q-STURM–LIOUVILLE PROBLEMS WITH EIGENPARAMETER DEPENDENT BOUNDARY CONDITIONS

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In this article, we study dissipative q-Sturm–Liouville problems with eigenvalue parameter contained in the boundary conditions. It is shown that the analysis of q-Sturm–Liouville problems on a finite closed interval carries over to regular problems involving the eigenvalue parameter in the boundary conditions at one end-point. For the considered problem, we obtain asymptotic formulae for eigenvalues and eigenfunctions.

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1. INTRODUCTION

Q-difference analysis (or quantum analysis) is a very interesting subject in mathematics. Quantum derivative, a type of derivative in which the concept of limit is not used, is regarded as one of the important issues of discrete mathematics. When the limit concept is not used, the functions which are not differentiable in the classical sense (manner) are added to the function class of interest. The functions which are not differentiable in the classical sense can be quantum-differentiable (the quantum derivations can be evaluated) ([13], [36]). There are various types of quantum analysis such as h-analysis (finite difference analysis), q-analysis and Hahn analysis ([19]).

It has been observed that the concept of q-derivative and q-integral defined by Jackson in the early 1900's has important applications in various fields such as quantum mechanics, particle physics, complex analysis and hypergeometric series. Specifically, q-difference equations have been widely used in mathematical physics problems, for dynamical systems and quantum models [1], for q-analogues of mathematical physics problems including heat and wave equations [26], for sampling theory of signal analysis [2, 3, 35]. For more information, we refer the reader to [24].

On the other hand, parameter dependent systems are of great interest to numerous problems in physics and engineering. Specially, such problems occur while solving the proper partial differential equations with boundary conditions having a directional derivative, by the Fourier method (the separation of variables) ([10]). There are a lot of studies about parameter dependent problems ([6–10, 20–23, 25, 27, 34, 39–43]).

In this study, we consider the two-point boundary value problem

$$l(y) := -\frac{1}{q} D_{q^{-1}} D_q y(x) + v(x)\, y(x), \; q \in (0,1), \; 0 \le x \le a < +\infty \tag{1.1}$$

$$\cos\alpha \; y(0) + \sin\alpha \; D_{q^{-1}} y(0) = 0, \; 0 \le \alpha < \pi, \tag{1.2}$$

$$-\left(\beta_1 y(a) - \beta_2 D_{q^{-1}} y(a)\right) = \lambda \left(\beta_1' y(a) - \beta_2' D_{q^{-1}} y(a)\right), \; \beta_1, \beta_2, \beta_1', \beta_2' \in \mathbb{R}, \tag{1.3}$$

where λ is spectral parameter, $v(x)$ is defined on $[0, a]$ and continuous at zero. This problem differs from the q-Sturm-Liouville problem only in the appearance of the eigenvalue parameter λ in the boundary condition at a. We shall assume that

$$\rho := \begin{vmatrix} \beta_1' & \beta_1 \\ \beta_2' & \beta_2 \end{vmatrix} = \beta_1' \beta_2 - \beta_2' \beta_1 > 0. \tag{1.4}$$

The setup of this paper is as follows: in Section 2, some preliminary concepts and results related to our subject matter are presented for the convenience of the reader. In Section 3, we introduce a special inner product in the Hilbert space and define linear operator on it. We study the properties of this operator. In Section 4, we obtain asymptotic formulae for eigenvalues and eigenfunctions. While proving our results, we use the machinery and methods of [16, 27, 34]

2. PRELIMINARIES

Let us introduce q-notations and results which we need throughout this paper. For a review of this topic, we direct the reader to the monographs [17], [33]. Let $0 < q < 1$, $A \subset \mathbb{R}$, $a \in \mathbb{C}$ and $y(x)$ be a complex-valued function on $x \in A$.

The q-difference operator D_q is defined by

$$D_q y(x) = \frac{y(qx) - y(x)}{\mu(x)}, \text{ for all } x \in A.$$

where $\mu(x) = (q-1)x$. The q-derivative at zero is defined by

$$D_q y(0) = \lim_{n\to\infty} \frac{y(q^n x) - y(0)}{q^n x}, \; x \in A,$$

if the limit exist and does not depend on x. A right inverse to D_q, the Jackson q-integration is given by

$$\int_0^x f(t)\, d_q t = x(1-q) \sum_{n=0}^{\infty} q^n f(q^n x), \; x \in A,$$

provided that the series converges, and

$$\int_a^b f(t)\, d_q t = \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t, \quad a, b \in A.$$

Let $L_q^2(0,a)$ be the space of all complex-valued functions defined on $[0,a]$ such that

$$\|f\| := \left(\int_0^a |f(x)|\, d_q x \right)^{1/2} < \infty.$$

The space $L_q^2(0,a)$ is a separable Hilbert space with the inner product

$$(f,g) := \int_0^a f(x)\, \overline{g(x)} d_q x, \quad f, g \in L_q^2(0,a).$$

For every $y, z \in D$ we have q-Lagrange's identity (see [14], [25], [12])

$$(ly, z) - (y, lz) = [y,z](a) - [y,z](0) \tag{2.1}$$

where $[y,z](x) := y(x)\, \overline{D_{q^{-1}} z(x)} - D_{q^{-1}} y(x)\, \overline{z(x)}$.

For $n \in \mathbb{N} = \{0, 1, 2, ...\}$, $\alpha, a_1, .., a_n \in \mathbb{C}$, the q-shifted factorial, the multiple q-shifted factorial and q-binomial coefficients are defined to be

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1} \left(1 - aq^k\right), \quad (a;q)_\infty = \prod_{k=0}^{\infty} \left(1 - aq^k\right),$$

$$(a_1, a_2, ..., a_k : q) = \prod_{j=1}^{k} (a_j; q)_n$$

and respectively ([16], [37]). The generalized q-shifted factorial is defined by

$$(a;q)_\nu = \frac{(a;q)_\infty}{(aq^\nu; q)_\infty}, \quad \nu \in \mathbb{R}.$$

The q-Gamma function is defined by

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty} (1-q)^{1-z}, \; z \in \mathbb{C}, \; |q| < 1 \text{ ([28], [32])}.$$

The third type of the q-Bessel functions of Jackson of order v, $v > -1$, is defined to be

$$J_v(z;q) = z^v \frac{(q^{v+1};q)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q;q)_n (q^{v+1};q)_n}, \; z \in \mathbb{C} \text{ ([30], [31])}.$$

$\cos(z;q)$ and $\sin(z;q)$ are defined by

$$\begin{aligned}
\cos(z;q) &= \sum_{n=0}^{\infty}(-1)^n \frac{q^{n^2}(z(1-q))^{2n}}{(q;q)_{2n}} \\
&= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\left(zq^{-1/2}(1-q)\right)^{1/2} J_{-1/2}\left(z(1-q)/\sqrt{q};q^2\right), \\
\sin(z;q) &= \sum_{n=0}^{\infty}(-1)^n \frac{q^{n(n+1)}(z(1-q))^{2n+1}}{(q;q)_{2n+1}} \\
&= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\left(z(1-q)^{1/2}\right) J_{1/2}\left(z(1-q);q^2\right) \ \text{([4])}.
\end{aligned}$$

Let $w_m^{(v)}, v > -1$, denote the positive zeros of $J_v\left(.;q^2\right)$ in an increasing order of m and

$$x_m = w_m^{(-1/2)}\frac{\sqrt{q}}{1-q},\ \ y_m = w_m^{(-1/2)}\frac{1}{1-q}$$

denote the positive zeros of $\cos(z;q)$ and $\sin(z;q)$ respectively ([15]). For sufficiently large m, we have

$$\begin{aligned}
x_m &= \frac{q^{-m+1/2}}{1-q}\left(1+O\left(q^m\right)\right), \\
y_m &= \frac{q^{-m}}{1-q}\left(1+O\left(q^m\right)\right) \ \text{([15])}.
\end{aligned}$$

Let A denote the linear operator acting in the Hilbert space H with the domain $Dom(A)$. We know that a complex number λ_0 is called an eigenvalue of an operator A if there exists a non-zero vector $z_0 \in Dom(A)$ satisfying the equation $Az_0 = \lambda_0 z_0$; here, z_0 is called an eigenvector of A for λ_0. The eigenvectors for λ_0 span a subspace of $Dom(A)$, called the eigenspace for λ_0 and the geometric multiplicity of λ_0 is the dimension of its eigenspace. The vectors $z_1, z_2, ..., z_k$ are called the associated vectors of the eigenvector z_0 if they belong to $Dom(A)$ and $Az_i = \lambda_0 z_i + z_{i-1}$, $i = 1, 2, ..., k$. The element $z \in Dom(A)$, $z \neq 0$ is called a root vector of the operator A corresponding to the eigenvalue λ_0, if all powers of A are defined on this element and $(A - \lambda_0 I)^n z = 0$ for some integer n. The set of all root vectors of A corresponding to the same eigenvalue λ_0 with the vector $z = 0$ forms a linear set N_{λ_0} and is called the root lineal. The dimension of the lineal N_{λ_0} is called the algebraic multiplicity of the eigenvalue λ_0 ([11], [12]). Let $P(z)$ denote a polynomial. Suppose that α is a zero of order h. Then we can write $P(z) = (z-a)^h P_h(z)$ with $P_h(\alpha) \neq 0$. A zero of order 1 is called a simple zero and is characterized by the conditions $P(\alpha) = 0,\ P'(\alpha) \neq 0$ ([5]).

3. AN OPERATOR-THEORETIC FORMULATION IN THE CONVENIENT HILBERT SPACE

Now, we introduce a special inner product in the Hilbert space $\mathcal{H} = L_q^2(0,a) \oplus \mathbb{C}$ and define linear operator A in it to remove the spectral parameter in the boundary condition and to use the tools of operator theory. We define a Hilbert space $\mathcal{H}$ with inner product

$$\begin{aligned} \langle F, G\rangle \quad &: = \int_0^a F_1(x)\,\overline{G_1(x)}\,d_qx + \frac{1}{\rho}F_2\overline{G_2}, \\ F \quad &= \begin{pmatrix} F_1(x) \\ F_2 \end{pmatrix},\ G = \begin{pmatrix} G_1(x) \\ G_2 \end{pmatrix} \in \mathcal{H}, \end{aligned}$$

where the constant ρ is defined above.

For convenience, we assume

$$\begin{aligned} R_a(y) &= \beta_1 y(a) - \beta_2 D_{q^{-1}} y(a), \\ R'_a(y) &= \beta'_1 y(a) - \beta'_2 D_{q^{-1}} y(a), \\ N_0(y) &= \cos\alpha\ y(0) + \sin\alpha\ D_{q^{-1}} y(0),\ 0 \le \alpha < \pi. \end{aligned}$$

We construct the operator $A : \mathcal{H} \to \mathcal{H}$ with domain

$$D(A) = \left\{ \begin{array}{c} F \in \mathcal{H} \mid F_1, D_{q^{-1}}F_1 \text{ are continuous in } [0,a],\ lF_1 \in L_q^2(0,a), \\ N_0(y) = 0,\ F_2 = R'_a(F_1) \end{array} \right\}$$

as

$$A(F) = \begin{pmatrix} lF_1 \\ -R_a(F_1) \end{pmatrix}.$$

Then, we can pose the problem (1.1)-(1.3) in $\mathcal{H}$ as $AF = \lambda F,\ F \in D(A)$, i.e., the problem (1.1)-(1.3) can be considered as the eigenvalue problem for the operator A. It follows that A is densely defined, symmetric and self-adjoint.

THEOREM 1. *The operator A is symmetric.*

Proof. Let $F, G \in D(A)$. Using q-Lagrange's identity (2.1), we obtain

$$\begin{aligned} \langle AF, G\rangle - \langle F, AG\rangle &= [F_1, G_1](a) - [F_1, G_1](0) \\ &\quad + \frac{1}{\rho}\left[\left(-R_a(F_1)\,\overline{(R'_a(G_1))}\right) - R'_a(F_1)\,\overline{(-R_a(G_1))}\right]. \end{aligned} \tag{3.1}$$

Furthermore,

$$\frac{1}{\rho}\left[\left(R'_a(F_1)\,\overline{(R_a(G_1))} - R_a(F_1)\,\overline{(R'_a(G_1))}\right)\right] =$$

$$\frac{1}{\rho}\left[\begin{array}{c} \left(\beta'_1 F_1(a) - \beta'_2 D_{q^{-1}}F_1(a)\right)\overline{\left(\beta_1 G_1(a) - \beta_2 D_{q^{-1}}G_1(a)\right)} \\ -\left(\beta_1 F_1(a) - \beta_2 D_{q^{-1}}F_1(a)\right)\overline{\left(\beta'_1 G_1(a) - \beta'_2 D_{q^{-1}}G_1(a)\right)} \end{array}\right]$$

$$= \frac{1}{\rho}\left(\beta_1'\beta_2 - \beta_1\beta_2'\right)\left[F_1(a)\overline{D_{q^{-1}}G_1(a)} - D_{q^{-1}}F_1(a)\overline{G_1(a)}\right]$$

$$= [F_1, G_1](a). \tag{3.2}$$

The short calculation gives

$$[F_1, G_1](0) = 0. \tag{3.3}$$

Finally, substituting (3.2) and (3.3) in (3.1) yield the required equality.

$$\langle AF, G\rangle - \langle F, AG\rangle = 0,\ \ F, G \in D(A).$$

□

Corollary 1. *The eigenvalues of problem (1.1)–(1.3) are real.*

The q-Wronskian of $y(x),\ z(x)$ is defined to be

$$W_q(y,z)(x) := y(x)\,D_q z(x) - z(x)\,D_q y(x),\quad x \in [0,a].$$

Theorem 2. *The Wronskian of any solution of Equation (1.1) is independent of x.*

Proof. Let $y(x)$ and $z(x)$ be two solutions of Equation (1.1). By q-Lagrange's identity (2.1), we have

$$(ly, z) - (y, lz) = [y,z](a) - [y,z](0).$$

Since $ly = \lambda y$ and $lz = \lambda z$, we have

$$\begin{aligned}(\lambda y, z) - (y, \lambda z) &= [y,z](a) - [y,z](0),\\ \left(\lambda - \overline{\lambda}\right)(y,z) &= [y,z](a) - [y,z](0).\end{aligned}$$

Since $\lambda \in \mathbb{R}$, we have $[y,z](a) = [y,z](0) = W_q(y,\overline{z})(0)$, *i.e.*, the Wronskian is independent of x. □

Corollary 2. *If $y(x)$ and $z(x)$ are both solutions of Equation (1.1), then either $W_q(y,z) = 0$ or $W_q(y,z) \neq 0$ for all $x \in [0,a]$.*

Theorem 3. *Any two solutions of Equation (1.1) are linearly dependent if and only if their Wronskian is zero.*

Proof. Let $y(x)$ and $z(x)$ be two linearly dependent solutions of Equation (1.1). Then, there exist a constant $c > 0$ such that $y(x) = c\ z(x)$. Hence

$$W_q(y,z) = \begin{vmatrix} y(x) & D_q y(x) \\ z(x) & D_q z(x) \end{vmatrix} = \begin{vmatrix} cz(x) & cD_q z(x) \\ z(x) & D_q z(x) \end{vmatrix} = 0.$$

Conversely, the Wronskian $W_q(y,z) = 0$ and therefore, $y(x) = c\ z(x)$, i.e., $y(x)$ and $z(x)$ are linearly dependent. □

Let ϕ_λ and χ_λ denote the solutions of Eq. (1.1) satisfying the conditions

$$\begin{aligned}\phi_\lambda(0) &= \sin\alpha,\ D_{q^{-1}}\phi_\lambda(0) = -\cos\alpha,\\ \chi_\lambda(a) &= \beta_2'\lambda+\beta_2,\ D_{q^{-1}}\chi_\lambda(a) = \beta_1'\lambda+\beta_1.\end{aligned}$$

Then by Eq. (3.2), we have

$$\begin{aligned}W(\lambda) \quad : &= W_q(\phi_\lambda,\chi_\lambda)(x) = W_q(\phi_\lambda,\chi_\lambda)\left(\frac{a}{q}\right) = [\phi_\lambda,\chi_\lambda](a)\\ &= \phi_\lambda(a)D_{q^{-1}}\chi_\lambda(a) - \chi_\lambda(a)D_{q^{-1}}\phi_\lambda(a)\\ &= \left(\beta_2'\lambda+\beta_2\right)\phi_\lambda(a) - \left(\beta_1'\lambda+\beta_1\right)D_{q^{-1}}\phi_\lambda(a)\\ &= \lambda\left(\beta_2'\phi_\lambda(a) - \beta_1'D_{q^{-1}}\phi_\lambda(a)\right) + \beta_2\phi_\lambda(a) - \beta_1 D_{q^{-1}}\phi_\lambda(a)\\ &= \lambda R_a'(\phi_\lambda) + R_a(\phi_\lambda) \qquad (3.4)\end{aligned}$$

$$\begin{aligned}R_a'(\chi_\lambda) &= \beta_2'D_{q^{-1}}\chi_\lambda(a) - \beta_1'\chi_\lambda(a)\\ &= \beta_2'\left(\beta_1'\lambda+\beta_1\right) - \beta_1'\left(\beta_2'\lambda+\beta_2\right)\\ &= \beta_2'\beta_1 - \beta_1'\beta_2\\ &= \rho \qquad (3.5)\end{aligned}$$

where ρ is given by (1.4).

THEOREM 4. *The eigenvalues of the problem (1.1)-(1.3) are the zeros of the function* $W(\lambda)$.

Proof. Let $W(\lambda_0) = 0$. Since $W_q(\phi_{\lambda_0},\chi_{\lambda_0}) = W(\lambda_0) = 0$, the functions $\phi_{\lambda_0},\chi_{\lambda_0}$ are linearly dependent, i.e., $\chi_{\lambda_0} = k\phi_{\lambda_0}$, for some $k \neq 0$. Since ϕ_{λ_0} satisfies the boundary condition (1.3), χ_{λ_0} satisfies too. Then, χ_{λ_0} is an eigenfunction of the problem corresponding to the eigenvalue λ_0. Therefore each zero of $W(\lambda)$ is eigenvalue.

Let $u_0(x)$ be any eigenfunction corresponding to eigenvalue λ_0 and $W(\lambda_0) \neq 0$. Then the functions $\phi_{\lambda_0},\chi_{\lambda_0}$ would be linearly independent on $[0,a]$. Therefore $u_0(x)$ may be represented in the form

$$u_0(x) = c_1\phi_{\lambda_0}(x) + c_2\chi_{\lambda_0}(x) \text{ for } x\in[0,a],$$

where at least one of the coefficients c_1 and c_2 is not zero. From the boundary condition (1.2) and (1.3), we obtain

$$c_1(\cos\alpha\ \phi_{\lambda_0}(0)+\sin\alpha\ D_{q^{-1}}\phi_{\lambda_0}(0)) + c_2(\cos\alpha\ \chi_{\lambda_0}(0)+\sin\alpha\ D_{q^{-1}}\chi_{\lambda_0}(0)) = 0$$

and

$$\begin{aligned}&c_1\left[\lambda_0\left(\beta_1'\phi_{\lambda_0}(a) - \beta_2'D_{q^{-1}}\phi_{\lambda_0}(a)\right) + \left(\beta_1\phi_{\lambda_0}(a) + \beta_2 D_{q^{-1}}\phi_{\lambda_0}(a)\right)\right]\\ &+c_2\left[\lambda_0\left(\beta_1'\chi_{\lambda_0}(a) - \beta_2'D_{q^{-1}}\chi_{\lambda_0}(a)\right) + \left(\beta_1\chi_{\lambda_0}(a) + \beta_2 D_{q^{-1}}\chi_{\lambda_0}(a)\right)\right] = 0.\end{aligned}$$

Using the definitions of ϕ_λ and χ_λ, we have

$$\begin{aligned}\cos\alpha\chi_{\lambda_0}(0)+\sin\alpha\, D_{q^{-1}}\chi_{\lambda_0}(0) &= -D_{q^{-1}}\phi_{\lambda_0}(0)\,\chi_{\lambda_0}(0)+\phi_{\lambda_0}(0)\,D_{q^{-1}}\chi_{\lambda_0}(0)\\ &= [\chi_{\lambda_0},\phi_{\lambda_0}](0)=W_q(\chi_{\lambda_0},\phi_{\lambda_0})(0)\end{aligned}$$

and similarly

$$\lambda_0\left(\beta_1'\phi_{\lambda_0}(a)-\beta_2' D_{q^{-1}}\phi_{\lambda_0}(a)\right)+\beta_1\phi_{\lambda_0}(a)+\beta_2 D_{q^{-1}}\phi_{\lambda_0}(a)=W(\lambda_0).$$

Hence the determinant of this system is

$$\begin{vmatrix} 0 & W_q(\chi_{\lambda_0},\phi_{\lambda_0})(0) \\ W(\lambda_0) & W(\lambda_0)\end{vmatrix} = -W_q(\chi_{\lambda_0},\phi_{\lambda_0})(0)\,W(\lambda_0)\neq 0.$$

Since $W(\lambda_0)\neq 0$ and $W_q(\chi_{\lambda_0},\phi_{\lambda_0})(0)\neq 0$, we have $c_1=c_2=0$, i.e., this system has only the trivial solution. Thus, we have contradiction. □

Lemma 1. *All eigenvalues λ_n are simple zeros of $W(\lambda)$.*

Proof. From q-Lagrange's identity (2.1), we have

$$(\lambda_n-\lambda)\int_0^a \phi_\lambda(x)\,\phi_{\lambda_n}(x)\,d_qx=[\phi_{\lambda_n},\phi_\lambda](a). \tag{3.6}$$

Recall that, for each zero of $W(\lambda)$,

$$\chi_{\lambda_n}(x)=k_n\phi_{\lambda_n}(x),\ x\in[0,a], \tag{3.7}$$

where $k_n\neq 0$ are real constants. Using this equality, we have

$$\begin{aligned}[\phi_\lambda,\phi_{\lambda_n}](a) &= -\frac{1}{k_n}[\chi_{\lambda_n},\phi_\lambda](a)\\ &= -\frac{1}{k_n}\left[\lambda_n R_a'(\phi_\lambda)+R_a(\phi_\lambda)\right]\\ &= -\frac{1}{k_n}\left[\lambda_n R_a'(\phi_\lambda)+\lambda R_a'(\phi_\lambda)-\lambda R_a'(\phi_\lambda)+R_a(\phi_\lambda)\right]\\ &= -\frac{1}{k_n}\left[W(\lambda)+(\lambda_n-\lambda)R_a'(\phi_\lambda)\right]\\ &= \frac{\lambda-\lambda_n}{k_n}\left[\frac{W(\lambda)}{\lambda-\lambda_n}-R_a'(\phi_\lambda)\right]. \qquad (3.8)\end{aligned}$$

Letting $\lambda\to\lambda_n$

$$\int_0^a(\phi_{\lambda_n}(x))^2\,d_qx=\frac{1}{k_n}\left[W'(\lambda_n)-R_a'(\phi_{\lambda_n})\right]. \tag{3.9}$$

Putting

$$R_a'(\phi_{\lambda_n})=\frac{\rho}{k_n} \tag{3.10}$$

in (3.9) , we get

$$W'(\lambda_n) = k_n \int_0^a (\phi_{\lambda_n}(x))^2 \, d_q x + \frac{\rho}{k_n} \neq 0.$$

□

Now, if we solve the operator equation

$$(\lambda I - A)\, y = F, \ \ F \in \mathcal{H}, \tag{3.11}$$

we get

$$y = R_\lambda(A)\, F = \begin{pmatrix} \langle \widetilde{G_{x,\lambda}}, \overline{F} \rangle \\ R'_a \left[\langle \widetilde{G_{x,\lambda}}, \overline{F} \rangle \right] \end{pmatrix}, \tag{3.12}$$

where

$$\widetilde{G_{x,\lambda}} = \begin{pmatrix} G(x,y,\lambda) \\ R'_a \left[G(x,y,\lambda) \right] \end{pmatrix} \tag{3.13}$$

and

$$G(x,y,\lambda) = \begin{cases} \frac{1}{W(\lambda)} \chi_\lambda(x)\, \phi_\lambda(y), & 0 \le y \le x \le a \\ \frac{1}{W(\lambda)} \phi_\lambda(x)\, \chi_\lambda(y), & 0 \le x \le y \le a \end{cases}. \tag{3.14}$$

By (3.12)-(3.14), we obtain the following facts.

i) $G(x,y,\lambda)$ satisfies the boundary conditions (1.2)-(1.3) for fixed $x \in [0,a]$.

ii) If λ is not a zero of $W(\lambda)$, then

$$R_\lambda(A)\, F \in D(A). \tag{3.15}$$

iii)

$$R_\lambda(A)(\lambda I - A)\, F = F \text{ for } F \in D(A). \tag{3.16}$$

iv)

$$\| R_\lambda(A)\, F \| \le \frac{1}{|v|} \| F \|, F \in \mathcal{H}, v = \operatorname{Im} \lambda \neq 0. \tag{3.17}$$

v) $R_\lambda(A)\, F$ is a meromorphic function in λ.

From (3.11), (3.12) and (3.15) with $\lambda = \pm i$, it may be concluded that A is a self-adjoint operator.

Using (3.7), (3.10) and (3.12) we have

$$\underset{\lambda=\lambda_n}{res}\, R_\lambda(A)\, F = c_n \Phi_n,$$

where

$$c_n := \langle F, \Phi_n \rangle, \ \ \Phi_n = \frac{1}{\left\| \begin{pmatrix} \phi_{\lambda_n}(x) \\ R'_a(\phi_{\lambda_n}) \end{pmatrix} \right\|} \begin{pmatrix} \phi_{\lambda_n}(x) \\ R'_a(\phi_{\lambda_n}) \end{pmatrix}.$$

Similarly,

$$\underset{\lambda=\lambda_n}{res} \langle R_\lambda (A) F, F \rangle = |c_n|^2 .$$

Now, we will prove the following eigenfunction expansion theorem.

THEOREM 5. i)

$$\int_0^a |F(x)|^2 d_q x = \sum_{n=0}^{\infty} |c_n|^2 , \ F \in \mathcal{H}.$$

ii) If $F \in D(A)$, then

$$F(x) = \sum_{n=0}^{\infty} c_n \Phi_n (x)$$

absolutely and uniformly convergent for $x \in [0, a]$.

Proof. i) If we apply the argument of [[18], Theorem 5.12] and using (3.16), we get the desired result.

ii) Using (i) and applying the argument of [[18], Theorem 5.15], we obtain the proof. □

4. ASYMPTOTIC FORMULAE FOR EIGENVALUES AND EIGENFUNCTIONS

In this section, we shall obtain asymptotic formulae for eigenvalues and eigenfunctions of the problem (1.1)-(1.3).

THEOREM 6. *Let $\lambda = s^2$. Then the characteristic function $W(\lambda)$ has the following asymptotic representations:*

Case I. $\beta_2' \neq 0, \ \alpha \neq 0$

$$\begin{aligned} W(\lambda) =& \beta_2' s^3 \sqrt{q} \sin\left(sq^{-1/2}a; q\right) \sin\alpha \\ &+ O\left(|s|^2 \exp\left(-\frac{\left(\log|s|\, aq^{-1/2}(1-q)\right)^2}{\log q}\right)\right), \end{aligned} \tag{4.1}$$

Case II. $\beta_2' \neq 0, \ \alpha = 0$

$$W(\lambda) = \beta_2' s^2 \cos\left(sq^{-1/2}a; q\right) + O\left(|s| \exp\left(-\frac{\left(\log|s|\, aq^{-1/2}(1-q)\right)^2}{\log q}\right)\right), \tag{4.2}$$

Case III. $\beta_2' = 0, \ \alpha \neq 0$

$$W(\lambda) = \beta_1' s^2 \cos(sa; q) \sin\alpha + O\left(|s| \exp\left(-\frac{\left(\log|s|\, a(1-q)\right)^2}{\log q}\right)\right), \tag{4.3}$$

Case VI. $\beta_2' = 0,\ \alpha = 0$

$$W(\lambda) = -\beta_1' s \sin(sa;q) + O\left(\exp\left(-\frac{(\log|s|\,a\,(1-q))^2}{\log q}\right)\right), \tag{4.4}$$

Proof. Let $\phi_\lambda(x)$ be the solution of Eq. (1.1). Then the following integral equation hold ([14]):

$$\phi_\lambda(x) = \sin\alpha\cos(sx;q) - \frac{\cos\alpha}{s}\sin(sx;q)$$
$$+\frac{q}{s}\int_0^x [\cos(sx;q)\sin(sqt;q) - \sin(sx;q)\cos(sqt;q)]\, v(qt)\,\phi(qt,\lambda)\, d_q t.$$

If $\sin\alpha \neq 0$, then we have

$$\phi_\lambda(x) = \sin\alpha\cos(sx;q) + O\left(|s|^{-1}\exp\left(-\frac{(\log|s|\,x\,(1-q))^2}{\log q}\right)\right). \tag{4.5}$$

If $\sin\alpha = 0$, then we have

$$\phi_\lambda(x) = -\frac{\cos\alpha}{s}\sin(sx;q) + O\left(|s|^{-2}\exp\left(-\frac{(\log|s|\,x\,(1-q))^2}{\log q}\right)\right). \tag{4.6}$$

Putting this equalities in the representation

$$W(\lambda) = \lambda\left[R_a'(\phi_\lambda) + R_a(\phi_\lambda)\right],$$

we obtain the desired results. The other cases may be considered analogically. □

Corollary 3. *The eigenvalues of the problem (1.1)–(1.3) are bounded below.*

Proof. Putting $s = it\ (t > 0)$ in the above formulae, we have that $W(-t^2) \to \infty$ as $t \to \infty$. Hence $W(\lambda) \neq 0$ for λ negative and sufficiently large. □

Theorem 7. *The eigenvalues $\lambda_n\,(n = 0,1,2,...)$ of the problem (1.1)-(1.3) have the following asymptotic representations for $n \to \infty$:*

Case I. $\beta_2' \neq 0,\ \alpha \neq 0$

$$s_n = \frac{q^{-n+1/2}}{a(1-q)}\left(1 + O\left(q^{n/2}\right)\right), \tag{4.7}$$

Case II. $\beta_2' \neq 0,\ \alpha = 0$

$$s_n = \frac{q^{-n+1}}{a(1-q)}\left(1 + O\left(q^{n/2}\right)\right), \tag{4.8}$$

Case III. $\beta_2' = 0,\ \alpha \neq 0$

$$s_n = \frac{q^{-n+1/2}}{a(1-q)}\left(1 + O\left(q^{n/2}\right)\right), \tag{4.9}$$

Case VI. $\beta_2' = 0,\ \alpha = 0$

$$s_n = \frac{q^{-n}}{a(1-q)}\left(1 + O\left(q^{n/2}\right)\right). \tag{4.10}$$

Proof. Let us consider the first case. Set $W(s) = W_1(s) + W_2(s)$ where

$$\begin{aligned} W_1(s) &= \beta_2' s^3 \sqrt{q} \sin\left(sq^{-1/2}a; q\right) \sin\alpha \\ W_2(s) &= O\left(|s| \exp\left(-\frac{\left(\log|s|\, aq^{-1/2}(1-q)\right)^2}{\log q}\right)\right). \end{aligned}$$

Let $\beta_n = \frac{\log \frac{y_n}{y_{n+1}}}{\log q}, n \in \mathbb{Z}^+$. Then $\beta_n \to 1$ as $n \to \infty$ and $\beta = \inf_{n \in \mathbb{Z}^+} \beta_n > 0$. Let $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ be sequences defined by

$$c_n = \begin{cases} \frac{\beta_n + \beta}{2}, & if\ \beta_n \neq \beta \\ \frac{\beta}{2}, & if\ \beta_n = \beta \end{cases},$$

and

$$\begin{aligned} d_1 &= \frac{\beta}{2}, \\ d_{n+1} &= \begin{cases} \frac{\beta_n - \beta}{2}, & if\ \beta_n \neq \beta \\ \frac{\beta}{2}, & if\ \beta_n = \beta \end{cases}, \end{aligned}$$

where $n \geq 1$. The set of annuli $\{A_n^s\}_{n=1}^{\infty}$ is defined to be

$$A_n^s = \left\{z \in \mathbb{C} : y_n q^{d_n} \leq |z| \leq y_n q^{-c_n}\right\}, n \geq 1$$

dividing the region $\left\{z \in \mathbb{C} : |z| \geq q^{\frac{\beta}{2}} y_1\right\}$.

We recall the Rouché theorem which asserts that if $f(z)$ and $g(z)$ are analytic inside and on a closed contour C, and $|f(z)| < |g(z)|$ on C, then $f(z)$ and $f(z) + g(z)$ have the same number zeros inside C, provided that each zero is counted according to their multiplicity [5].

Now, we can apply the Rouché theorem on A_n^s. It is readily shown that $|W_2(s)| < |W_1(s)|$ on A_n^s in similar way as in [16].

Hence if $\lambda_0 < \lambda_1 < \lambda_2 < \ldots$ are the zeros of $W(\lambda)$ and $s_n^2 = \lambda_n$, we have

$$s_n = \frac{q^{-n+1/2}}{a(1-q)} + \delta_n$$

for sufficiently large n. By substituting this in (4.1), we have $\delta_n = O\left(q^{n/2}\right)$, which completes the proof of Case I. The other cases may be considered analogically. $\square$

Recall that $\phi_{\lambda_n}(x)$ is an eigenvalue according to eigenvalue λ_n. Using (4.7), (4.5) and (1.2), we have

$$\phi_n(x) = \left(\int_0^a \cos^2\left(\frac{q^{-n+1/2}}{a(1-q)}x;q\right) d_qx\right)^{-1/2} \cos\left(\frac{q^{-n+1/2}}{a(1-q)}x;q\right)(1+O(q^n))$$

in the first case. Proof is similar to [16].

In Case 2,

$$\phi_n(x) = \left(\int_0^a \sin^2\left(\frac{q^{-n+1}}{a(1-q)}x;q\right) d_qx\right)^{-1/2} \sin\left(\frac{q^{-n+1}}{a(1-q)}x;q\right)(1+O(q^n)).$$

In Case 3,

$$\phi_n(x) = \left(\int_0^a \cos^2\left(\frac{q^{-n+1/2}}{a(1-q)}x;q\right) d_qx\right)^{-1/2} \cos\left(\frac{q^{-n+1/2}}{a(1-q)}x;q\right)(1+O(q^n)).$$

In Case 4,

$$\phi_n(x) = \left(\int_0^a \sin^2\left(\frac{q^{-n}}{a(1-q)}x;q\right) d_qx\right)^{-1/2} \sin\left(\frac{q^{-n}}{a(1-q)}x;q\right)(1+O(q^n)).$$

All this asymptotic approximations are hold uniformly on $\{xq^n : n \in \mathbb{N}\}$ for each $x \in [0,a]$.

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