LARGE STARS WITH FEW COLORS

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A recent question in generalized Ramsey theory is that for fixed positive integers $s \leq t$, at least how many vertices can be covered by the vertices of no more than s monochromatic members of the family $\mathcal{F}$ in every edge coloring of K_n with t colors. This is related to an old problem of Chung and Liu: for graph G and integers $1 \leq s < t$ what is the smallest positive integer $n = R_{s,t}(G)$ such that every coloring of the edges of K_n with t colors contains a copy of G with at most s colors. We answer this question when G is a star and s is either $t-1$ or $t-2$ generalizing the well-known result of Burr and Roberts.

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1. INTRODUCTION

Ramsey theory is an area of combinatorics which uses techniques from many branches of mathematics and is currently among the most active areas in combinatorics. Let $G_1, \ldots, G_c$ be graphs. The *Ramsey number* denoted by $r(G_1, \ldots, G_c)$ is defined to be the least number p such that if the edges of the complete graph K_p are arbitrarily colored with c colors, then for some i the spanning subgraph whose edges are colored with the i-th color contains G_i. More information about the Ramsey numbers of known graphs can be found in the survey [13].

There are various types of Ramsey numbers that are important in the study of classical Ramsey numbers and also hypergraph Ramsey numbers. A question recently proposed by Gyárfás *et al.* in [6] is that for fixed positive integers $s \leq t$, at least how many vertices can be covered by the vertices of no more than s monochromatic members of the family $\mathcal{F}$ in every edge coloring of K_n with t colors. This is related to an old problem of Chung and Liu [4]: for a given graph G and for fixed $1 \leq s < t$, find the smallest $n = R_{s,t}(G)$ such that in every t-coloring of the edges of K_n there is a copy of G colored with at most s colors. Note that for $s = 1$ this is the same Ramsey number $r_t(G)$. Several

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problems and interesting conjectures was presented in [6]. A basic problem here is to find the largest s-colored element of $\mathcal{F}$ that can be found in every t-coloring of K_n. The answer for matchings when $s = t - 1$ was given in [6]; every t-coloring of K_n contains a $(t-1)$-colored matching of size k provided that $n \geq 2k + [\frac{k-1}{2^{t-1}-1}]$. One can say more; we can guarantee the existence of a $(t-1)$-colored path on $2k$ vertices instead of a matching of size k. This was proved for $t = 2, 3, 4, 5$ in [5], [12], [10], [9], respectively and in general in [1]. The paper [8] contains similar results for linear forests. For complete graphs the problem was partially answered in [4] and [7]. Naturally, for these graphs the answer is very few known and there are many open problems. For stars, when $s = 1$ it is the well-know result of Burr and Roberts [2], and when $s = t - 1 = 2$ it was determined in [3].

In this paper, we find the value of $R_{s,t}(G)$ when G is a star and s is either $t-1$ or $t-2$. This will generalize the results of [2] and [3]. The paper is organized as follows. In section 2, we give the upper bound and lower bound of $R_{t-l,t}(K_{1,n})$ for given integer $l \geq 1$. In sections 3 and 4, we determine the values of $R_{t-1,t}(K_{1,n})$ and $R_{t-2,t}(K_{1,n})$, respectively. We only concerned with undirected simple finite graphs and for the vertex v of G the set of edges adjacent to v in G is denoted by $E_G(v)$.

2. SOME BOUNDS

In this section, we find some bounds for $R_{t-l,t}(K_{1,n})$. The *Turán number* $ex(H,p)$ is the maximum number of edges in a graph on p vertices which is H-free i.e., it does not have H as a subgraph. It is easily seen that $ex(K_{1,n},p) \leq \frac{p(n-1)}{2}$. This fact yields an upper bound for $R_{t-l,t}(K_{1,n})$ as we see in the following theorem.

THEOREM 2.1. *Suppose that $t' = [t/l]$. Then $R_{t-l,t}(K_{1,n}) \leq p$ for $p > \frac{t'n-1}{t'-1}$.*

Proof. Consider an edge coloring of K_p with t colors. Divide these t colors into $t' = [t/l]$ classes each of which contains l colors except the last one which may contains more colors. There exist l colors with at most $\left[\frac{1}{t'}\binom{p}{2}\right]$ edges. Thus the remaining $t-l$ colors appear on at least $\binom{p}{2} - \left[\frac{1}{t'}\binom{p}{2}\right]$ edges and the existence of $K_{1,n}$ with these $t-l$ colors is guaranteed if

$$\binom{p}{2} - \left[\frac{1}{t'}\binom{p}{2}\right] > \frac{p(n-1)}{2}.$$

So if $p > \frac{t'n-1}{t'-1}$, the above inequality is fulfilled and there exists a $K_{1,n}$ with at most $t-l$ colors. $\square$

The next theorem gives a lower bound for $R_{t-l,t}(K_{1,n})$.

THEOREM 2.2. *Let* $y = \left[\frac{t(n-l+1)-l}{t-l}\right]$. *Then* $R_{t-l,t}(K_{1,n}) > y - \epsilon$ *where* $\epsilon = 1$ *if* y *is odd and* $\epsilon = 0$, *otherwise.*

Proof. Let $p = y - \epsilon$. It is sufficient to give an edge coloring of K_p such that the set of colors appearing on the edges of every $K_{1,n}$ contains at least $t-l+1$ colors. By Vizing's theorem, there exists a proper edge coloring of K_p with $p-1$ colors. Let $p-1 = qt + r$, $0 \le r \le t-1$. We partition the above $p-1$ colors into t classes each of which contains $q = \left[\frac{p-1}{t}\right]$ colors except the last one which may contains $(p-1) - q(t-1)$ colors. Every $K_{1,n}$ contains at least $t-l+1$ colors if

$$n > (t-l-1)q + p - 1 - (t-1)q = (p-1) - lq.$$

The above inequality holds if $\frac{p-1}{t} \ge \frac{p-n-1}{l} + 1$ or equivalently, $p \le \frac{t(n-l+1)-l}{t-l}$ as asserted in Theorem 2.2. So there is no $K_{1,n}$ with at most $t-l$ colors, that is, $R_{t-l,t}(K_{1,n}) > p$. $\square$

Combining Theorems 2.1 and 2.2, we have an approximation of the value of $R_{t-l,t}(K_{1,n})$. For the small values of l this approximation is closer to the exact value. In particular, for $l = 1, 2$, we have the following corollaries.

CORROLARY 2.3. *Let* $x = \left[\frac{nt-1}{t-1}\right]$. *Then*

$$x \le R_{t-1,t}(K_{1,n}) \le x + 1.$$

In particular, when x *is even,* $R_{t-1,t}(K_{1,n}) = x + 1$.

CORROLARY 2.4. *Let* $t \ge 4$, $t' = [t/2]$ *and* $x = \left[\frac{nt'-1}{t'-1}\right]$. *Then*

$$x - 2 \le R_{t-2,t}(K_{1,n}) \le x + 1.$$

In particular, when $\left[\frac{t(n-1)-2}{t-2}\right]$ *is even,* $x - 1 \le R_{t-2,t}(K_{1,n}) \le x + 1$.

Remark. Let x be odd. Consider the complete graph K_x with its vertices $v_1, \ldots, v_x$ respectively placed on a circle. For v_x, there exists corresponding matching M_{v_x} containing $(x-1)/2$ parallel edges

$$v_1 v_{x-1}, v_2 v_{x-2}, \ldots, v_{(x-1)/2} v_{(x+1)/2}.$$

Order these edges as above. Similarly, for each vertex v_i, $1 \le i \le x-1$, there exists the matching M_{v_i} containing $(x-1)/2$ ordered edges. These matchings are used to construct certain edge colorings of K_x, for example in the proof of following lemmas.

Lemma 2.5. *Suppose that q is even and $x-1=tq$. There exists an edge coloring of K_x with t colors such that the set of all neighbors of every vertex contains q edges of any color.*

Proof. Partition the vertices of K_x as a single vertex v_x plus q classes $T_1,\dots,T_q$ where T_i contains t vertices say $v_{i_1},\dots,v_{i_t}$. Set $q/2$ classes $T_1,\dots,T_{q/2}$ on one side of v_x and $q/2$ classes $T_{q/2+1},\dots,T_q$ on the other side of v_x (see (a) of figure 1). For each vertex v_{i_j}, $1\le j\le t$ and $1\le i\le q$, color all $(x-1)/2$ parallel edges in $M_{v_{ij}}$ with color j. Moreover, for vertex v_x, color the edge $v_{i_j}v_{(q+1-i)_j}$ in M_{v_x} with j. The result is a coloring of K_x with the property that the set of all neighbors of every vertex contains q edges of any color, as desired. □

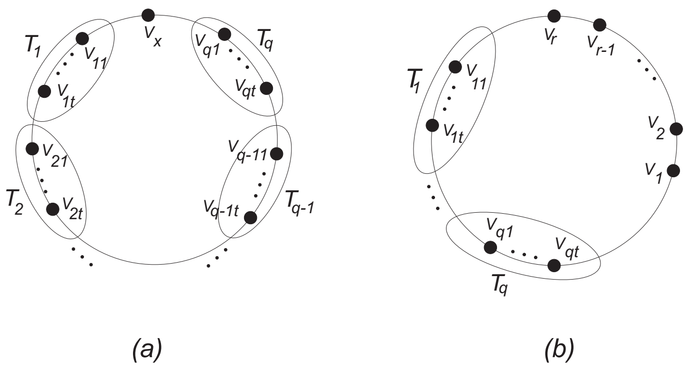

Fig. 1 – Partitions of the vertices of K_x

Lemma 2.6. *Suppose that $x=tq+r$ is odd and $2\le r\le t-1$. There exists an edge coloring of K_x with t colors such that the set of all neighbors of every vertex contains at least q edges of any color.*

Proof. Partition the vertices of K_x as $v_1,v_2,\dots,v_r$ plus q classes $T_1,\dots,T_q$ where T_i, $1\le i\le q$, contains t vertices say $v_{i_1},\dots,v_{i_t}$ (see (b) of figure 1). For each vertex v_{i_j} color all $(x-1)/2$ parallel edges in $M_{v_{i_j}}$ with color j. Moreover, for vertex v_r (also v_1) color the parallel edges in M_{v_r} (also in M_{v_1}) with $1,2,\dots,t$ alternatively (also $t,t-1,\dots,1$ alternatively). Color the remaining edges i.e., parallel edges corresponding to $v_2,\dots,v_{r-1}$ arbitrarily. The result is a coloring of the edges of K_x with the property that for any vertex, each color appears on at least q edges, as desired. □

3. THE VALUE OF $R_{t-1,t}(K_{1,n})$

In this section, using Corollary 2.3, we determine the exact value of $R_{t-1,t}(K_{1,n})$.

THEOREM 3.1. *Suppose that* $x = \left[\frac{nt-1}{t-1}\right]$ *and* $q = [\frac{x}{t}]$. *Then*

$$R_{t-1,t}(K_{1,n}) = \begin{cases} x & \text{if } x = tq+1 \text{ for } x, q \text{ odd,} \\ x+1 & \text{otherwise.} \end{cases}$$

Proof. First note that since $x = [\frac{nt-1}{t-1}]$, then $\frac{nt-1}{t-1} - 1 < x \leq \frac{nt-1}{t-1}$, or equivalently $x - x/t + 1/t \leq n < x - x/t + 1$ and so $n = x - [x/t] = x - q$. If x is even, then by Corollary 2.3, $R_{t-1,t}(K_{1,n}) = x + 1$. So we may assume that x is odd. We consider three cases as follows.

Case 1. $x = tq + 1$, where q is odd.

Consider an edge coloring of K_x with t colors. Suppose first that any color appears on q edges adjacent to every vertex. Consider a color c, then the subgraph induced by the edges with color c is q-regular and so the sum of degrees of its vertices is equal to the odd number xq, a contradiction. Thus there exists a vertex v and a color c with the property that c appears on at most $q-1$ edges adjacent to v. Then there are at least $x-1-(q-1) = x-q = n$ edges adjacent to v such that c does not appear on these edges. Hence there exists a subgraph $K_{1,n}$ without color c in K_x i.e., $R_{t-1,t}(K_{1,n}) \leq x$ and so by Corollary 2.3, $R_{t-1,t}(K_{1,n}) = x$.

Case 2. $x = tq + 1$, where q is even.

In the coloring of K_x given by Lemma 2.5, every $K_{1,n}$ contains all t colors i.e., $R_{t-1,t}(K_{1,n}) > x$ and so by Corollary 2.3, $R_{t-1,t}(K_{1,n}) = x + 1$.

Case 3. $x = tq + r$, where $2 \leq r \leq t - 1$.

In the coloring of K_x given by Lemma 2.6, every $K_{1,n}$ contains all t colors i.e., $R_{t-1,t}(K_{1,n}) > x$ and so by Corollary 2.3, $R_{t-1,t}(K_{1,n}) = x + 1$. □

As a corollary, we have the value of standard Ramsey number $r_2(K_{1,n})$ (see [13]).

CORROLARY 3.2. $r_2(K_{1,n}) = 2n - \epsilon$ *where* $\epsilon = 1$ *if* n *is even and* $\epsilon = 0$, *otherwise.*

4. THE VALUE OF $R_{t-2,t}(K_{1,n})$

In this section, we determine $R_{t-2,t}(K_{1,n})$. Corollary 2.4 gives a lower bound and an upper bound for $R_{t-2,t}(K_{1,n})$ for $t \geq 4$. Let us first settle the case $t = 3$. It is also a special case of multi-color Ramsey numbers for stars obtained in [2].

LEMMA 4.1. *There exists an edge coloring of K_{3n-2} with 3 colors such that every vertex contains exactly $n-1$ edges from each color.*

Proof. If $3n-2$ is even, then Vizing's Theorem gives a proper edge coloring of K_{3n-2} with $3n-3$ colors. Divide these $3n-3$ colors into 3 new color classes each of which contains $n-1$ colors to get the desired coloring of K_{3n-2} with 3 colors. Thus we may assume that $3n-2$ is odd. Then K_{3n-2} has $3n-2$ matchings each of which contains $(3n-3)/2$ parallel edges. For every vertex, color the corresponding parallel edges with 1, 2 and 3 respectively to get the desired coloring. □

THEOREM 4.2. *It holds $R_{1,3}(K_{1,n}) = 3n-1$.*

Proof. Consider an arbitrary edge coloring of K_{3n-1} with 3 colors 1, 2, 3 and a vertex v. Suppose that 3 is a color with maximum number of edges adjacent to v. So two colors 1 and 2 appear on at most $2[\frac{3n-2}{3}]$ edges adjacent to v. It is easily seen that $3n-2-2[\frac{3n-2}{3}] \geq n$ and so we have a $K_{1,n}$ with color 3, i.e. $R_{1,3}(K_{1,n}) \leq 3n-1$. To prove $R_{1,3}(K_{1,n}) \geq 3n-1$, apply Lemma 4.1. In this coloring of K_{3n-2} every $K_{1,n}$ contains at least 2 colors and so $R_{1,3}(K_{1,n}) > 3n-2$. □

For general case $t \geq 4$, we let R stands for $R_{t-2,t}(K_{1,n})$, $t' = [t/2]$ and $x = [\frac{nt'-1}{t'-1}]$.

LEMMA 4.3. *Suppose that $x-2 = tq+r$ where $0 \leq r \leq t-1$ and l is a natural number. Then $x-l-2q < n$ iff $t > (2r+4)/l$ when t is even and $t > 1+(2q+2r+4)/l$, otherwise.*

Proof. Since $n = x - [\frac{x}{t'}]$, we have $x-l-2q < n$ iff $[\frac{x}{t'}] < 2q+l$ or equivalently, $\frac{x}{t'} < 2q+l$. So $x-l-2q < n$ iff $t > (2r+4)/l$ when t is even and $t > 1+(2q+2r+4)/l$, otherwise. □

Theorem 4.4, states the necessary and sufficient conditions for R being $x+1$.

Theorem 4.4. *Suppose that* $x-2=tq+r$ *where* $0\leq r\leq t-1$. *Then* $R=x+1$ *iff one the following conditions holds.*

(a) $r=t-1>2q+4$ *and* x *is even.*

(b) $r=t-1>2q+4$ *and* x *and* t *are odd.*

(c) $r=t-1$, x *is odd and* t *and* $q+1$ *are even.*

(d) $r<t-2$ *and* $t>2r+4$ *is even.*

(e) $r<t-2$ *and* $t>2q+2r+5$ *is odd.*

Proof. We first suppose that x is even and we consider three cases as follows.

Case 1.1. $r=t-1$.

Note that since $x-1=t(q+1)$ is odd, t can't be even. Let $t>2q+5$. To prove $R=x+1$, using Corollary 2.4, it is enough to give a coloring of K_x with t colors such that every $K_{1,n}$ contains at least $t-1$ colors. By Vizing's Theorem, there exists a proper edge coloring of K_x with $x-1$ colors. We partition these $x-1$ colors into t color classes each of which contains $q+1$ colors to get a coloring of K_x with t colors. Then every $K_{1,n}$ contains at least $t-1$ colors iff $x-1-2(q+1)<n$ which holds by the assertion and Lemma 4.3 for $l=3$. Now let $t\leq 2q+5$. Suppose that an arbitrary edge coloring of K_x with t colors is given. For each vertex v, there are least two colors that appear on at most $2(q+1)$ edges of $E_G(v)$, since $x-1=t(q+1)$. Using Lemma 4.3 for $l=3$, at least n edges of $E_G(v)$ are colored with the remaining $t-2$ colors, that is, $R\leq x$.

Case 1.2. $r=t-2$.

We now prove $R\neq x+1$ by showing that $R\leq x$. Suppose that an arbitrary edge coloring of K_x with t colors is given. For each vertex v, there are two colors that appear on at most $2q+1$ edges of $E_G(v)$, since $x-1=t(q+1)-1$. Using Lemma 4.3 for $l=2$, at least n edges of $E_G(v)$ are colored with the remaining $t-2$ colors, that is, there exists a $K_{1,n}$ with at most $t-2$ colors.

Case 1.3. $r<t-2$.

Let either $t>2r+4$ be even or $t>2q+2r+5$ be odd. To prove $R=x+1$, it is enough to give a coloring of K_x with t colors such that every $K_{1,n}$ contains at least $t-1$ colors. By Vizing's Theorem, there is a proper edge coloring of K_x with $x-1$ colors. We partition these $x-1$ colors into $t-r-1$ color classes each of which contains q colors plus $r+1$ color classes each of which contains $(q+1)$ colors to get a coloring of K_x with t colors. Then every $K_{1,n}$ contains at least $t-1$ colors iff $x-1-2q<n$ which holds by the assertion and Lemma 4.3 for $l=1$, that is, $R>x$.

Now suppose that either $t\leq 2r+4$ or $t\leq 2q+2r+5$ is odd. Suppose that an arbitrary edge coloring of K_x is given. For each vertex v, there are two colors that appear on at most $2q$ edges of $E_G(v)$, since $x-1 = tq+r+1 < t(q+1)-1$. Hence by the assertion and Lemma 4.3 for $l = 1$, at least n edges of $E_G(v)$ are colored with the remaining $t-2$ colors, that is, $R \leq x$.

Now suppose that x is odd. We consider three cases as follows.

Case 2.1. $r = t-1$.

Let either $t > 2q+5$ be odd or both t and $q+1$ be even. We show that $R = x+1$. Note that since $x-1 = t(q+1)$ is even, if t is odd, then $q+1$ is even. By Lemma 2.5, there exists an edge coloring of K_x with t colors such that for each vertex v, $E_G(v)$ contains $q+1$ edges of any color. What is left is similar to the Case 1.1. If $t \leq 2q+5$ is odd and $q+1$ is even, similar argument as in the Case 1.1 yields $R \leq x$. Assume that $q+1$ is odd and hence t is even. Suppose that an arbitrary edge coloring of K_x is given. If for each vertex v, $E_G(v)$ contains $q+1$ edges of any color, the induced subgraph on the edges with a fixed color is $(q+1)$-regular with x vertices, a contradiction. So there exists a vertex v and a color c such that $E_G(v)$ contains at most q edges with color c. So there are two colors that appear on at most $2q+1$ edges of $E_G(v)$. Since $x-1-(2q+1) \geq n$, at least n edges of $E_G(v)$ are colored with the remaining $t-2$ colors, that is, $R \leq x$.

Case 2.2. $r = t-2$.

By the same argument as the Case 1.2, we get $R \leq x$.

Case 2.3. $r < t-2$.

Let either $t > 2r+4$ be even or $t > 2q+2r+5$ be odd. By Lemma 2.6, there exists an edge coloring of K_x such that for each vertex v, $E_G(v)$ contains at least q edges of any color. What is left is similar to the Case 1.3. □

Theorem 4.5, states the necessary and sufficient conditions for R being x.

THEOREM 4.5. *Suppose that $x-2 = tq+r$ where $0 \leq r \leq t-1$. Then $R = x$ iff one the following conditions holds.*

(a) *$r = t-1$ and x and $q+1$ are odd.*

(b) *$r < t-2$ and $t \leq 2r+4$ is even.*

(c) *$r < t-2$ and $q+r+3 < t \leq 2q+2r+5$ is odd.*

Proof. Let $p = x-1$, then $p-1 = tq+r$. We first suppose that p is even and consider three cases as follows.

Case 1.1. $r = t-1$.

Let t be even and $q+1$ be odd. By Theorem 4.4, $R \leq x$. By Vizing's Theorem there exists a proper edge coloring of K_p with $p-1$ colors. We partition these $p-1$ colors into $t-1$ classes each of which contains $q+1$ colors plus a class which contains q colors to get a coloring of K_p with t colors. Then every $K_{1,n}$ contains at least $t-1$ colors iff $p-1-(2q+1)<n$ which holds by the assertion and Lemma 4.3 for $l=3$, that is, $R>p=x-1$ and so $R=x$.

If both t and $q+1$ are even then $R>x$ by Theorem 4.4. Note that the case when both t and $q+1$ are odd is impossible, since $p=t(q+1)$ is even.

Assume that t is odd and $q+1$ is even. If $t>2q+5$, then $R\neq x$ by Theorem 4.4. Let $t\leq 2q+5$ be odd and $q+1$ be even. Suppose that an arbitrary edge coloring of $G=K_p$ with t colors is given. For each vertex v, there are two colors that appear on at most $2q+1$ edges of $E_G(v)$, since $p-1=t(q+1)-1$. Hence by Lemma 4.3 for $l=3$, at least n edges of $E_G(v)$ are colored with the remaining $t-2$ colors, that is, $R\leq p=x-1$.

Case 1.2. $r=t-2$.

Suppose that an arbitrary edge coloring of $G=K_p$ with t colors is given. For each vertex v, there are two colors that appear on at most $2q$ edges of $E_G(v)$, since $p-1=t(q+1)-2$. Hence by Lemma 4.3 for $l=2$, at least n edges of $E_G(v)$ are colored with the remaining $t-2$ colors, that is, $R\leq p=x-1$.

Case 1.3. $r<t-2$.

Let either $t\leq 2r+4$ be even or $q+r+3<t\leq 2q+2r+5$ be odd. By Theorem 4.4, $R\leq x$. By Vizing's Theorem, there exists a proper edge coloring of K_p with $p-1$ colors. We partition these $p-1$ colors into $t-r$ color classes each of which contains q colors plus r color classes each of which contains $q+1$ colors to get a coloring of K_p with t colors. Then every $K_{1,n}$ contains at least $t-1$ colors iff $p-1-2q<n$ which holds by the assertion and Lemma 4.3 for $l=2$, that is, $R>p=x-1$ and so $R=x$.

If either $t>2r+4$ is even or $t>2q+2r+5$ is odd, then $R\neq x$ by Theorem 4.4. Assume that $t\leq q+r+3$ is odd. Suppose that an arbitrary edge coloring of $G=K_p$ with t colors is given. For each vertex v, there are two colors that appear on at most $2q$ edges of $E_G(v)$, since $p-1<t(q+1)-2$. Hence by the assertion and Lemma 4.3 for $l=2$, at least n edges of $E_G(v)$ are colored with the remaining $t-2$ colors, that is, $R\leq p=x-1$.

Now suppose that p is odd. We consider three cases as follows.

Case 2.1. $r=t-1$.

So t and $q+1$ are odd. If $t>2q+5$, then by Theorem 4.4, $R=x+1$. Now let $t\leq 2q+5$ be odd. Suppose that an arbitrary edge coloring of $G=K_p$ with t colors is given. For each vertex v, there are two colors that appear on at most $2q+1$ edges of $E_G(v)$, since $p-1=t(q+1)-1$. Hence by Lemma 4.3 for $l=3$, at least n edges of $E_G(v)$ are colored with the remaining $t-2$ colors, that is, $R\leq p=x-1$.

Case 2.2. $r=t-2$.

By the same arguments as the Case 1.2, we get $R\leq p=x-1$.

Case 2.3. $r < t - 2$.

Let either $t \leq 2r + 4$ be even or $q + r + 3 < t \leq 2q + 2r + 5$ be odd. If $r = 0$ and t is even, then $t = 4$ and so $x = [\frac{nt'-1}{t'-1}] = 2n - 1$, which is impossible. By Lemmas 2.5 and 2.6, there exists an edge coloring of $G = K_p$ with t colors such that for each vertex v, $E_G(v)$ contains at least q edges of any color. What is left is similar to Case 1.3. □

Theorem 4.6, states the necessary and sufficient conditions for R being $x - 1$.

THEOREM 4.6. *Suppose that $x - 2 = tq + r$ where $0 \leq r \leq t - 1$. Then $R = x - 1$ iff one the following conditions holds.*

(a) $r = 1$, $\frac{2q+9}{3} < t \leq q + 4$ *is odd and* x *is even.*
(b) $r = 1$, $\frac{2q+9}{3} < t \leq q + 4$ *is odd and* x *is odd.*
(c) $1 < r < t - 2$ *and* $\frac{2q+2r+7}{3} < t \leq q + r + 3$ *is odd.*
(d) $r = t - 2$ *and either* t *is even or* $t > \frac{2q+2r+7}{3}$ *is odd.*

Proof. Let $p = x - 2$, then $p = tq + r$. We first suppose that x is even and consider five cases as follows.

Case 1.1. $r = 0$.

If either t is even or $t > 2q + 5$ is odd, then $R \neq x - 1$ by Theorems 4.4 and 4.5. If $q + 3 < t \leq 2q + 5$ is odd, then $R \neq x - 1$, by Theorem 4.5. Now let $t \leq q + 3$ be odd. Note that q is even in this case. Suppose that an arbitrary edge coloring of $G = K_p$ with t colors is given. For each vertex v, there are two colors that appear on at most $2q - 1$ edges of $E_G(v)$, since $p - 1 = x - 3 = tq - 1$. Hence by the assertion and Lemma 4.3 for $l = 2$, at least n edges of $E_G(v)$ are colored with the remaining $t - 2$ colors, that is, $R \leq p = x - 2$ and so by Corollary 2.4, $R = x - 2$.

Case 1.2. $r = 1$.

Let $\frac{2q+9}{3} < t \leq q + 4$ be odd. Since $t \leq q + 4$, by Theorems 4.4 and 4.5, $R \leq x - 1$. By Vizing's Theorem, there exists a proper edge coloring of K_p with $p - 1$ colors. We partition these $p - 1$ colors into t color classes each of which contains q colors to get a coloring of K_p with t colors. Then every $K_{1,n}$ contains at least $t - 1$ colors iff $x - 3 - 2q = p - 1 - 2q < n$ which holds by the assertion and Lemma 4.3 for $l = 3$, that is, $R > p = x - 2$ and hence $R = x - 1$. If either $t > q + 4$ is odd or t is even, then $R \neq x - 1$, by Theorems 4.4 and 4.5. Now let $t \leq \frac{2q+9}{3}$ be odd. Suppose that an arbitrary edge coloring of $G = K_p$ with t colors is given. For each vertex v, there are two colors that appear on at most $2q$ edges of $E_G(v)$, since $p - 1 = x - 3 = tq$. Hence by the assertion and Lemma 4.3 for $l = 3$, at least n edges of $E_G(v)$ are colored with the remaining $t - 2$ colors, that is, $R \leq p = x - 2$.

Case 1.3. $1 < r < t - 2$.

Let $\frac{2q+2r+7}{3} < t \leq q+r+3$ be odd. Since $t \leq q+r+3$, by Theorems 4.4 and 4.5, $R \leq x-1$. By Vizing's Theorem, there exists a proper edge coloring of K_p with $p-1$ colors. We partition these $p-1$ colors into $t-r+1$ color classes each of which contains q colors plus $r-1$ color classes each of which contains $q+1$ colors to get a coloring of K_p with t colors. Then every $K_{1,n}$ contains at least $t-1$ colors iff $x-3-2q = p-1-2q < n$ which holds by the assertion and Lemma 4.3 for $l = 3$, that is, $R > p = x-2$ and so $R = x-1$.

If either $t > q+r+3$ is odd or t is even, then $R \neq x-1$ by Theorems 4.4 and 4.5. Now let $t \leq \frac{2q+2r+7}{3}$ be odd. Suppose that an arbitrary edge coloring of $G = K_p$ with t colors is given. For each vertex v, there are two colors that appear on at most $2q$ edges of $E_G(v)$, since $p-1 = x-3 = tq+r-1 < t(q+1)-3$. Hence by the assertion and Lemma 4.3 for $l = 3$, at least n edges of $E_G(v)$ are colored with the remaining $t-2$ colors, that is, $R \leq p = x-2$.

Case 1.4. $r = t - 2$.

Let either t be even or $t > \frac{2q+2r+7}{3}$ be odd. By Theorems 4.4 and 4.5, $R \leq x-1$. By Vizing's Theorem, there exists a proper edge coloring of K_p with $p-1$ colors. We partition these $p-1$ colors into $t-3$ color classes each of which contains $q+1$ colors plus 3 color classes each of which contains q colors to get a coloring of K_p with t colors. Then every $K_{1,n}$ contains at least $t-1$ colors iff $x-3-2q = p-1-2q < n$ which holds by the assertion and Lemma 4.3 for $l = 3$, that is, $R > p = x-2$. Therefore $R = x-1$.

Now let $t \leq \frac{2q+2r+7}{3}$ be odd. Suppose that an arbitrary edge coloring of $G = K_p$ with t colors is given. For each vertex v, there are two colors that appear on at most $2q$ edges of $E_G(v)$, since $p-1 = x-3 = t(q+1)-3$. Hence by the assertion and Lemma 4.3 for $l = 3$, at least n edges of $E_G(v)$ are colored with the remaining $t-2$ colors, that is, $R \leq p = x-2$.

Case 1.5. $r = t - 1$.

Hence t is odd. If $t > 2q+5$, then by Theorem 4.4, $R \neq x-1$. Now let $t \leq 2q+5$ be odd. Using Lemma 4.3 for $l = 3$, we have $x-3-2q \geq n$ and so for each edge coloring of $G = K_p$ with t colors, n edges of $E_G(v)$ are colored with at most $t-2$ colors, that is $R \leq p = x-2$.

Now suppose that x is odd. We consider five cases as follows.

Case 2.1. $r = 0$.

The proof is similar to the Case 1.1. Note that when t is odd, q can't be even.

Case 2.2. $r = 1$.

Let $\frac{2q+9}{3} < t \leq q+4$ be odd and q be even. By Lemma 2.5, there exists an edge coloring of K_p with t colors such that for every vertex v, $E_G(v)$ contains

q edges of any color. What is left is similar to Case 1.2. Note that the case when both t and $q+1$ are odd is impossible.

Case 2.3. $1 < r < t-2$.

Let $\frac{2q+2r+7}{3} < t \leq q+r+3$ be odd. By Lemma 2.6, there exists an edge coloring of K_p with t colors such that for every vertex v, $E_G(v)$ contains q edges of any color. What is left is similar to Case 1.3.

Case 2.4. $r = t-2$.

Let either t be even or $t > \frac{2q+2r+7}{3}$ be odd. By Lemma 2.6, there exists an edge coloring of K_p with t colors such that for every vertex v, $E_G(v)$ contains q edges of any color. What is left is similar to Case 1.4.

Case 2.5. $r = t-1$.

If either t is even or $t > 2q+5$ is odd, then $R \neq x-1$ by Theorems 4.4 and 4.5. What is left is similar to the Case 1.5. □

Corrolary 4.7. *$R = x-2$ if and only if none of the conditions stated in Theorems 4.4, 4.5 and 4.6 holds.*

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