ON THE MEAN VALUE OF DEDEKIND SUMS OVER PRIMITIVE ROOTS OF AN ODD PRIME

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The main purpose of this paper is using the analytic method and the properties of primitive roots $\bmod\ p$ (an odd prime) to study the computational problem of one kind Dedekind sums, and give an interesting computational formula for it.

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1. INTRODUCTION

Let q be a natural number and h an integer prime to q. The classical Dedekind sums

$$S(h,q)=\sum_{a=1}^{q}\left(\left(\frac{a}{q}\right)\right)\left(\left(\frac{ah}{q}\right)\right),$$

where

$$((x))=\begin{cases} x-[x]-\frac{1}{2}, & \text{if } x \text{ is not an integer;}\\ 0, & \text{if } x \text{ is an integer,}\end{cases}$$

describes the behaviour of the logarithm of the eta-function (see [7], [8]) under modular transformations. Many authors have studied the arithmetical properties of $S(h,q)$, and obtained many interesting results, some of them can be found in [2]–[10]. For example, J. B. Conrey *et al.* [4] studied the mean value distribution of $S(h,k)$, and proved the asymptotic formula

$$\text{(1)}\qquad \sum_{h=1}^{k}{}' |S(h,k)|^{2m}=f_m(k)\left(\frac{k}{12}\right)^{2m}+O\left(\left(k^{\frac{9}{5}}+k^{2m-1+\frac{1}{m+1}}\right)\cdot \ln^3 k\right),$$

where $\sum_{h}{}'$ denotes the summation over all h such that $(k,h)=1$, and

$$\sum_{m=1}^{\infty}\frac{f_m(n)}{n^s}=2\cdot\frac{\zeta^2(2m)}{\zeta(4m)}\cdot\frac{\zeta(s+4m-1)}{\zeta^2(s+2m)}\cdot\zeta(s).$$

C. Jia [5] improved the error terms in (1) to $O\left(k^{2m-1}\ln^3 k\right)$ for all integer $m \geq 2$. W. Zhang [10] established a close contact between $S(h,q)$ and the mean square value of Dirichlet L-functions (see the Lemma 1 below).

Perhaps the most famous property of Dedekind sums is the reciprocity formula (see references [2], [3] and [6]):

$$S(h,k) + S(k,h) = \frac{h^2+k^2+1}{12hk} - \frac{1}{4} \tag{2}$$

for all $(h,k)=1$, $h>0$ and $k>0$.

An interesting three term version of (2) was also discovered by H. Rademacher and E. Grosswald [8].

Let p be an odd prime, $A(p)$ denotes the set of all primitive roots mod p in the interval $[1, p-1]$. In this paper, we consider the computational problem of the mean value

$$\sum_{a\in A(p)} S(a,p). \tag{3}$$

About this problem, it seems that no one had studied it until now; at least we have not seen any related result before. The problem is interesting because it can reflect some new distribution properties of Dedekind sums $S(a,p)$ in a special number set. In this paper, we are using the analytic method and the properties of characters mod p to give several interesting computational formulae for (3) with some special primes p. That is, we shall prove the following two conclusions:

THEOREM 1. *Let p be an odd prime with $p \equiv 1 \bmod 4$, $A(p)$ denotes the set of all primitive roots* mod p *in the interval $[1, p-1]$. Then we have the identity*

$$\sum_{a\in A(p)} S(a,p) = 0.$$

THEOREM 2. *Let q be an odd prime such that $p = 2q+1$ is also an odd prime. Then we have the identity*

$$\sum_{a\in A(p)} S(a,p) = \frac{(p-1)(p-2)}{12p} - \frac{1}{2}h_p^2,$$

where h_p denotes the class number of the quadratic field $\mathbb{Q}(\sqrt{-p})$.

Some notes: It is clear that our theorem 2 reveals a close relationship between the class number h_p, the Dedekind sums and primitive roots mod p. That is, the class number h_p can be represented by $\sum_{a\in A(p)} S(a,p)$.

It is very difficult to generalize the result in our theorem 2 to a high power. In fact this time, the problem we mentioned involving the fourth power mean of the Dirichlet L-functions, its exact calculation formula can not be obtained.

For general odd prime $p = 2 \cdot l + 1$ (where l is an odd composite number), whether there exists an exact computational formula for (3) is an open problem.

2. SEVERAL LEMMAS

To complete the proof of our theorem, we need to prove several lemmas. Hereinafter, we shall use some properties of characters mod q and Dirichlet L-functions. All of these can be found in reference [1], so they will not be repeated here.

Lemma 1. *Let $q > 2$ be an integer, then for any integer a with $(a, q) = 1$, we have the identity*

$$S(a,q) = \frac{1}{\pi^2 q} \sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(a)|L(1,\chi)|^2,$$

where $L(1, \chi)$ denotes the Dirichlet L-function corresponding to the character χ mod d.

Proof. See Lemma 2 of [10]. □

Lemma 2. *Let p be an odd prime. Then for any integer c with $(c, p) = 1$, we have the identity*

$$\frac{\phi(p-1)}{p-1} \sum_{h|p-1} \frac{\mu(h)}{\phi(h)} \sum_{\substack{k=1 \\ (h,k)=1}}^{h} e\left(\frac{k\ ind\ c}{h}\right) = \begin{cases} 1, & \text{if } c \text{ is a primitive root mod } p, \\ 0, & \text{otherwise}, \end{cases}$$

where ind c denotes the index of c relative to some fixed primitive root of p, and $\mu(n)$ is the Möbius function.

Proof. See Proposition 2.2 of reference [11]. □

Lemma 3. *Let p be an odd prime with $p = 2^b \cdot l + 1$ and $(2, l) = 1$, χ be any odd character* mod *p with $\chi(n) = e\left(\frac{r\ ind\ n}{2s}\right)$, where $2s|p-1$, $1 \le r \le 2s$ and $(r, 2s) = 1$. Then we have the identity*

$$\sum_{h|p-1} \frac{\mu(h)}{\phi(h)} \sum_{\substack{k=1 \\ (h,k)=1}}^{h} \sum_{a=1}^{p-1} \chi(a) e\left(\frac{k\ ind\ a}{h}\right) = \begin{cases} 0, & \text{if } b \ge 2; \\ -\frac{\mu(s)}{\phi(s)} \cdot (p-1), & \text{if } b = 1. \end{cases}$$

Proof. It is clear that if χ is an odd character $\bmod\ p$, then for any integer n with $(n,p)=1$, we must have $\chi(n)=e\left(\frac{r\ \text{ind}\ n}{2s}\right)$ for some $2s|p-1$, $1\le r\le 2s$ and $(r,2s)=1$. Otherwise, χ is not an odd character mod p. Now for $p=2^b\cdot l+1$ with $(2,l)=1$ and $b\ge 2$, from the definition and the orthogonality properties of the characters mod p we have

$$\sum_{a=1}^{p-1}\chi(a)e\left(\frac{k\ \text{ind}\ a}{h}\right)=\sum_{a=1}^{p-1}\chi(a)\chi_{k,h}(a)=\begin{cases} p-1, & \text{if } \overline{\chi}=\chi_{k,h};\\ 0, & \text{if } \overline{\chi}\ne\chi_{k,h}.\end{cases}\tag{4}$$

If $\overline{\chi}=\chi_{k,h}$ in (4), then $\chi_{h,k}$ must be an odd character mod p. So this time, we have $h=2^b\cdot d$, where $d|l$. In this case, we have $\mu(h)=\mu\left(2^b\cdot d\right)=0$. Thus,

$$\frac{\mu(h)}{\phi(h)}\sum_{\substack{k=1\\(h,k)=1}}^{h}\sum_{a=1}^{p-1}\chi(a)e\left(\frac{k\ \text{ind}\ a}{h}\right)=\sum_{a=1}^{p-1}\chi(a)\chi_{k,h}(a)=0.\tag{5}$$

If $h=2^u\cdot d$ with $0\le u\le b-1$ and $d|l$, then $\chi_{k,h}$ must be an even character mod p, so in this case, $\chi\chi_{k,h}$ is also an odd character mod p. Therefore, we have

$$\sum_{a=1}^{p-1}\chi(a)e\left(\frac{k\ \text{ind}\ a}{h}\right)=0.\tag{6}$$

If $b=1$, then $\chi\chi_{k,h}$ be a principal character mod p if and only if $h=2s$ and $k=2s-r$. In the other case, the character sums is 0. So we have

$$\sum_{h|p-1}\frac{\mu(h)}{\phi(h)}\sum_{\substack{k=1\\(h,k)=1}}^{h}\sum_{a=1}^{p-1}\chi(a)e\left(\frac{k\ \text{ind}\ a}{h}\right)=\frac{\mu(2s)}{\phi(2s)}\cdot(p-1)=-\frac{\mu(s)}{\phi(s)}\cdot(p-1).\tag{7}$$

Combining (5), (6) and (7) we may immediately deduce Lemma 3. $\square$

3. PROOF OF THE THEOREMS

In this section, we shall complete the proof of our theorems. First from Lemma 1 with $q=p$, an odd prime, we have (or see [12])

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}|L(1,\chi)|^2=\frac{\pi^2(p-1)}{p}\sum_{a=1}^{p-1}\left(\frac{a}{p}-\frac{1}{2}\right)^2=\frac{\pi^2}{12}\cdot\frac{(p-1)^2(p-2)}{p^2}\tag{8}$$

and

$$S(a,p) = \frac{1}{\pi^2}\cdot\frac{p}{p-1}\cdot\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(a)|L(1,\chi)|^2. \tag{9}$$

If $p = 2^b \cdot l + 1$ with $b \geq 2$ and $(2, l) = 1$, then from (9), Lemma 2 and Lemma 3 we have

$$\begin{aligned}
\sum_{a\in A(p)} S(a,p) &= \frac{1}{\pi^2}\cdot\frac{p}{p-1}\cdot\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\sum_{a\in A(p)} \chi(a)|L(1,\chi)|^2\\
&= \frac{1}{\pi^2}\cdot\frac{p}{p-1}\cdot\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\frac{\phi(p-1)}{p-1}\sum_{h|p-1}\frac{\mu(h)}{\phi(h)}\sum_{\substack{k=1\\(h,k)=1}}^{h}\sum_{a=1}^{p-1} e\left(\frac{k \text{ ind } a}{h}\right)\chi(a)|L(1,\chi)|^2\\
&= \frac{1}{\pi^2}\cdot\frac{p\phi(p-1)}{(p-1)^2}\sum_{h|p-1}\frac{\mu(h)}{\phi(h)}\sum_{\substack{k=1\\(h,k)=1}}^{h}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\left(\sum_{a=1}^{p-1} e\left(\frac{k \text{ ind } a}{h}\right)\chi(a)\right)|L(1,\chi)|^2\\
&= 0.
\end{aligned}$$

This proves Theorem 1.

If $p = 2\cdot l + 1$ with $(2, l) = 1$, then from (9), Lemma 2 and Lemma 3 we have

(10)

$$\begin{aligned}
\sum_{a\in A(p)} S(a,p) &= \frac{1}{\pi^2}\cdot\frac{p}{p-1}\cdot\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\sum_{a\in A(p)} \chi(a)|L(1,\chi)|^2\\
&= \frac{1}{\pi^2}\cdot\frac{p\phi(p-1)}{(p-1)^2}\sum_{h|p-1}\frac{\mu(h)}{\phi(h)}\sum_{\substack{k=1\\(h,k)=1}}^{h}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\left(\sum_{a=1}^{p-1} e\left(\frac{k \text{ ind } a}{h}\right)\chi(a)\right)|L(1,\chi)|^2\\
&= \frac{1}{\pi^2}\cdot\frac{p\phi(p-1)}{(p-1)^2}\sum_{h|l}\frac{\mu(2h)}{\phi(2h)}\sum_{\substack{k=1\\(2h,k)=1}}^{2h}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\left(\sum_{a=1}^{p-1} e\left(\frac{k \text{ ind } a}{2h}\right)\chi(a)\right)|L(1,\chi)|^2\\
&= \frac{-1}{\pi^2}\cdot\frac{p\phi(p-1)}{p-1}\sum_{h|l}\frac{\mu(h)}{\phi(h)}\sum_{\substack{k=1\\(2h,k)=1}}^{2h} |L(1,\chi_{k,2h})|^2.
\end{aligned}$$

If $l = q$ is an odd prime, let χ_2 denote the Legendre symbol mod p. Then for prime $p = 2q + 1$, the number of all odd characters mod p is $\frac{1}{2}\phi(p) = q$. It is clear that for any $1 \le k \le 2q$ with $(2q,k)=1$, $\chi_{k,2q}$ is an odd character $\bmod\ p$, and the number of all $\chi_{k,2q}$ $(1 \le k \le 2q$ and $(k,2q)=1)$ is $\phi(2q)=q-1$, because it does not contain Legendre symbol mod p. So we have the identity

$$\sum_{\substack{k=1\\(2q,k)=1}}^{2q} |L(1,\chi_{k,2q})|^2 = \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} |L(1,\chi)|^2 - |L(1,\chi_2)|^2. \tag{11}$$

From (8), (10) and (11) we have

$$\sum_{a\in A(p)} S(a,p) = \frac{-1}{\pi^2}\cdot\frac{p\phi(p-1)}{p-1}\sum_{h|l}\frac{\mu(h)}{\phi(h)}\sum_{\substack{k=1\\(2h,k)=1}}^{2h} |L(1,\chi_{k,2h})|^2$$

$$= \frac{-1}{\pi^2}\cdot\frac{p\phi(p-1)}{p-1}\frac{\mu(q)}{\phi(q)}\sum_{\substack{k=1\\(2q,k)=1}}^{2q} |L(1,\chi_{k,2q})|^2 - \frac{1}{\pi^2}\cdot\frac{p\phi(p-1)}{p-1}\cdot|L(1,\chi_2)|^2$$

$$= \frac{1}{\pi^2}\cdot\frac{p\phi(p-1)}{p-1}\frac{1}{\phi(q)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} |L(1,\chi)|^2 - \frac{1}{\pi^2}\cdot\frac{p\phi(p-1)}{p-1}\cdot\frac{q}{\phi(q)}\cdot|L(1,\chi_2)|^2$$

$$= \frac{1}{\pi^2}\cdot\frac{p\phi(p-1)}{p-1}\frac{1}{\phi(q)}\frac{\pi^2(p-1)^2(p-2)}{12p^2} - \frac{1}{\pi^2}\cdot\frac{p\phi(p-1)}{p-1}\cdot\frac{q}{\phi(q)}|L(1,\chi_2)|^2$$

$$= \frac{(p-1)(p-2)}{12p} - \frac{1}{2}h_p^2,$$

where we have used the identity $|L(1,\chi_2)| = \frac{\pi}{\sqrt{p}}\cdot h_p$ (This is the formula (15) in Chapter 6 of [13]), and h_p denotes the class number of the quadratic field $\mathbf{Q}(\sqrt{-p})$.

This completes the proof of Theorem 2.

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