THE CONNECTIVITY OF THE PRIME INDEX GRAPH OF NON-ABELIAN FINITE SIMPLE GROUPS

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Let G be a group. The prime index graph of G, denoted by $\Pi(G)$, is an undirected graph whose vertices are all subgroups of G and two distinct comparable subgroups H and K are adjacent if and only if $[H : K]$ or $[K : H]$ is prime. In this paper among other results, it is shown that the prime index graph of a finite simple group G is connected if and only if G is isomorphic to $A_5, PSL_2(11), PSL_3(2), PSL_3(3)$ or $PSL_2(2^{2^n})$, where $n \leq 4$.

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1. INTRODUCTION

Throughout this paper all groups are assumed to be finite. The notation and terminology used along the paper are standard in Group Theory, can be found in [9], for instance.

We denote the complete graph of order n by K_n. A star graph S_k is the complete bipartite graph $K_{1,k}$. We use $GF(q)$ to denote a finite field of order q, where for a prime number p and a natural number m, $q = p^m$. Semidirect product of groups G and H are denoted by $G \rtimes H$.

The investigation of graphs associated to algebraic structures and the study of the properties of algebraic structures using graphs are two interesting areas in the algebraic graph theory. In the last two decades, many authors worked in this field. (See, for instance, [1, 2, 4].)

In [4] it is studied a graph called *subgroup graph* of a group, as a graph whose vertices are all subgroups of the group and two subgroups H_1 and H_2 are adjacent if and only if $H_1 \leqslant H_2$ and there is no subgroup K such that $H_1 \lneqq K \lneqq H_2$. They view the subgroup lattice of a group as a graph and investigate the planarity of this graph.

Recently, Akbari *et al.* [3] introduced a new graph called *prime index graph*.

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Let G be a group. The prime index graph of G, denoted by $\Pi(G)$, is an undirected graph whose vertices are all subgroups of G and two distinct comparable subgroups H and K are adjacent, $H \sim K$, if and only if $[H : K]$ or $[K : H]$ is prime.

In fact, the prime index graph is a subgraph of the subgroup graph. So it is interesting to study the properties of this graph.

In [3], the authors proved that for every group G, $\Pi(G)$ is a bipartite graph and the girth of $\Pi(G)$ is either 4 or ∞. Also, they showed that $\Pi(G)$ is a complete graph if and only if G is a cyclic group of prime order or $|G| = rs$, for some primes r and s.

About the connectivity of the prime index graph, it has been proved that the prime index graph of a solvable group is connected and moreover, the prime index graph of the symmetric group of degree n is connected if and only if $n \leq 5$ (see [3]). In this paper, we investigate the connectivity of prime index graphs on the simple groups. As a corollary, we examine the non-solvable groups whose prime index graphs are disconnected.

2. LEMMAS

First we state some lemmas without proof and use them in our proofs.

Lemma 2.1 ([3, Theorem 8]). *Let G be a group and N be a normal subgroup of G. If $\Pi(G)$ is a connected graph, then $\Pi(N)$ and $\Pi(G/N)$ are connected graphs.*

Lemma 2.2 ([3, Theorem 7]). *Let G be a finite solvable group. Then $\Pi(G)$ is connected.*

The following lemmas are due to Dickson [6].

Lemma 2.3. *Let $q = 2^f \geq 4$. Then the maximal subgroups of $PSL_2(q)$ are:*

1) *The semidirect product of $\mathbb{Z}_2^f$ and $\mathbb{Z}_{q-1}$, that is, the stabilizer of a point of the projective line;*

2) $D_{2(q-1)}$;

3) $D_{2(q+1)}$;

4) *$PSL_2(q_0)$, where $q = q_0^r$ for some prime r and $q_0 \neq 2$.*

Lemma 2.4. *Let p be a prime number. Then the maximal subgroups of $PSL_2(p)$ are:*

1) *The Frobenius group of order* $p(p-1)/2$*;*

2) D_{p-1}*;*

3) D_{p+1}*;*

4) A_4*, where* $p \equiv 3, 13, 27, 3 \pmod{40}$*,* $Sym(4)$*, where* $p \equiv \pm 1 \pmod 8$ *or* A_5*, where* $p \equiv \pm 1 \pmod{10}$*.*

LEMMA 2.5 ([7, Theorem 1]). *Let* G *be a non-abelian finite simple group with* $H \leq G$ *and* $[G : H] = r^a$*, where* r *is a prime number. One of the following holds:*

a) $G = A_n$ *and* $H \cong A_{n-1}$ *with* $n = r^a$*;*

b) $G = PSL_n(q)$ *and* H *is the stabilizer of a line or hyperplane. Then* $[G : H] = \dfrac{q^n - 1}{q - 1} = r^a$ *(Note that* n *must be prime);*

c) $G = PSL_2(11)$ *and* $H \cong A_5$*;*

d) $G = M_{23}$ *and* $H \cong M_{22}$ *or* $G = M_{11}$ *and* $H \cong M_{10}$*;*

e) $G = PSU_4(2) \cong PSp_4(3) \cong B_4(3)$ *and* H *is the parabolic subgroup of index* 27*.*

Remark 2.1. Assume that for every maximal subgroup M of a finite group G, $\Pi(M)$ is connected and also there exists a maximal subgroup M' such that $[G : M']$ is prime. For every $H_1, H_2 < G$, there exist the maximal subgroups M_1, M_2 of G such that $H_1 \leq M_1$ and $H_2 \leq M_2$. Since $\Pi(M_1)$ and $\Pi(M_2)$ are connected, there exist a path between H_1 and $\{e\}$, and a path between H_2 and $\{e\}$. So there exists a path between H_1 and H_2. Therefore, $\Pi(G)$ is connected.

LEMMA 2.6 ([13], Zsigmondy's Theorem). *Let* a *and* n *be integers greater than* 1*. Then there exists a prime divisor* r *of* $a^n - 1$ *such that* r *does not divide* $a^j - 1$ *for all* j*,* $0 < j < n$*, except in the following cases:*

a) $n = 2,\ a = 2^s - 1$*, where* $s > 0$*;*

b) $n = 6,\ a = 2$*.*

Such a prime divisor is called a primitive prime divisor of $a^n - 1$*.*

Remark 2.2. Let t be an odd prime number. Since $\gcd(2, t) = 1$ one has $t \mid 2^{t-1} - 1$. If for a given natural number a, t divides $2^{t^a} - 1$, then it is easy to see that t divides $2^{\gcd(t^a, t-1)} - 1$ and hence $t \mid 1$, a contradiction. Therefore $\gcd(t, 2^{t^a} - 1) = 1$.

Lemma 2.7. *Let $n > 2$ be an integer and $q > 3$ be a prime power. If $SL_n(q)$ contains a maximal subgroup of a prime index, then both $\frac{q^n-1}{q-1}$ and n are prime numbers.*

Proof. Assume that $N \leq SL_n(q)$ is a maximal subgroup of a prime index. If $Z = Z(SL_n(q)) \nleq N$, then since N is a maximal subgroup of G, we have $ZN = SL_n(q)$ and hence, $N \trianglelefteq SL_n(q)$. It is well known that for any $N \triangleleft SL_n(q)$, we have $N \leq Z$. So by the above fact, we can conclude that $Z(SL_n(q)) = SL_n(q)$, which is a contradiction. Thus $Z \leq N$ and $N/Z \leq PSL_n(q)$. So $[SL_n(q) : N]$ is prime if and only if $[PSL_n(q) : N/Z]$ is prime. Thus Lemma 2.5 shows that if $[SL_n(q) : N]$ is prime, then both $\frac{q^n-1}{q-1}$ and n are prime numbers, as desired. $\square$

In the following, let

$$T_1 = \left\{ \begin{pmatrix} 1 & t \\ 0 & I \end{pmatrix} \ : \ t \in (GF(q))^{n-1} \right\},$$

$$T_2 = \left\{ \begin{pmatrix} \det(A)^{-1} & 0 \\ 0 & A \end{pmatrix} \ : \ A \in GL_{n-1}(q) \right\}$$

and $T_3 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \ : \ B \in SL_{n-1}(q) \right\}$. It is obvious that $T_1 \cong (GF(q))^{n-1}$, $T_2 \cong GL_{n-1}(q)$ and $T_3 \cong SL_{n-1}(q)$.

Remark 2.3. Assume that $\gcd(n, q-1) = 1$. Under our assumptions, it is known that $PSL_n(q) = SL_n(q)$ acts on $P(V) = \{[v] : v \in (GF(q))^n\}$, where for every $v \in (GF(q))^n$, $[v] = \{av : a \in GF(q) - \{0\}\}$. Since $M = T_1 \rtimes T_2 = (SL_n(q))_{[v_1]}$ and $SL_n(q)$ acts primitively on $P(V)$, we conclude that $SL_n(q)$ contains a maximal subgroup which is isomorphic to $M \cong (GF(q))^{n-1} \rtimes GL_{n-1}(q)$.

Lemma 2.8. *Let $n > 2$ be a prime number and N be a maximal subgroup of $M = T_1 \rtimes T_2$ of the prime index.*

1) *Let $n > 3$ and $q = p^m$, where p is a prime number and $m \in \mathbb{N}$. Then $T_1, T_3 \leq N$.*

2) *Let $m = 3^\alpha, n = 3$ and $q = 2^m$. Then $T_3 \leq N$ and $T_1 \leq N$.*

Proof. (1) Suppose, contrary to our claim, that $T_1 \nleq N$. Then $T_1 N = M$ and hence, $\frac{N}{T_1 \cap N} \cong \frac{M}{T_1} \cong GL_{n-1}(q)$. Thus N contains an element of order $q^{n-1} - 1$, namely x. We can see at once that $\langle x \rangle$ acts fixed point freely on $T_1 \cap N$ by conjugation. Consequently, we have $O(x)$ divides $|T_1 \cap N| - 1 = p^a - 1$

and hence, $m(n-1) \mid a$. On the other hand, $[M : N] = [T_1N : N] = [T_1 : T_1 \cap N] \neq 1$. Hence $|T_1 \cap N| = p^a < p^{m(n-1)}$, which is a contradiction. The result is $T_1 \leq N$.

Now, to obtain a contradiction, suppose that $T_3 \not\leq N$. Then since $T_1 \rtimes T_3 \trianglelefteq M$, we have $N(T_1 \rtimes T_3) = M$. Therefore, $[M : N] = [N(T_1 \rtimes T_3) : N] = [T_1 \rtimes T_3 : (T_1 \rtimes T_3) \cap N]$ and hence, by Dedekind's modular law (see [11, p. 15]), we get $[M : N] = [T_1 \rtimes T_3 : T_1 \rtimes (T_3 \cap N)] = \dfrac{|SL_{n-1}(q)|}{|T_3 \cap N|}$. Thus since $n-1$ is not prime, Lemma 2.7 shows that $[M : N]$ is not prime, which is a contradiction. So $T_3 \leq N$.

(2) The same reasoning as (1) shows that $T_1 \leq N$. Now suppose, contrary to our claim, that $T_3 \not\leq N$. Since $T_1 \rtimes T_3 \trianglelefteq M$, we have $N(T_1 \rtimes T_3) = M$. Thus $[M : N] = [T_1 \rtimes T_3 : T_1 \rtimes (T_3 \cap N)] = \dfrac{|SL_{n-1}(q)|}{|T_3 \cap N|}$ is prime. From this, we get that $\dfrac{(2^{3^\alpha})^2 - 1}{2^{3^\alpha} - 1}$ is prime by Lemma 2.7. But $2 + 1 \mid 2^{3^\alpha} + 1$. So $2^{3^\alpha} + 1 = 3$ and hence $2^{3^\alpha} = 2$, a contradiction. $\square$

Lemma 2.9. *Let $n > 3$ be a prime number or $n = 3$ and $q = 2^{3^\alpha}$. Then $T_1 \rtimes T_3$ doesn't contain any maximal subgroup of a prime index.*

Proof. On the contrary, suppose that there exists the maximal subgroup N of $T_1 \rtimes T_3$ of a prime index. If $T_1 \leq N$, then $[\dfrac{T_1 \rtimes T_3}{T_1} : \dfrac{N}{T_1}] = [T_1 \rtimes T_3 : N]$ is prime and $\dfrac{N}{T_1} \leq \dfrac{T_1 \rtimes T_3}{T_1} \cong SL_{n-1}(q)$. But by Lemma 2.7, $SL_{n-1}(q)$ doesn't contain any maximal subgroup of a prime index, a contradiction. So $T_1 \not\leq N$. As in the proof of Lemma 2.8, we get a contradiction. These contradictions complete the proof. $\square$

3. MAIN RESULTS

Theorem 3.1. *Let G be a non-abelian finite simple group. The prime index graph of G is connected if and only if G is isomorphic to*

$$A_5, PSL_2(11), PSL_3(2), PSL_3(3)$$

or $PSL_2(2^{2^n})$, where $n \leq 4$.

Proof. If $\Pi(G)$ is connected, then G contains a subgroup M of a prime index. So M is a maximal subgroup of G and hence, it is sufficient to analyze the groups listed in Lemma 2.5. We consider these possibilities in the following cases:

(I) By [3, p. 7], $\Pi(A_n)$ is connected if and only if $n \leq 5$.

(II) Let $G = PSL_n(q)$, where both $\frac{q^n-1}{q-1}$ and n are primes. The proof has been divided into several cases:

Case 1. Assume that nq is odd and $n = 2l+1 > 3$ is prime, where $l \in \mathbb{N}$. By [9, p. 70], $SO_n(q)$ is a maximal subgroup of $PSL_n(q)$. By comparing the orders of $SO_n(q)$ and $PSL_n(q)$, we have $[PSL_n(q) : SO_n(q)]$ is not prime. Thus they are not adjacent in $\Pi(PSL_n(q))$. It is well known that $(SO_n(q))' = B_l(q)$ is a simple group and $[SO_n(q) : B_l(q)] = 2$. Now, let $N \leq SO_n(q)$ be a maximal subgroup of $SO_n(q)$ such that $N \neq B_l(q)$. Then it is easy to see that $NB_l(q) = SO_n(q)$ and hence, $[SO_n(q) : N] = [B_l(q) : N \cap B_l(q)]$ is not prime, by Lemma 2.5. Thus $SO_n(q) \sim B_l(q)$ is a connected component of $\Pi(SO_n(q))$. Therefore, considering the orders of maximal subgroups of $PSL_n(q)$ (see [9]) shows that $SO_n(q) \sim B_l(q)$ forms a connected component of $\Pi(PSL_n(q))$. From this, $\Pi(PSL_n(q))$ is disconnected.

If $n = 3$ and $q \neq 3$, then $SO_3(q) \cong PGL_2(q)$ is a maximal subgroup of $PSL_3(q)$ by [5, p. 378]. Since $\dfrac{11^3-1}{10} = 133$ is not prime, $\Pi(PSL_3(11))$ is disconnected. So, let $q \neq 11$. It is easy to see that $PGL_2(q)$ has a normal subgroup H such that $H \cong PSL_2(q)$ and H is adjacent to $PGL_2(q)$. By Lemma 2.5, there is no subgroup of $PSL_2(q)$ that is adjacent to $PSL_2(q)$. Also, according to Mitchell's results [8] and by comparing the orders of subgroups of $PSL_3(q)$ and the order of $PSL_2(q)$, we can get that $PSL_3(q)$ does not contain any subgroup except $PGL_2(q)$ which is adjacent to $PSL_2(q)$ and hence $\Pi(PSL_3(q))$, where $q \neq 3$, is disconnected.

If $n = 3$ and $q = 3$, then by [12], the maximal subgroups of $PSL_3(3)$ are isomorphic to $Sym(4)$ or solvable groups of orders 39 and 432. Therefore, Remark 2.1 shows that $\Pi(PSL_3(3))$ is connected.

Case 2. If $n = 2$, then $[PSL_2(q) : M] = \dfrac{q^2-1}{q-1} = q+1$ is a prime number, namely p. This forces $q = 2^s$, where $s = 2^m$, $m \in \mathbb{N}$. So, p is a Fermat prime. Suppose that $\Pi(PSL_2(q))$ is connected. According to Lemma 2.3 and [11], all maximal subgroups of $PSL_2(2^{2^m})$ except $PSL_2(2^{2^{m-1}})$ (if $m \geq 2$) are solvable and hence their prime index graphs are connected by Lemma 2.2. Also, $PSL_2(2^{2^m})$ and $PSL_2(2^{2^{m-1}})$ are not adjacent in $\Pi(PSL_2(2^{2^m}))$. Let $m > 4$. If $2^{2^{m-1}}+1$ is not prime, then Lemma 2.5 allows us to deduce that $PSL_2(2^{2^{m-1}})$ is an isolated vertex and hence, $\Pi(PSL_2(2^{2^m}))$ is disconnected. Now assume that $2^{2^{m-1}}+1$ is prime. Then again $PSL_2(2^{2^{m-2}})$ is a maximal subgroup of $PSL_2(2^{2^{m-1}})$. Since the other maximal subgroups of $PSL_2(2^{2^m})$ are solvable and $PSL_2(2^{2^{m-2}})$ is non-solvable, we deduce that if $PSL_2(2^{2^m})$ contains a maximal subgroup L such that $PSL_2(2^{2^{m-2}}) \lesssim L$. Then $L \lesssim PSL_2(2^{2^{m-1}})$.

Now, applying the above argument guarantees that either $PSL_2(2^{2^{m-2}})$ is an isolated vertex in $\Pi(PSL_2(2^{2^m}))$ or $2^{2^{m-2}}+1$ is prime. We continue the above process about the maximal subgroups $PSL_2(2^{2^{i-1}})$ of $PSL_2(2^{2^i})$, where $i \leq m$. Since $2^{2^5}+1$ is not a prime number, we obtain that $PSL_2(2^{2^5})$ is an isolated vertex. If $m > 4$, then $\Pi(PSL_2(2^{2^m}))$ is disconnected. Also, if $m \leq 4$, then repeating the above argument shows that $\Pi(PSL_2(2^{2^m}))$ is connected.

Case 3. Assume that $n = u$ is an odd prime, $q = 2^m > 2$ and $\dfrac{2^{mu}-1}{2^m-1}$ is prime. Let $m = u^\alpha t$, where $\alpha \geq 0$ and $\gcd(u,t) = 1$. By Lemma 2.6, there exists a primitive prime divisor r of $2^{u^{\alpha+1}}-1$. Since $2^{u^{\alpha+1}}-1$ divides $2^{mu}-1$, we have $r \mid 2^{mu}-1$. Also, since $\gcd(m, u^{\alpha+1}) = u^\alpha$, we conclude that $r \nmid 2^m - 1$. So $r \mid \dfrac{2^{mu}-1}{2^m-1} = s$, where s is a primitive prime divisor of $2^{mu}-1$. This implies that $r = s$. It follows that $mu = u^{\alpha+1}$ and hence, $m = u^\alpha$. Therefore, $PSL_u(2^{u^\alpha}) = SL_u(2^{u^\alpha})$ by Remark 2.2. By Remark 2.3, $M = T_1 \rtimes T_2$ is a maximal subgroup of $SL_u(2^{u^\alpha})$. Note that $[SL_u(2^{u^\alpha}) : M] = p$, $T_3 \trianglelefteq T_2$ and

$$T_2/T_3 \cong K, \text{ where } K = \left\{ \begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & I \end{pmatrix} : a \in (GF(2^m)) - \{0\} \right\} \cong \mathbb{Z}_{2^m-1}.$$

Since $K \cong \mathbb{Z}_{2^m-1}$ is cyclic, there is a normal series $K_0 = \{e\} \trianglelefteq K_1 \trianglelefteq \cdots \trianglelefteq K_{t-1} \trianglelefteq K_t = K$ of the subgroups of K such that $\dfrac{|K_{i+1}|}{|K_i|}$ is prime. Note that $T_3 \rtimes K \cong T_2$. So there is a series $T_3 = T_3 \rtimes K_0 \trianglelefteq T_3 \rtimes K_1 \trianglelefteq \cdots \trianglelefteq T_3 \rtimes K_{t-1} \trianglelefteq T_3 \rtimes K_t = T_3 \rtimes K = T_2$ between T_3 and T_2 such that $[T_3 \rtimes K_{i+1} : T_3 \rtimes K_i] = \frac{|T_3||K_{i+1}|}{|T_3||K_i|} = \frac{|K_{i+1}|}{|K_i|}$ is prime. Therefore,

$$(*) \quad H_0 = T_1 \rtimes T_3 \sim H_1 = T_1 \rtimes (T_3 \rtimes K_1) \sim \cdots H_{t-1} = T_1 \rtimes (T_3 \rtimes K_{t-1}) \sim H_t = T_1 \rtimes T_2$$

is a path in $\Pi(SL_u(2^m))$. Let

$$B = \{L \mid L \leq T_1 \rtimes T_2 \text{ and } T_1 \rtimes T_3 \trianglelefteq L\} \cup \{SL_u(2^m)\},$$

and let $L_0 \in B$. Then $H_i \in B$. Since $\dfrac{T_1 \rtimes T_2}{T_1 \rtimes T_3}$ is isomorphic to a cyclic group, the argument given for $(*)$ shows that B forms a connected subgraph of $\Pi(SL_u(2^m))$, say Γ. Now, we choose a path in Γ. Without loss of generality, we choose $(*)$. Suppose that there exists a subgroup of $SL_u(2^m)$, say H, such that $H \sim H_i$, for some i, $0 \leq i \leq t$.

a) If $H_i \leq H$ and $[H : H_i]$ is prime, then there exists a maximal subgroup of $SL_u(2^m)$, say W, such that $H \leq W$. Since considering the orders of maximal subgroups of $SL_2(2^m)$ shows that every maximal subgroup of $SL_2(2^m)$ containing $T_1 \rtimes T_3$ is isomorphic to M, we deduce that M and W are isomorphic. Let $\phi : M \longrightarrow W$ be a group isomorphism. Let j be a minimal value between 0 and

t such that $T_1 \rtimes T_3 \leq \phi(H_j)$. If $j \neq 0$, then $T_1 \rtimes T_3 \nleq \phi(H_{j-1})$. From the above argument, we know that $\phi(H_{j-1}) \trianglelefteq \phi(H_j)$ and hence $(T_1 \rtimes T_3)\phi(H_{j-1}) = \phi(H_j)$. So $[T_1 \rtimes T_3 : \phi(H_{j-1}) \cap (T_1 \rtimes T_3)] = [\phi(H_j) : \phi(H_{j-1})]$ is prime, contrary to Lemma 2.9. So, we thus have $j = 0$ that is $T_1 \rtimes T_3 \leq \phi(H_0)$. Moreover, $T_1 \rtimes T_3 \leq \phi(H_0) = \phi(T_1 \rtimes T_3) \trianglelefteq \phi(M) = W$ and hence $T_1 \rtimes T_3 = \phi(T_1 \rtimes T_3)$. Thus $T_1 \rtimes T_3 \trianglelefteq H \leq W \leq N_{SL_u(2^m)}(T_1 \rtimes T_3) = M$. So $H \leq M$. This gives $H \in B$.

b) Let $H \leq H_i$ and $[H_i : H]$ be prime. Clearly, $T_1 \trianglelefteq T_1 \rtimes T_2$, $T_1 \trianglelefteq H_i$. If $T_1 \nleq H$, then $T_1 H = H_i$ and this yields that $[T_1 : T_1 \cap H] = [H_i : H] = 2$. Thus $H \trianglelefteq H_i$. If $T_3 \nleq H$, then $T_3 H = H_i$. Therefore $[T_3 : T_3 \cap H] = [H_i : H]$ is prime, which is a contradiction with Lemma 2.7. So $T_3 \leq H$ and hence applying the argument as the proof of Lemma 2.8 leads us to a contradiction. Thus $T_1 \leq H$. If $T_3 \leq H$, then $H \in B$. Therefore the theorem follows by applying the same method as the above. If $T_3 \nleq H$, then $H(T_1 \rtimes T_3) = H_i$. Consequently, $[H_i : H] = [(T_1 \rtimes T_3) : (T_1 \rtimes T_3) \cap H] = [T_3 : T_3 \cap H]$. It follows that $[T_3 : T_3 \cap H]$ is prime, which is a contradiction with Lemma 2.7. Thus $H \in B$.

These show that Γ and their conjugates form a connected component and hence $\Pi(SL_u(2^m))$ is disconnected.

Case 4. Assume that $q = 2$, $n \neq 2, 3$. Then the proof runs as Case 3. Note that $PSL_2(2) \cong Sym(3)$ is not simple. Also according to [12], we can see that the maximal subgroups of $PSL_3(2)$ are isomorphic to $Sym(4)$ or a solvable group of order 21 and $[PSL_3(2) : Sym(4)] = 7$. Hence Lemma 2.2 and Remark 2.1 show that $\Pi(PSL_3(2))$ is connected.

(III) Let $G = PSL_2(11)$. Then $M \cong A_5$. Since $[PSL_2(11) : A_5]$ is prime, $PSL_2(11)$ and A_5 are adjacent. By Lemmas 2.4, all maximal subgroups of $PSL_2(11)$ except A_5 are solvable. So Lemma 2.2 forces the prime index graphs of all maximal subgroups of $PSL_2(11)$ to be connected. Therefore, Remark 2.1 shows that $\Pi(PSL_2(11))$ is connected.

(IV) Let $G = M_{23}$. Then the maximal subgroup M of M_{23} is adjacent to M_{23} if and only if $M \cong M_{22}$. But M_{22} is a simple group, so we can conclude by Lemma 2.5 that M_{22} contains no maximal subgroup of a prime index. Therefore, all maximal subgroups of M_{23} which are isomorphic to M_{22} and M_{23} form a connected component of $\Pi(M_{23})$, that is a star graph S_k for some natural number k. This forces $\Pi(M_{23})$ to be disconnected.

(V) Let $G = M_{11}$. Then a maximal subgroup M of M_{11} is adjacent to M_{11} if and only if $M \cong M_{10}$. We know that the derived subgroup of M_{10} is the simple group A_6. Also, $[M_{10} : A_6] = 2$. For every maximal subgroup L of M_{10}, $|L| \in \{16, 20, 72, 360\}$ and for every maximal subgroup N of M_{11}, either

$N \cong M_{10}$ or $|N| \in \{48, 120, 144, 660\}$, see [12]. Thus Lemma 2.5 guarantees that there is no path between M_{11} and $\{e\}$ in $\Pi(M_{11})$. Therefore, $\Pi(M_{11})$ is disconnected. $\square$

COROLLARY 3.1. *If G is a non-solvable finite group with at least a non-abelian composition factor K/H such that*

$$K/H \ncong A_5, PSL_2(11), PSL_3(2), PSL_3(3)$$

and $PSL_2(2^{2^n})$, where $n \leq 4$, then $\Pi(G)$ is disconnected.

Proof. It is straightforward from Theorem 3.1 and Lemma 2.1. $\square$

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