1}^\infty = \{\sqrt{1-2^{-k}}c_k\}_{k=1}^\infty$, we have $U = U^*$ and $T = U^2$. Let $\{\delta_k\}_{k=1}^\infty$ be the standard basis of $\ell^2(\mathbb{N})$ and let S be the frame operator of $\{U\delta_k\}_{k=1}^\infty = \{\sqrt{1-2^{-k}}\delta_k\}_{k=1}^\infty$. Then

$$Sf = \sum_{k=1}^{\infty}\langle f, U\delta_k\rangle U\delta_k = U\sum_{k=1}^{\infty}\langle U^*f, \delta_k\rangle\delta_k = UU^*f = Tf, \quad f \in \ell^2(\mathbb{N}),$$

i.e., $S = T$. □

Motivated by Example 2.5, we can characterize the case that a frame has a representation $\{T^n\varphi\}_{n=0}^{\infty}$, where T is a frame operator. Indeed, we show that positive and invertible operators are a characteristic of frame operators:

PROPOSITION 2.6. *Let $T \in B(\mathcal{H})$. Then the followings are equivalent:*

(i) *T is positive and invertible.*

(ii) *T is the frame operator for a frame.*

Proof. To prove (i) $\Rightarrow$ (ii) , consider the bounded and surjective operator $U : \mathcal{H} \to \mathcal{H}$ such that $T = UU^*$. Let $\{e_k\}_{k=1}^{\infty}$ denote an orthonormal basis for $\mathcal{H}$, and let $f_k = Ue_k$ for each $k \in \mathbb{N}$. Then $\{f_k\}_{k=1}^{\infty}$ is a frame and its frame operator T because

$$Tf = UU^*f = \sum_{k=1}^{\infty}\langle f, Ue_k\rangle Ue_k = \sum_{k=1}^{\infty}\langle f, f_k\rangle f_k, \quad f \in \mathcal{H}.$$

This proves (ii). The implication (ii) $\Rightarrow$ (i) is clear. □

In the following proposition we provide a necessary condition for

$$\{S^n g\}_{n\geq 0, g\in\mathcal{G}}$$

to be a frame, where $\mathcal{G} \subset \mathcal{H}$ is a countable set.

PROPOSITION 2.7. *Assume that $\{f_k\}_{k=1}^{\infty}$ is a frame with lower frame bound A and frame operator S. If $\mathcal{G}$ is a countable subset of $\mathcal{H}$ and $\{S^n g\}_{n\geq 0, g\in\mathcal{G}}$ is a frame for $\mathcal{H}$, then $A < 1$.*

Proof. Since $A\langle f, f\rangle \leq \langle Sf, f\rangle$, we get $A\|f\| \leq \|Sf\|$ for all $f \in \mathcal{H}$. Therefore,

$$\langle S^2 f, f\rangle = \langle Sf, Sf\rangle = \|Sf\|^2 \geq A^2\|f\|^2 = A^2\langle f, f\rangle,$$

and then $A^2\|f\| \leq \|S^2 f\|$ for all $f \in \mathcal{H}$. By Induction, we conclude that for each positve integer m,

$$A^m\|f\| \leq \|S^m f\|, \quad f \in \mathcal{H}.$$

Since $\{S^n g\}_{n\geq 0, g\in\mathcal{G}}$ is a frame for $\mathcal{H}$, we get $\|S^m f\| \to 0$ as $m \to \infty$ for all $f \in \mathcal{H}$ by [[3], Theorem 7]. Then $A^m \to 0$ as $m \to \infty$, and this leads to get $A < 1$. □

Remark 2.8. Suppose that $\{f_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$ with lower bound A. Let S be the frame operator for $\{f_k\}_{k=1}^{\infty}$ such that $V \subset \mathcal{H}$ is an invariant subspace under S. If there exists $\lambda \in [0,1)$ such that $\|S\varphi\| \le \lambda\|\varphi\|$ for all $\varphi \in V$, then $\{S^n\varphi\}_{n=0}^{\infty}$ is a Bessel sequence for all $\varphi \in V$. Indeed, for all $f \in \mathcal{H}$ and $\varphi \in V$, we have that

$$\sum_{n=0}^{\infty} |\langle f, S^n\varphi\rangle|^2 \le \|f\|^2 \sum_{n=0}^{\infty} \|S^n\varphi\|^2 \le \|f\|^2 \sum_{n=0}^{\infty} \lambda^{2n} = \frac{\|f\|^2}{1-\lambda^2}.$$

It follows from [[3], Theorem 7] that for any unitary operator $T : \mathcal{H} \to \mathcal{H}$ and any set of vectors $G \subseteq \mathcal{H}$, $\{T^n g\}_{g\in\mathcal{G}, n\ge 0}$ is not a frame.

PROPOSITION 2.9. *Let $\{e_k\}_{k=1}^{\infty}$ and $\{\delta_k\}_{k=1}^{\infty}$ denote two orthonormal bases for a Hilbert space $\mathcal{H}$, and consider the mixed frame operator*

$$T : \mathcal{H} \to \mathcal{H}, \quad Tf = \sum_{k=1}^{\infty} \langle f, e_k\rangle \delta_k.$$

Then $\{T^n\varphi\}_{n=0}^{\infty}$ cannot be a frame for $\mathcal{H}$ for any $\varphi \in \mathcal{H}$.

Proof. Since $Te_j = \delta_j$ for all $j \in \mathbb{N}$, the operator T maps the orthonormal basis $\{e_k\}_{k=1}^{\infty}$ onto the orthonormal basis $\{\delta_k\}_{k=1}^{\infty}$. Therefore T is unitary. By [[3], Corollary 2], we conclude that $\{T^n\varphi\}_{n=0}^{\infty}$ is not a frame for $\mathcal{H}$ for any $\varphi \in \mathcal{H}$. $\square$

By use of Theorem 1.2 we get some useful results related to iterative actions of a mixed frame operator:

COROLLARY 2.10. *Suppose that $\{e_k\}_{k=1}^{\infty}$ and $\{\delta_k\}_{k=1}^{\infty}$ are orthonormal bases for $\mathcal{H}$. The following statements hold:*

(i) *Let $\{Ue_k\}_{k=1}^{\infty}$ be a Riesz basis for $\mathcal{H}$ and $Gf := \sum_{k=1}^{\infty} \langle f, \delta_k\rangle Ue_k$ for all $f \in \mathcal{H}$, where $U \in GL(\mathcal{H})$ is a bounded bijective operator. If $\{G^n\varphi\}_{n=0}^{\infty}$ is a frame for some $\varphi \in \mathcal{H}$, then $\|U\| \ge 1$.*

(ii) *Let $\{Ue_k\}_{k=1}^{\infty}$ and $\{V\delta_k\}_{k=1}^{\infty}$ be two frames for $\mathcal{H}$ and*

$$Gf := \sum_{k=1}^{\infty} \langle f, V\delta_k\rangle Ue_k$$

for all $f \in \mathcal{H}$, where $U, V : \mathcal{H} \to \mathcal{H}$ are bounded surjective linear operators. If $\{G^n\varphi\}_{n=0}^{\infty}$ is a frame for $\mathcal{H}$, then $\|U\|\|V\| \ge 1$.

Proof. (i) We define the operator $T : \mathcal{H} \to \mathcal{H}$ by $Tf = \sum \langle f, \delta_k\rangle e_k$. It is clear that T is isometric, and $Gf = UTf$ for all $f \in \mathcal{H}$. Therefore, $\|G\| \le \|U\|$.

On the other hand, [[3], Theorem 9] shows that $\|G\| \geq 1$, which yields the result.

(ii) Let T as in (i). Therefore $G = UTV^*$, and we get $\|G\| \leq \|U\|\|V\|$. Hence, $\|U\|\|V\| \geq 1$ by [[3], Theorem 9]. □

COROLLARY 2.11. *Suppose that* $\{e_k\}_{k=1}^{\infty}$ *and* $\{\delta_k\}_{k=1}^{\infty}$ *are two orthonormal bases for a Hilbert space* $\mathcal{H}$.

(i) *Let* $\{f_k\}_{k=1}^{\infty}$ *be a Parseval frame for* $\mathcal{H}$ *and let* T *be the mixed frame operator defined by* $Tf = \sum_{k=1}^{\infty}\langle f, f_k\rangle e_k$. *If* $\{T^n\varphi\}_{n=0}^{\infty}$ *is a frame for* $\mathcal{H}$ *for some* $\varphi \in \mathcal{H}$, *then* T *is not a surjective operator.*

(ii) *Let* $\{U\delta_k\}_{k=1}^{\infty}$ *be a frame for* $\mathcal{H}$ *and* $Tf = \sum_{k=1}^{\infty}\langle f, U\delta_k\rangle e_k$, *where* $U : \mathcal{H} \to \mathcal{H}$ *is a bounded surjective linear operator. If* $\{T^n\varphi\}_{n=0}^{\infty}$ *is a frame for* $\mathcal{H}$ *for some* $\varphi \in \mathcal{H}$, *then* $U^*U \neq I$, *i.e.,* U *is not isometric.*

Proof. (i) Since $\{f_k\}_{k=1}^{\infty}$ is a Parseval frame, we have

$$\|Tf\|^2 = \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 = \|f\|^2$$

for all $f \in \mathcal{H}$. Then $T^*T = I$. If we suppose that T is surjective, then T is unitary. Using [[3], Corollary 2], we conclude that $\{T^n\varphi\}_{n=0}^{\infty}$ is not a frame for $\mathcal{H}$. For part *(ii)*, if $U^*U = I$ and U is surjective, then U will be a unitary operator. Since $TU\delta_k = e_k$ for all $k \in \mathbb{N}$, we get TU is unitary. Therefore T is unitary, and then $\{T^n\varphi\}_{n=0}^{\infty}$ cannot be a frame for $\mathcal{H}$. □

In the case of normal operators, we have the following result for infinite dimensional Hilbert spaces:

LEMMA 2.12. *Suppose that* $T : \mathcal{H} \to \mathcal{H}$ *is a normal operator and* $\varphi \in \mathcal{H}$ *such that* $\{T^n\varphi\}_{n=0}^{\infty}$ *is a frame for* $\mathcal{H}$. *Then* $\|T\| = 1$.

Proof. Using [[2], Theorem 5.7], we have $T = \sum_{j=0}^{\infty}\lambda_j P_j$, where each P_j is a rank one orthogonal projection such that $\sum_j P_j = I$, $P_jP_i = 0$ for all $j \neq i$, and $|\lambda_j| < 1$ for all $j \in \mathbb{N}$. Since $\sum_j P_j = I$, we have that $\|f\|^2 = \sum_j \|P_jf\|^2$ for all $f \in \mathcal{H}$. Therefore

$$\|Tf\|^2 = \sum_j |\lambda_j|^2\|P_jf\|^2 \leq \sum_j \|P_jf\|^2 = \|f\|^2, \quad f \in \mathcal{H}.$$

Therefore $\|T\| \leq 1$. On the other hand, we have $\|T\| \geq 1$ by [[3], Theorem 9], which leads to the desired result. □

PROPOSITION 2.13. *Let* $T \in B(\mathcal{H})$ *and* $\varphi \in \mathcal{H}$ *be such that* $\{T^n\varphi\}_{n=0}^{\infty}$ *is a frame for* $\mathcal{H}$.

(i) *There exists a countable set* $\mathcal{G} \subset \mathcal{H}$ *such that* $\{V^n\psi\}_{\psi\in\mathcal{G},n\geq 0}$ *is a tight frame for* $\mathcal{H}$, *where* $V = \|T\|^{-1}T$.

(ii) *If* T *is a normal operator, then there exists a countable set* $\mathcal{G} \subset \mathcal{H}$ *such that* $\{(TT^*)^n\psi\}_{\psi\in\mathcal{G},n\geq 0}$ *is a tight frame for* $\mathcal{H}$.

Proof. (i) By using of [[3], Theorems 7, 9], we have $\|T\| \geq 1$ and $(T^*)^n f \to 0$ for all $f \in \mathcal{H}$ as $n \to \infty$. Since $\|V\| = 1$ and $(V^*)^n f \to 0$ for all $f \in \mathcal{H}$ as $n \to \infty$, the result follows from [[3], Theorem 8].

In order to prove (ii), since $\{T^n\varphi\}_{n=0}^{\infty}$ is a frame and T is normal, Lemma 2.12 leads us to get $\|T\| = 1$, and then $\|TT^*\| = 1$. On the other hand, we have $\|(TT^*)^n f\| = \|T^n(T^*)^n f\| \leq \|T\|^n\|(T^*)^n f\| = \|(T^*)^n f\| \to 0$, for all $f \in \mathcal{H}$ as $n \to \infty$. Therefore, the result follows from [[3], Theorem 8]. □

Remark 2.14. Consider a linearly independent frame sequence $\{f_k\}_{k\in\mathbb{Z}}$ in a Hilbert space $\mathcal{H}$ which spans an infinite dimensional subspace. By using [[7], Proposition 2.1] and [[8], Proposition 2.3], there exists a linear invertible operator $T : \mathrm{span}\{f_k\}_{k\in\mathbb{Z}} \to \mathrm{span}\{f_k\}_{k\in\mathbb{Z}}$ such that $Tf_k = f_{k+1}$. However, if $\{f_k\}_{k\in\mathbb{Z}}$ is a frame sequence and the operator T is bounded, it has a unique extension to a bounded operator $\widetilde{T} : \overline{\mathrm{span}}\{f_k\}_{k\in\mathbb{Z}} \to \overline{\mathrm{span}}\{f_k\}_{k\in\mathbb{Z}}$ such that

$$\widetilde{T}\Big(\sum_{k\in\mathbb{Z}} c_k f_k\Big) = \sum_{k\in\mathbb{Z}} c_k f_{k+1}, \quad \{c_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z}).$$

By using previous remark and operator representation of dual frames, we can construct a frame in terms of its frame operator:

Proposition 2.15. *Let* $\{f_k\}_{k\in\mathbb{Z}} = \{T^k f_0\}_{k\in\mathbb{Z}}$ *be a frame for* $\mathcal{H}$ *for some bounded, invertible and self-adjoint operator* $T : \mathcal{H} \to \mathcal{H}$ *with the frame operator* S. *Assume that* $V \in B(\mathcal{H})$ *and* $\{V^k f_m\}_{k\in\mathbb{Z}}$ *is a dual frame of* $\{f_k\}_{k\in\mathbb{Z}}$ *for some* $m \in \mathbb{Z}$. *Then* $\{S^k f_0\}_{k\in\mathbb{Z}}$ *is a frame for* $\mathcal{H}$, *whenever* T *is an isometry.*

Proof. We let $V^k f_m = g_k$ for all $k \in \mathbb{Z}$. It is clear that $Tf_k = f_{k+1} = T^{k+1}f_0$ for all $k \in \mathbb{Z}$ and $Tf = \sum_{k\in\mathbb{Z}}\langle f, g_k\rangle f_{k+1}$ for all $f \in \mathcal{H}$. On the other hand, by [[7], Lemma 3.3], $V = (T^*)^{-1}$. Since T is self-adjoint, we have $Tf = \sum_{k\in\mathbb{Z}}\langle f, T^{-k}f_m\rangle T^{k+1}f_0$, for all $f \in \mathcal{H}$. If T is an isometry, *i.e.*, $T^*T = I$, then $T = T^{-1}$, and therefore we get

$$Tf = \sum_{k\in\mathbb{Z}}\langle f, T^{k+m}f_0\rangle T^{k+1}f_0 = T^{m+1}\sum_{k\in\mathbb{Z}}\langle f, T^k f_0\rangle T^k f_0 = T^{m+1}Sf,$$

for all $f \in \mathcal{H}$. Hence, $T^m = S$, and we infer that $\{S^k f_0\}_{k\in\mathbb{Z}}$ is a frame for $\mathcal{H}$. □

It can be an interesting question whether the converse of Proposition 2.15 holds. We know that if $\{S^k f_0\}_{k\in\mathbb{Z}}$ is a tight frame for $\mathcal{H}$, [[7], Corollary 2.7] shows that the frame operator S is an isometry. It is still an open question whether T is an isometry or not.

Suppose that T is a bounded bijective operator on $\mathcal{H}$, and $f_0 \in \mathcal{H}$ such that $\{T^n f_0\}_{n\in\mathbb{Z}}$ is a frame for $\mathcal{H}$. We get that $TST^* = S$, where S is the frame operator for $\{T^n f_0\}_{n\in\mathbb{Z}}$. Indeed,

$$TST^*f = \sum_{n\in\mathbb{Z}} \langle T^*f, T^n f_0\rangle T^{n+1} f_0 = \sum_{n\in\mathbb{Z}} \langle f, T^{n+1} f_0\rangle T^{n+1} f_0 = Sf$$

In particular, T is similar to a unitary operator.

Proposition 2.16. *Let $T \in GL(\mathcal{H})$ and $\varphi \in \mathcal{H}$ such that $\{T^n\varphi\}_{n\in\mathbb{Z}}$ is a frame for $\mathcal{H}$ with frame bounds A, B and frame operator S. Let $U := S^{-1/2}TS^{1/2}$ and $\psi = S^{-1/2}\varphi$. Then $\{U^n\psi\}_{n\in\mathbb{Z}}$ is a frame for $\mathcal{H}$ with bounds AB^{-1} and BA^{-1}.*

Proof. It is clear that $TST^* = S$ and U is unitary (see [[9], Lemma 4.4]). Since $U^n = S^{-1/2}T^nS^{1/2}$ for all $n \in \mathbb{Z}$, we have $\sum_{n\in\mathbb{Z}} |\langle f, U^n\psi\rangle|^2 = \sum_{n\in\mathbb{Z}} |\langle S^{-1/2}f, T^n\varphi\rangle|^2$. Then

$$\frac{A}{B}\|f\|^2 \le A\|S^{-1/2}f\|^2 \le \sum_{n\in\mathbb{Z}} |\langle f, U^n\psi\rangle|^2 \le B\|S^{-1/2}f\|^2 \le \frac{B}{A}\|f\|^2, \quad f \in \mathcal{H}.$$

□

As a minor modification in [[9], Corollary 4.5], we also obtain the following result:

Proposition 2.17. *Let $T \in GL(\mathcal{H})$ and $\varphi \in \mathcal{H}$ such that $\{T^n\varphi\}_{n\in\mathbb{Z}}$ is a frame for $\mathcal{H}$ with frame bounds A, B. Then*

$$\sqrt{\frac{A}{B}}\|f\| \le \|T^nf\| \le \sqrt{\frac{B}{A}}\|f\|, \ \sqrt{\frac{A}{B}}\|f\| \le \|(T^*)^nf\| \le \sqrt{\frac{B}{A}}\|f\|, \ n \in \mathbb{Z}, \ f \in \mathcal{H}.$$

In particular, if $\{T^n\varphi\}_{n\in\mathbb{Z}}$ is a tight frame, then T^n and $(T^*)^n$ are isometric for all $n \in \mathbb{Z}$.

Proof. Let S denote the frame operator of $\{T^n\varphi\}_{n\in\mathbb{Z}}$ and let

$$U := S^{-1/2}TS^{1/2}.$$

Since T is invertible, we infer that U is unitary. Hence, for $f \in \mathcal{H}$ and $n \in \mathbb{Z}$ we have

$$\frac{1}{\sqrt{B}}\|f\| \le \|U^nS^{-1/2}f\| \le \frac{1}{\sqrt{A}}\|f\|.$$

Therefore

$$\sqrt{\frac{A}{B}}\|f\| \leq \|S^{1/2}U^nS^{-1/2}f\| = \|T^nf\| = \|S^{1/2}U^nS^{-1/2}f\| \leq \sqrt{\frac{B}{A}}\|f\|.$$

A similar calculation applies to $\|(T^*)^nf\|$. $\square$

Let $T \in GL(\mathcal{H})$. Similarly as in [9], we define the set

$$\mathcal{V}_{\mathbb{Z}}(T) := \Big\{ f \in \mathcal{H} : \ \{T^nf\}_{n\in\mathbb{Z}} \text{ is a frame for } \mathcal{H} \Big\}.$$

Proposition 4.11 of [9] shows that from one vector $\varphi \in \mathcal{V}_{\mathbb{Z}}(T)$ (if it exists) we obtain all vectors in $\mathcal{V}_{\mathbb{Z}}(T)$. Indeed,

$$\mathcal{V}_{\mathbb{Z}}(T) = \Big\{ V\varphi : \ V \in GL(\mathcal{H}) \text{ and } VT = TV \Big\}.$$

PROPOSITION 2.18. *Assume that $T \in GL(\mathcal{H})$, $\varphi \in \mathcal{V}_{\mathbb{Z}}(T)$ and V is a unitary operator such that $VT = TV$. Let S and $\widetilde{S}$ be the frame operators for $\{T^n\varphi\}_{n\in\mathbb{Z}}$ and $\{T^nV\varphi\}_{n\in\mathbb{Z}}$, respectively. Then $\{(\widetilde{S})^nf\}_{n\in\mathbb{Z}}$ is a frame for $\mathcal{H}$ if and only if $\{S^nV^*f\}$ is a frame for $\mathcal{H}$. In other words, $f \in \mathcal{V}_{\mathbb{Z}}(\widetilde{S})$ if and only if $V^*f \in \mathcal{V}_{\mathbb{Z}}(S)$.*

Proof. For each $f \in \mathcal{H}$, we have

$$\begin{aligned}\widetilde{S}f &= \sum_{n\in\mathbb{Z}}\langle f, T^nV\varphi\rangle T^nV\varphi = \sum_{n\in\mathbb{Z}}\langle f, VT^n\varphi\rangle VT^n\varphi \\ &= V\sum_{n\in\mathbb{Z}}\langle V^*f, T^n\varphi\rangle T^n\varphi = VSV^*f.\end{aligned}$$

As V is unitary, we get $(\widetilde{S})^n = VS^nV^*$ and $V^*(\widetilde{S})^n = S^nV^*$ which immediately yields the desired conclusion. $\square$

3. FRAME REPRESENTATION OF THE FORM $\{A_NT^N\varphi\}_{N=0}^{\infty}$

In this section, we generalize some results in the recent papers [8, 10] which have been proved by Christensen *et al.* We consider frames of the form $\{f_k\}_{k=1}^{\infty} = \{a_nT^nf_1\}_{n=0}^{\infty}$ for some scalars $a_n \neq 0$ with $\sup_n \left|\dfrac{a_n}{a_{n+1}}\right| < \infty$ and a bounded linear operator $T : \operatorname{span}\{f_k\}_{k=1}^{\infty} \to \mathcal{H}$. Using [10], we define $\mathcal{T}_\omega : \ell^2(\mathbb{N}_0) \to \ell^2(\mathbb{N}_0)$ by $\mathcal{T}_\omega\{c_i\}_{i=0}^{\infty} = \Big(0, \frac{a_0}{a_1}c_0, \frac{a_1}{a_2}c_1, \cdots\Big)$. The following theorem was proved in [10]:

THEOREM 3.1. *Let* $\{a_n\}_{n=0}^{\infty}$ *be a sequence of non-zero scalars with* $\sup_n \left|\frac{a_n}{a_{n+1}}\right| < \infty$*, and let* $\{f_k\}_{k=1}^{\infty} = \{a_n T^n f_1\}_{n=0}^{\infty}$ *be a linearly independent frame for an infinite-dimensional Hilbert space* $\mathcal{H}$*, where* $T : \mathrm{span}\{f_k\}_{k=1}^{\infty} \to \mathcal{H}$ *is a linear operator. Then* T *is bounded if and only if* $\mathcal{N}_U$ *is invariant under* $\mathcal{T}_\omega$*.*

The condition $\sup_n \left|\frac{a_n}{a_{n+1}}\right| < \infty$ is indeed necessary for frames of the form $\{a_n T^n \varphi\}_{n=0}^{\infty}$ when $T \in B(\mathcal{H})$.

PROPOSITION 3.2. *Assume that* $T \in B(\mathcal{H})$ *such that* $\{a_n T^n \varphi\}_{n=0}^{\infty}$ *is a frame for some* $\varphi \in \mathcal{H}$ *and some non-zero scalars* $\{a_n\}_{n=0}^{\infty}$*. Then*

$$\sup_n \left|\frac{a_n}{a_{n+1}}\right| < \infty.$$

Proof. Let A and B be frames bounds of $\{f_k\}_{k=1}^{\infty} = \{a_n T^n \varphi\}_{n=0}^{\infty}$. Using that $\sqrt{A} \le \|f_k\| \le \sqrt{B}$ for all $k \in \mathbb{N}$, we get

$$\|f_k\|\|T\| \ge \|Tf_k\| = \left\|\frac{a_{k-1}}{a_k} f_{k+1}\right\| \ge \left|\frac{a_{k-1}}{a_k}\right|\sqrt{A} \ge \left|\frac{a_{k-1}}{a_k}\right|\sqrt{\frac{A}{B}}\|f_k\|.$$

Then $\sup_n \left|\frac{a_n}{a_{n+1}}\right| \le \sqrt{\frac{B}{A}}\|T\|$ as desired. □

If $T : \mathcal{H} \to \mathcal{H}$ is a linear operator and $\{f_k\}_{k=1}^{\infty} = \{a_n T^n \varphi\}_{n=0}^{\infty}$ is a frame (with frame bounds A and B) for some $\varphi \in \mathcal{H}$ and some non-zero scalars $\{a_n\}_{n=0}^{\infty}$ with $\sup_n \left|\frac{a_n}{a_{n+1}}\right| < \infty$, then we have

$$\|Tf_k\| = \left\|\frac{a_{k-1}}{a_k} f_{k+1}\right\| \le \left|\frac{a_{k-1}}{a_k}\right|\sqrt{B} \le \sqrt{\frac{B}{A}}\|f_k\|, \quad k \in \mathbb{N}.$$

In this case T may be unbouded (see Proposition 3.5). Using [[8],Proposition 2.5], we can obtain the following result for a frame in the form $\{a_n T^n \varphi\}_{n=0}^{\infty}$.

PROPOSITION 3.3. *Assume that* $T \in B(\mathcal{H})$ *such that* $\{a_n T^n \varphi\}_{n=0}^{\infty}$ *is a frame for some* $\varphi \in \mathcal{H}$ *and some non-zero scalars* $\{a_n\}_{n=0}^{\infty}$*. Then* T *has closed range and* $\mathcal{R}_T = \overline{\mathrm{span}}\{a_n T^{n+1} \varphi\}_{n=0}^{\infty}$*.*

Proof. Using [[4], Theorem 5.5.1], the synthesis operator

$$U : \ell^2(\mathbb{N}_0) \to \mathcal{H}, \quad U(c_0, c_1, c_2, ...) = \sum_{i=0}^{\infty} c_i a_i T^i \varphi$$

is surjective. Letting $x \in \mathcal{H}$ there exists $(c_0, c_1, c_2, ...) \in \ell^2(\mathbb{N}_0)$ such that $x = \sum_{i=0}^{\infty} c_i a_i T^i \varphi$. Therefore

$$Tx = \sum_{i=0}^{\infty} c_i a_i T^{i+1} \varphi \in \overline{\text{span}}\{a_i T^{i+1}\varphi\}_{i=0}^{\infty}.$$

Therefore $\mathcal{R}_T \subseteq \mathcal{K} := \overline{\text{span}}\{a_i T^{i+1}\varphi\}_{i=0}^{\infty}$. On the other hand, $\{a_i T^{i+1}\varphi\}_{i=0}^{\infty}$ is a frame for $\mathcal{K}$, and then its synthesis operator is surjective. Letting $x \in \mathcal{K}$, there is $(c_0, c_1, c_2, ...) \in \ell^2(\mathbb{N}_0)$ such that $x = \sum_{i=0}^{\infty} c_i a_i T^{i+1}\varphi = T\sum_{i=0}^{\infty} c_i a_i T^i \varphi \in \mathcal{R}_T$. Therefore $\mathcal{R}_T = \overline{\text{span}}\{a_n T^{n+1}\varphi\}_{n=0}^{\infty}$, *i.e.*, T has closed range. $\square$

The following proposition generalize a result in [5, 6], where we characterize the availability of the representation $\{f_k\}_{k=1}^{\infty} = \{a_n T^n f_1\}_{n=0}^{\infty}$.

PROPOSITION 3.4. *Let $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ be sequences in $\mathcal{H}$ such that each $f \in \mathcal{H}$ has the convergent expansion*

$$f = \sum_{k=1}^{\infty} \langle f, g_k\rangle f_k. \tag{3.1}$$

Suppose that $\{a_n\}_{n=0}^{\infty}$ is a sequence of non-zero scalars such that for any $f \in \mathcal{H}$ the series $\sum_{k=1}^{\infty} \langle f, g_k\rangle \frac{a_{k-1}}{a_k} f_{k+1}$ converges. Then $\{f_k\}_{k=1}^{\infty} = \{a_n T^n f_1\}_{n=0}^{\infty}$ for some $T \in B(\mathcal{H})$ if and only if

$$f_{j+1} = \frac{a_j}{a_{j-1}} \sum_{k=1}^{\infty} \langle f_j, g_k\rangle \frac{a_{k-1}}{a_k} f_{k+1}, \quad j \in \mathbb{N}. \tag{3.2}$$

Proof. Assume that $\{f_k\}_{k=1}^{\infty}$ can be represented as $\{a_n T^n f_1\}_{n=0}^{\infty}$ for some $T \in B(\mathcal{H})$. Then $Tf_k = \frac{a_{k-1}}{a_k} f_{k+1}$ for all $k \in \mathbb{N}$. By applying T on (3.1), we get

$$Tf = \sum_{k=1}^{\infty} \langle f, g_k\rangle Tf_k = \sum_{k=1}^{\infty} \langle f, g_k\rangle \frac{a_{k-1}}{a_k} f_{k+1}, \quad f \in \mathcal{H}.$$

Letting $f = f_j$ in the above expression, it follows that $\frac{a_{j-1}}{a_j} f_{j+1} = \sum_{k=1}^{\infty} \langle f_j, g_k\rangle \frac{a_{k-1}}{a_k} f_{k+1}$, and we get (3.2).

For the opposite implication, suppose that (3.2) holds. Define the linear operator

$$T : \mathcal{H} \to \mathcal{H}, \quad Tf = \sum_{k=1}^{\infty} \langle f, g_k\rangle \frac{a_{k-1}}{a_k} f_{k+1}, \quad f \in \mathcal{H}.$$

By uniform boundedness principle, T is bounded. Then by (3.2) we conclude that $Tf_j = \sum_{k=1}^{\infty} \langle f_j, g_k\rangle \frac{a_{k-1}}{a_k} f_{k+1} = \frac{a_{j-1}}{a_j} f_{j+1}$ for all $j \in \mathbb{N}$. Therefore $\{f_k\}_{k=1}^{\infty} = \{a_n T^n f_1\}_{n=0}^{\infty}$. $\square$

Motivated by Proposition 2.6 in [8] and with a small change in its proof, we can obtain the following result which generalizes it.

Proposition 3.5. *Assume that the frame $\{f_k\}_{k=1}^{\infty}$ is linearly independent, contains a Riesz basis and has finite and strictly positive excess. Let $T : \mathcal{H} \to \mathcal{H}$ be a linear operator such that $\{f_k\}_{k=1}^{\infty} = \{a_n T^n f_1\}_{n=0}^{\infty}$ for some non-zero scalars $\{a_n\}_{n=0}^{\infty}$ with $\sup_n \left|\frac{a_n}{a_{n+1}}\right| < \infty$ and $\inf_n \left|\frac{a_n}{a_{n+1}}\right| > 0$. Then T is unbounded.*

Proof. Let $\delta := \inf_n \left|\frac{a_n}{a_{n+1}}\right|$ and $\gamma := \sup_n \left|\frac{a_n}{a_{n+1}}\right|$. By assumption there exists $m \in \mathbb{N}$ such that $\{f_k\}_{k=m+1}^{\infty}$ is a Riesz basis for $\mathcal{K} := \overline{\text{span}}\{f_k\}_{k=m+1}^{\infty}$ and $\{f_k\}_{k=m}^{\infty}$ is an overcomplete frame for $\mathcal{K}$. Since $0 < \delta \leq \gamma < \infty$, we infer that $\left\{\frac{a_{k-1}}{a_k} f_{k+1}\right\}_{k=m}^{\infty}$ is a Riesz basis for $\mathcal{K}$, and we denote its lower Riesz basis bound by A. For each $n \in \mathbb{N}$, let A_n denote the optimal lower Riesz basis bound for the finite sequence $\{f_k\}_{k=m}^{m+n-1}$. Since $\{f_k\}_{k=m}^{\infty}$ is a linearly independentan and overcomplete frame, it follows $A_n \to 0$ as $n \to \infty$ by Proposition 7.2.1 in [4]. Let $n \in \mathbb{N}$, then there exists a non-zero sequence $\{c_k\}_{k=m}^{m+n-1}$ such that

$$\Big\| \sum_{k=m}^{m+n-1} c_k f_k \Big\|^2 \leq (A_n + \frac{1}{n}) \sum_{k=m}^{m+n-1} |c_k|^2.$$

Then

$$\begin{aligned}
\Big\| T \sum_{k=m}^{m+n-1} c_k f_k \Big\|^2 &= \Big\| \sum_{k=m}^{m+n-1} c_k \frac{a_{k-1}}{a_k} f_{k+1} \Big\|^2 \\
&\geq A \sum_{k=m}^{m+n-1} |c_k|^2 \\
&\geq \frac{A}{A_n + \frac{1}{n}} \Big\| \sum_{k=m}^{m+n-1} c_k f_k \Big\|^2.
\end{aligned}$$

If T is bounded, then it follows from the above inequlity that $\|T\| \geq \frac{A}{A_n + \frac{1}{n}}$. Since $\frac{A}{A_n + \frac{1}{n}} \to \infty$ as $n \to \infty$, we obtain a contradiction. □

4. SOME AUXILIARY RESULTS: PERTURBATION OF A FRAME $\{T^N \varphi\}_{N=0}^{\infty}$

Motivated by some results about perturbations of frames of the form $\{T^n \varphi\}_{n=0}^{\infty}$ in [5], we give some results by restricting ourself to perturb a frame

$\{T^n\varphi\}_{n=0}^{\infty}$ with elements from a subspace on which T acts as a contraction. We also state some stability results obtained by considering perturbations of operators belonging to an invariant subspace.

PROPOSITION 4.1. *Assume that $\{T^n\varphi\}_{n=0}^{\infty}$ is a Riesz sequence for some $T \in B(\mathcal{H})$ and some $\varphi \in \mathcal{H}$, and let A denote a lower Riesz bound. Suppose that $V \subset \mathcal{H}$ is invariant under T and that there exists $\mu \in [0,1)$ such that $\|T\psi\| \leq \mu\|\psi\|$. Then $\{T^n(\varphi+\psi)\}_{n=0}^{\infty}$ is a Riesz sequence for all $\psi \in V$ for which $\|\psi\| < (1-\mu)\sqrt{A}$.*

Proof. It is clear that $\sum_{n=0}^{\infty} \|T^n\psi\|^2 < \infty$ for all $\psi \in V$. By [[11], Theorem 2.14] it is sufficient to show that $\sum_{n=0}^{\infty} \|T^n(\varphi+\psi) - T^n\varphi\| \|S^{-1}T^n\varphi\| < 1$, where S is frame operator for $\{T^n\varphi\}_{n=0}^{\infty}$. Since $\|S^{-1}T^n\varphi\| \leq 1/\sqrt{A}$, we have

$$\sum_{n=0}^{\infty} \|T^n(\varphi+\psi) - T^n\varphi\| \|S^{-1}T^n\varphi\| \leq \frac{\|\psi\|}{\sqrt{A}} \sum_{n=0}^{\infty} \mu^n = \frac{\|\psi\|}{(1-\mu)\sqrt{A}} < 1,$$

as desired. □

A similar approach as in the proof of Proposition 3.3 in [5] yields the following result.

PROPOSITION 4.2. *Let $\{a_n\}_{n=0}^{\infty}$ be a bounded sequence of scalars. Assume that $\{a_nT^n\varphi\}_{n=0}^{\infty}$ is a frame for some bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ and some $\varphi \in \mathcal{H}$, and let A denote a lower frame bound. Suppose that $V \subset \mathcal{H}$ is invariant under T and that there exists $\mu \in [0,1)$ such that $\|T\psi\| \leq \mu\|\psi\|$. Then the following hold:*

(i) *$\{a_nT^n(\varphi+\psi)\}_{n=0}^{\infty}$ is a frame sequence for all $\psi \in V$.*

(ii) *$\{a_nT^n(\varphi+\psi)\}_{n=0}^{\infty}$ is a frame for all $\psi \in V$ for which $\sup_n \|a_n\psi\| < \sqrt{A(1-\mu^2)}$.*

We now provide a perturbation result which can be used to construct a frame with representation $\{a_nT^n\varphi\}_{n=0}^{\infty}$.

PROPOSITION 4.3. *Let $T \in B(\mathcal{H})$ and $\varphi, \psi \in \mathcal{H}$. Assume that $\{a_n\}_{n=0}^{\infty}$ is sequence of non-zero scalars such that $\{a_nT^n\varphi\}_{n=0}^{\infty}$ is a frame for $\mathcal{H}$ with lower bound A and $\{a_{n+1}T^n\psi\}_{n=0}^{\infty}$ is a Bessel sequence for $\mathcal{H}$ with Bessel bound B. If $\sup_n \left|\frac{a_n}{a_{n+1}}\right| < \sqrt{\frac{A}{B}}$, then $\{a_nT^n(\varphi+\psi)\}_{n=0}^{\infty}$ is a frame for $\mathcal{H}$.*

Proof. Let $\{c_n\}_{n=0}^{\infty} \in \ell^2(\mathbb{N}_0)$ and $\alpha := \sup_n \left|\dfrac{a_n}{a_{n+1}}\right|$. By assumption, we have

$$\begin{aligned}\Big\|\sum_{n=0}^{\infty} c_n(a_nT^n\varphi - a_nT^n(\varphi+\psi))\Big\|^2 &= \Big\|\sum_{n=0}^{\infty} c_na_nT^n\psi\Big\|^2 \\ &= \sup_{\|f\|=1}\Big|\Big\langle \sum_{n=0}^{\infty} c_na_nT^n\psi, f\Big\rangle\Big|^2 \\ &= \sup_{\|f\|=1}\Big|\sum_{n=0}^{\infty} c_n\frac{a_n}{a_{n+1}}\langle a_{n+1}T^n\psi, f\rangle\Big|^2 \\ &\le \sum_{n=0}^{\infty}\Big|c_n\frac{a_n}{a_{n+1}}\Big|^2 \sup_{\|f\|=1}\sum_{n=0}^{\infty}\big|\langle a_{n+1}T^n\psi, f\rangle\big|^2 \\ &\le \alpha^2 B\sum_{n=0}^{\infty}|c_n|^2.\end{aligned}$$

Hence, [[4], Theorem 22.1.1] implies that the desired result. $\square$

Here $\mathcal{B}$ denotes the set of bounded linear operators $T : \mathcal{H} \to \mathcal{H}$ for which there exist $\lambda_T \in [0,1)$ and an invariant subspace $V_T \subset \mathcal{H}$ under T such that $\|T\varphi\| \le \lambda_T\|\varphi\|$ for all $\varphi \in V_T$. In the following proposition I is a countable index set and $\{g_j\}_{j\in I}$ is a sequence in $\mathcal{H}$.

PROPOSITION 4.4. *Suppose that $T, W \in \mathcal{B}$ and $\{g_j\}_{j\in I} \subseteq V_W \cap V_T$. Let $\{W^ng_j\}_{n\ge0,j\in I}$ be a Riesz sequence with frame operator S, and $\{T^ng_j\}_{n\ge0,j\in I}$ be a Bessel sequence for $\mathcal{H}$. Assume that $\sum_{j\in I}\|g_j\|^2 < \dfrac{1-\lambda^2}{2\|S^{-1}\|}$, where $\lambda :=$* $\max\{\lambda_W, \lambda_T\}$. *Then $\{T^ng_j\}_{n\ge0,j\in I}$ is a Riesz sequence.*

Proof. By assumptions, we have

$$\|Wg_j\| \le \lambda\|g_j\|, \quad \|Tg_j\| \le \lambda\|g_j\|, \quad j \in I.$$

Then

$$\begin{aligned}\sum_{j\in I}\sum_{n=0}^{\infty}\|W^ng_j - T^ng_j\|\|S^{-1}W^ng_j\| &\le \sum_{j\in I}\sum_{n=0}^{\infty}\|W^ng_j - T^ng_j\|\|S^{-1}\|\|W^ng_j\| \\ &\le \sum_{j\in I}\sum_{n=0}^{\infty}(\|W^ng_j\|+\|T^ng_j\|)\|S^{-1}\|\|W^ng_j\| \\ &\le 2\|S^{-1}\|\sum_{j\in I}\sum_{n=0}^{\infty}\lambda^{2n}\|g_j\|^2\end{aligned}$$

$$= \frac{2\|S^{-1}\|}{1-\lambda^2} \sum_{j\in I} \|g_j\|^2 < 1.$$

Therefore, [[11], Theorem 2.14] leads to the desired result. □

PROPOSITION 4.5. *Let $T, W \in \mathcal{B}$ and $\varphi \in V_T \cap V_W$. Suppose that $\{T^n\varphi\}_{n=0}^\infty$ is a frame for $\mathcal{H}$ with lower frame bound A and $\{W^n\varphi\}_{n=0}^\infty$ is a Bessel sequence for $\mathcal{H}$. Let $2\|\varphi\| < \sqrt{A(1-\lambda^2}$, where $\lambda := \max\{\lambda_T, \lambda_W\}$. Then $\{W^n\varphi\}_{n=0}^\infty$ is a frame for $\mathcal{H}$.*

In the case where $\{T^n\varphi\}_{n=0}^\infty$ is a Riesz sequence with lower bound A, then $\{T^n\varphi + W^n\varphi\}_{n=0}^\infty$ is a Riesz sequence, whenever $\|\varphi\| < \sqrt{A(1-\lambda^2)}$.

Proof. By assumptions, we have

$$\sum_{n=0}^{\infty} \|T^n\varphi - W^n\varphi\|^2 \leq 2\Big(\sum_{n=0}^{\infty} \|T^n\varphi\|^2 + \sum_{n=0}^{\infty} \|W^n\varphi\|^2\Big)$$
$$\leq 4\|\varphi\|^2 \sum_{n=0}^{\infty} \lambda^{2n} = \frac{4\|\varphi\|^2}{1-\lambda^2} < A.$$

We conclude by [[4], Corollary 22.1.5] that $\{W^n\varphi\}_{n=0}^\infty$ is a frame for $\mathcal{H}$. If $\{T^n\varphi\}_{n=0}^\infty$ be a Riesz sequence, then

$$\Big\|\sum_{n=0}^{\infty} c_n(T^n\varphi - (T^n\varphi + W^n\varphi))\Big\|^2 = \Big\|\sum_{n=0}^{\infty} c_n W^n\varphi\Big\|^2$$
$$\leq \sum_{n=0}^{\infty} |c_n|^2 \sum_{n=0}^{\infty} \|W^n\varphi\|^2$$
$$\leq \frac{\|\varphi\|^2}{1-\lambda^2} \sum_{n=0}^{\infty} |c_n|^2.$$

Therefore, the result follows from [[4], Theorem 22.3.2]. □

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