ON THE ORDER OF MAGNITUDE OF SOME ARITHMETICAL FUNCTIONS UNDER DIGITAL CONSTRAINT II

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Let $q \geqslant 2$ be an integer and $S_q(n)$ denote the sum of the digits in base q of a positive integer n. We look for an estimate of the average of some multiplicative arithmetical functions under a constraint on the sum of digits in view of some estimates along consecutive integers.

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1. **INTRODUCTION**

Throughout this paper, we denote by $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$ the sets of positive integers, non negative integers, integers, real and complex numbers respectively. Given a real number x, $\lfloor x \rfloor$ denotes the greatest integer $\leqslant x$ and $e(x) = e^{2i\pi x}$. The greatest common divisor of two integers a and b will be denoted by (a, b).

Let $n \in \mathbb{N}_0$ and let $q \geqslant 2$ be an integer, then there exists a unique sequence $(a_j(n))_{j\in\mathbb{N}_0} \in \{0, 1, ..., q-1\}^{\mathbb{N}_0}$ such that

$$n = \sum_{k=0}^{\infty} a_k(n) q^k. \tag{1.1}$$

The right hand side of (1.1) is called the q-ary expansion of n. The sum of digits to base q is denoted by

$$S_q(n) = \sum_{k=0}^{\infty} a_k(n),$$

that we shall write $S(n)$ as long as this does not make a confusion.
A function $f : \mathbb{N}_0 \to \mathbb{C}$ is said to be completely q-additive if $f(0) = 0$ and

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$$f(aq^k + b) = f(a) + f(b) \text{ for any integers } a \geqslant 1,\ k \geqslant 1 \text{ and } 0 \leqslant b < q^k.$$

Such functions were introduced by Gelfond [10] and further studied by Delange [8], Bésineau [6], Coquet [7], Kàtai [12] and others. Thanks to (1.1), a function f is completely q-additive if and only if $f(0) = 0$ and

$$f(n) = \sum_{k=0}^{\infty} f(a_k(n)).$$

Hence, a completely q-additive function is completely determined by its values on the set $\{0, 1, ..., q-1\}$. A typical example of a completely q-additive function is the sum of digits function S_q.

Another kind of arithmetic functions is the multiplicative ones, i.e. that satisfy $f(1) = 1$ and whenever a and b are coprime integers, then $f(ab) = f(a)f(b)$ (see [5, Chapter 2] for further information). Indeed, we shall focus on the following functions depending on a positive integer n :

- the number of positive divisors function, $\tau : n \longmapsto \sum_{t|n} 1$.
- The sum of the s^{th} powers of all the positive divisors function (for $s \in \mathbb{R}$), $\sigma_s : n \longmapsto \sum_{t|n} \left(\frac{n}{t}\right)^s$. In particular, $\sigma_0 = \tau$.
- The number of positive integers $\leqslant n$ and relatively prime to n, $\varphi : n \longmapsto \sum_{\substack{k=1 \\ (k,n)=1}}^{n} 1$.
- The Möbius function,

$$\mu : n \longmapsto \begin{cases} 1 & \text{if } n = 1 \\ (-1)^r & \text{if } n = p_1 \dots p_r \text{ is a product of distinct primes} \\ 0 & \text{otherwise.} \end{cases}$$

- The non principal Dirichlet character modulo 4,

$$\chi : n \longmapsto \begin{cases} 0 & \text{if } 2|n \\ (-1)^{\frac{n-1}{2}} & \text{else .} \end{cases}$$

- The number of representations of n as the sum of two integral squares denoted by $r(n)$.

Except for the last one, all these functions are multiplicative and it can be shown (see [11, Chapter 16] for instance) that

$$\varphi(n) = \sum_{t|n} \mu(t) \frac{n}{t},$$

$$r(n) = 4 \sum_{t|n} \chi(t).$$

Indeed, the function $\frac{r(n)}{4}$ is multiplicative so that $r(n)$ is "almost multiplicative".

For every $(r,a,a') \in \mathbb{Z}^3$, q,h,m and $m' \geqslant 2$ such that $(m,q-1)=(m',q-1)=1$, we define the following functions that depend on n,q,r,h,a,m,a' and m' (but we will omit the latter ones for briefness)

$$\hat{\tau}(n) = \sum_{\substack{t\mid n\\ t\equiv r \mod h\\ S(t)\equiv a \mod m\\ S(t+1)\equiv a' \mod m'}} 1,$$

$$\hat{\sigma}_s(n) = \sum_{\substack{t\mid n\\ t\equiv r \mod h\\ S(t)\equiv a \mod m\\ S(t+1)\equiv a' \mod m'}} \left(\frac{n}{t}\right)^s, \text{ for } s\in\mathbb{R},$$

$$\hat{\varphi}(n) = \sum_{\substack{t\mid n\\ t\equiv r \mod h\\ S(t)\equiv a \mod m\\ S(t+1)\equiv a' \mod m'}} \mu(t)\frac{n}{t},$$

$$\hat{r}(n) = 4\sum_{\substack{t\mid n\\ t\equiv r \mod h\\ S(t)\equiv a \mod m\\ S(t+1)\equiv a' \mod m'}} \chi(t).$$

Note that we can always reduce the study to the case $(m,q-1)=1$. In fact, if $d=(m,q-1)$, we set $m_1=\frac{m}{d}$ then we get $S(n)\equiv n \bmod d$. For every residue class h modulo m represented by ℓ modulo d, we have

$$\{n\in\mathbb{N}:\ S(n)\equiv h \bmod m\} = \bigcup_{\substack{0\leqslant \ell<d\\ \ell\equiv h \mod d}} \{\ell+td:\ t\in\mathbb{N},\ S(\ell+td)\equiv h \bmod m_1\}.$$

Similarly, the assumption $(m',q-1)=1$ is also not restrictive.

In order to detect the congruences, we shall use the classic orthogonality relation

$$(1.2)\ \frac{1}{m}\sum_{j=0}^{m-1} e\left(\frac{j(a-b)}{m}\right) = \begin{cases} 1 & \text{if } a\equiv b \bmod m\\ 0 & \text{else.}\end{cases} \qquad (m\in\mathbb{N},\ a,b\in\mathbb{Z})$$

For the sake of simplicity, we ought to set

$$\mathfrak{F}^* = \{0,\dots,m-1\}\times\{0,\dots,m'-1\}\setminus\{(0,0)\}.$$

$$\mathfrak{G}^* = \{0,\dots,h-1\}\times\{0,\dots,m-1\}\times\{0,\dots,m'-1\}\setminus\{(0,0,0)\}.$$

Basically, we shall need the following result proved by Gelfond in [10].

Theorem A (Gelfond, 1968). *Let* $q \geqslant 2$, $m \geqslant 2$ *be integers such that* $(m, q-1) = 1$ *and* $\delta \in \mathbb{R}$. *Then we have for* $1 \leqslant h \leqslant m-1$

$$\left|\sum_{n=1}^{N} e\left(\delta n + \frac{h}{m} S_q(n)\right)\right| = O_q\left(N^{\lambda}\right), \text{ as } N \longrightarrow +\infty$$

where $\lambda = \dfrac{1}{2\log q} \log \dfrac{q \sin\left(\frac{\pi}{2m}\right)}{\sin\left(\frac{\pi}{2mq}\right)} < 1.$

A similar estimate was made by Aloui-Mauduit-Mkaouar [4] under a constraint over the sum of digits of consecutive integers.

Theorem B (Aloui-Mauduit-Mkaouar, 2017). *Let* $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ *and let* $q \geqslant 2$ *be an integer such that* $(q-1)(\alpha+\beta) \in \mathbb{R} \setminus \mathbb{Z}$. *Then, for any real number* x *sufficiently large we have*

$$B(x, \alpha, \beta, \gamma) = \sum_{n \leqslant x} e\left(\alpha S(n) + \beta S(n+1) + \gamma n\right) \ll x^{\lambda} \log x,$$

where λ *is the constant defined in Theorem* **A** *and the implicit constant depends only on* q.

Recently, the first author [3] extended this result by estimating the exponential sum

$$\sum_{n \leq N} e(\alpha_0 S_q(n) + \ldots + \alpha_k S_q(n+k)),$$

where $k \in \mathbb{N}$ and $(\alpha_0, \ldots, \alpha_k) \in \mathbb{R}^{k+1}$, making a new proof of the Thue-Morse sequence recursion formula.

Mkaouar and Wannès [14] used an improved result of Drmota, Mauduit and Rivat [9], in addition to the classic Abel's summation formula, to prove interesting results about the average of some additive functions (namely, the number of distinct prime factors ω and the total number of prime factors Ω of a positive integer n) under digital constraints.

We follow a similar path and we benefit of some results of [1, 2] in order to estimate $\sum_{n \leqslant x} \hat{\tau}(n)$, $\sum_{n \leqslant x} \hat{\sigma}_s(n)$, $\sum_{n \leqslant x} \hat{\varphi}(n)$ and $\sum_{n \leqslant x} \hat{r}(n)$ (representing the averages of the functions $\hat{\tau}$, $\hat{\sigma}_s$, $\hat{\varphi}$ and $\hat{r}$ respectively) in accordance with the study made in [11, Chapter 18].

2. AVERAGE OF $\hat{\tau}$

THEOREM 2.1. *Let q, h, m and m' be integers $\geqslant 2$ such that $(m, m') = (m, q-1) = (m', q-1) = 1$, let $(r, a, a') \in \mathbb{Z}^3$. Then we have*

$$\sum_{n\leqslant x} \hat{\tau}(n) = \frac{1}{hmm'}\ x\log x + \frac{\alpha+\gamma+\gamma(r,h)-1}{hmm'}\ x + O\left(x^{\frac{1+\lambda}{2}}\log x\right),$$

$$\text{as } x \longrightarrow +\infty,$$

where γ stands for Euler-Mascheroni's constant defined by the equation

$$\gamma = \lim_{x\longrightarrow+\infty}\left(\sum_{n\leq x}\frac{1}{n} - \log x\right),$$

$\gamma(r,h)$ is the constant defined in [13], λ is the constant stated in Theorem A and

$$\alpha = \sum_{u=0}^{h-1}\sum_{(\ell,k)\in\mathfrak{F}^*} e\left(-\frac{ur}{h} - \frac{\ell a}{m} - \frac{ka'}{m'}\right)\left(\int_1^{+\infty}\left(\sum_{t\leqslant v} e\left(\frac{u}{h}t + \frac{\ell}{m}S(t) + \frac{k}{m'}S(t+1)\right)\right)\frac{\mathrm{d}v}{v^2}\right).$$

Proof. Given x large enough, we have

$$\sum_{n\leqslant x}\hat{\tau}(n) = \sum_{\substack{t,j\\ tj\leqslant x\\ t\equiv r \bmod h\\ S(t)\equiv a \bmod m\\ S(t+1)\equiv a' \bmod m'}} 1 = A_1 + A_2 - A_3,$$

where

$$A_1 = \sum_{\substack{t\leqslant\sqrt{x}\\ t\equiv r \bmod h\\ S(t)\equiv a \bmod m\\ S(t+1)\equiv a' \bmod m'}} \sum_{j\leqslant\frac{x}{t}} 1,$$

$$A_2 = \sum_{j\leqslant\sqrt{x}} \sum_{\substack{t\leqslant\frac{x}{j}\\ t\equiv r \bmod h\\ S(t)\equiv a \bmod m\\ S(t+1)\equiv a' \bmod m'}} 1$$

and

$$A_3 = \sum_{\substack{t\leqslant\sqrt{x}\\ t\equiv r \bmod h\\ S(t)\equiv a \bmod m\\ S(t+1)\equiv a' \bmod m'}} \sum_{j\leqslant\sqrt{x}} 1.$$

Using (1.2), we write

$$\begin{aligned}
(2.1)\quad A_1 &= \sum_{\substack{t\leqslant\sqrt{x}\\ t\equiv r \mod h\\ S(t)\equiv a \mod m\\ S(t+1)\equiv a' \mod m'}} \left\lfloor \frac{x}{t} \right\rfloor \\
&= \frac{1}{mm'} \sum_{\substack{t\leqslant\sqrt{x}\\ t\equiv r \mod h}} \left\lfloor \frac{x}{t} \right\rfloor \sum_{\ell=0}^{m-1}\sum_{k=0}^{m'-1} e\left(\frac{\ell}{m}(S(t)-a)+\frac{k}{m'}(S(t+1)-a')\right) \\
&= \frac{1}{mm'}\, x \sum_{\substack{t\leqslant\sqrt{x}\\ t\equiv r \mod h}} \frac{1}{t} + \frac{1}{hmm'}\, x \sum_{u=0}^{h-1} \sum_{(\ell,k)\in\mathfrak{F}^*} e\left(-\frac{ur}{h}-\frac{\ell a}{m}-\frac{ka'}{m'}\right) \\
&\quad \times \sum_{t\leqslant\sqrt{x}} \frac{e\left(\frac{u}{h}t+\frac{\ell}{m}S(t)+\frac{k}{m'}S(t+1)\right)}{t} + O\left(\sqrt{x}\ \right) \\
&= \frac{1}{mm'}\, x\ H(\sqrt{x},r,h) + \frac{1}{hmm'}\, x \sum_{u=0}^{h-1} \sum_{(\ell,k)\in\mathfrak{F}^*} e\left(-\frac{ur}{h}-\frac{\ell a}{m}-\frac{ka'}{m'}\right) S_{u,\ell,k}(x) \\
&\quad + O\left(\sqrt{x}\ \right),
\end{aligned}$$

with

$$S_{u,\ell,k}(x) = \sum_{t\leqslant\sqrt{x}} \frac{e\left(\frac{u}{h}t+\frac{\ell}{m}S(t)+\frac{k}{m'}S(t+1)\right)}{t},$$

for each $u \in \{0,\ldots,h-1\}$ and $(\ell,k)\in\mathfrak{F}^*$.
Following [13], we find

$$(2.2)\quad \frac{1}{mm'}\, x\ H\left(\sqrt{x},r,h\right) \quad = \quad \frac{1}{2hmm'}\, x\log x + \frac{\gamma(r,h)}{hmm'}\ x + O\left(\sqrt{x}\ \right).$$

Then, using Abel's summation formula, we obtain

$$\begin{aligned}
S_{u,\ell,k}(x) &= \sum_{t\leqslant\sqrt{x}} \frac{e\left(\frac{u}{h}t+\frac{\ell}{m}S(t)+\frac{k}{m'}S(t+1)\right)}{t} \\
&= \frac{1}{\sqrt{x}} B\left(\sqrt{x},\frac{\ell}{m},\frac{k}{m'},\frac{u}{h}\right) + \int_1^{\sqrt{x}} B\left(v,\frac{\ell}{m},\frac{k}{m'},\frac{u}{h}\right)\frac{dv}{v^2}.
\end{aligned}$$

But, $(m,m')=1$ so that $(q-1)\left(\frac{\ell}{m}+\frac{k}{m'}\right)\in\mathbb{R}\setminus\mathbb{Z}$ for every couple $(\ell,k)\in\mathfrak{F}^*$, as seen in the proof of [4, Théorème 2.2], so thanks to Theorem B the integral is absolutely convergent and

$$\left| B\left(v,\frac{\ell}{m},\frac{k}{m'},\frac{u}{h}\right)\right| \frac{1}{v^2} \quad \ll \quad v^{\lambda-2}\log v.$$

Hence, we get

$$\int_1^{\sqrt{x}} B\left(v, \frac{\ell}{m}, \frac{k}{m'}, \frac{u}{h}\right) \frac{dv}{v^2} = \alpha_{u,\ell,k} + O\left(x^{\frac{\lambda-1}{2}} \log x\right), \tag{2.3}$$

with

$$\alpha_{u,\ell,k} = \int_1^{+\infty} \left(\sum_{t \leqslant v} e\left(\frac{u}{h}t + \frac{\ell}{m}S(t) + \frac{k}{m'}S(t+1)\right)\right) \frac{dv}{v^2},$$

for every $u \in \{0, \ldots, h-1\}$ and $(\ell, k) \in \mathfrak{F}^*$.

Using Theorem B again, we get

$$\frac{1}{\sqrt{x}} \sum_{t \leqslant \sqrt{x}} e\left(\frac{u}{h}t + \frac{\ell}{m}S(t) + \frac{k}{m'}S(t+1)\right) = O\left(x^{\frac{\lambda-1}{2}} \log x\right). \tag{2.4}$$

Gathering (2.3) and (2.4), we find

$$S_{u,\ell,k}(x) = \alpha_{u,\ell,k} + O\left(x^{\frac{\lambda-1}{2}} \log x\right).$$

Considering (2.2), we go back to (2.1) in order to obtain

$$A_1 \quad = \quad \frac{1}{2hmm'}\, x \log x + \frac{\alpha + \gamma(r,h)}{hmm'}\, x + O\left(x^{\frac{\lambda+1}{2}} \log x\right), \tag{2.5}$$

where $\alpha = \sum_{u=0}^{h-1} \sum_{(\ell,k) \in \mathfrak{F}^*} e\left(-\frac{ur}{h} - \frac{\ell a}{m} - \frac{ka'}{m'}\right) \alpha_{u,\ell,k}$.

Next, we write

$$\begin{aligned} A_2 \quad &= \quad \frac{1}{mm'} \sum_{j \leqslant \sqrt{x}} \sum_{\substack{t \leqslant \frac{x}{j} \\ t \equiv r \mod h}} \sum_{\ell=0}^{m-1} \sum_{k=0}^{m'-1} e\left(\frac{\ell}{m}(S(t) - a) + \frac{k}{m'}(S(t+1) - a')\right) \\ &= \quad \frac{1}{mm'} \sum_{j \leqslant \sqrt{x}} \sum_{\substack{t \leqslant \frac{x}{j} \\ t \equiv r \mod h}} 1 + \frac{1}{hmm'} \sum_{u=0}^{h-1} \sum_{(\ell,k) \in \mathfrak{F}^*} e\left(-\frac{ur}{h} - \frac{\ell a}{m} - \frac{ka'}{m'}\right) \\ &\times \quad \sum_{j \leqslant \sqrt{x}} \sum_{t \leqslant \frac{x}{j}} e\left(\frac{u}{h}t + \frac{\ell}{m}S(t) + \frac{k}{m'}S(t+1)\right). \end{aligned}$$

Easily, [5, Theorem 3.2] shows that

$$\begin{aligned} \frac{1}{mm'} \sum_{j \leqslant \sqrt{x}} \sum_{\substack{t \leqslant \frac{x}{j} \\ t \equiv r \mod h}} 1 &= \frac{1}{mm'} \sum_{j \leqslant \sqrt{x}} \left(\frac{x}{jh} + O(1)\right) \\ &= \frac{1}{2hmm'}\, x \log x + \frac{\gamma}{hmm'}\, x + O\left(\sqrt{x}\ \right). \end{aligned} \tag{2.6}$$

Following Theorem B, we get for $u \in \{0, \dots, h-1\}$ and $(\ell, k) \in \mathfrak{F}^*$ (we recall that in this case $(q-1)\left(\frac{\ell}{m} + \frac{k}{m'}\right) \in \mathbb{R} \setminus \mathbb{Z}$),

$$\begin{aligned} \sum_{j \leqslant \sqrt{x}} \sum_{t \leqslant \frac{x}{j}} e\left(\frac{u}{h} t + \frac{\ell}{m} S(t) + \frac{k}{m'} S(t+1)\right) &\ll \sum_{j \leqslant \sqrt{x}} \left(\frac{x}{j}\right)^{\lambda} \log x \\ &\ll x^{\frac{1+\lambda}{2}} \log x. \end{aligned} \tag{2.7}$$

The last bound follows from [5, Theorem 3.2]. Combining (2.6) and (2.7) gives

$$A_2 \quad = \quad \frac{1}{2hmm'}\, x \log x + \frac{\gamma}{hmm'}\, x + O\left(x^{\frac{1+\lambda}{2}} \log x\right). \tag{2.8}$$

Finally, thanks to Theorem B, we get

$$\begin{aligned} A_3 &= \frac{1}{mm'} \sum_{\substack{t \leqslant \sqrt{x} \\ t \equiv\ r \mod h}} \left(\sqrt{x} + O(1)\right) + \frac{1}{hmm'} \left(\sqrt{x} + O(1)\right) \\ &\quad \times \sum_{u=0}^{h-1} \sum_{(\ell,k) \in \mathfrak{F}^*} \sum_{t \leqslant \sqrt{x}} e\left(\frac{u}{h}(t-r) + \frac{\ell}{m}(S(t)-a) + \frac{k}{m'}(S(t+1)-a')\right) \\ &= \frac{1}{hmm'}\, x + O\left(x^{\frac{\lambda+1}{2}} \log x\right). \end{aligned} \tag{2.9}$$

Gathering (2.5), (2.8) and (2.9) together, we get the desired conclusion. □

3. AVERAGE OF $\hat{\sigma}_S$

The subcase $s = 0$ was already considered in the previous paragraph. So we first assume that $s > 0$ and deal carefully with the case $s = 1$.

Theorem 3.1. *Let q, h, m and m' be integers $\geqslant 2$ such that $(m, m') = (m, q-1) = (m', q-1) = 1$, let $(r, a, a') \in \mathbb{Z}^3$ and $s > 0$ be a real number. Then*

$$\sum_{n \leqslant x} \hat{\sigma}_s(n) = \frac{\frac{\zeta(s+1)}{s+1} + \beta}{hmm'}\, x^{s+1} + \begin{cases} O(x \log x) & \text{if } s = 1 \\ O(x^z) & \text{if } s \neq 1 \end{cases}, \quad \text{as } x \longrightarrow +\infty,$$

where ζ stands for the Riemann zeta function defined in [5, Theorem 3.2], $z = \max(1, s)$ and

$$\begin{aligned} \beta = \sum_{(u,\ell,k) \in \mathfrak{G}^*} e\left(-\frac{ur}{h} - \frac{\ell a}{m} - \frac{ka'}{m'}\right) \left(\int_1^{+\infty} \left(\sum_{t \leqslant v} e\left(\frac{u}{h} t + \frac{\ell}{m} S(t)\right.\right.\right. \\ \left.\left.\left. + \frac{k}{m'} S(t+1)\right)\right) \frac{\mathrm{d}v}{v^{s+2}}\right). \end{aligned}$$

Proof. First, we may write

$$\begin{aligned}\sum_{n\leqslant x}\hat{\sigma}_s(n) &= \sum_{n\leqslant x}\sum_{\substack{t\mid n\\ t\equiv r \mod h\\ S(t)\equiv a \mod m\\ S(t+1)\equiv a' \mod m'}}\left(\frac{n}{t}\right)^s\\ &= \sum_{\substack{t\leqslant x\\ t\equiv r \mod h\\ S(t)\equiv a \mod m\\ S(t+1)\equiv a' \mod m'}}\sum_{\substack{n\leqslant x\\ n=jt}}\left(\frac{n}{t}\right)^s \qquad (3.1)\\ &= \sum_{\substack{t\leqslant x\\ t\equiv r \mod h\\ S(t)\equiv a \mod m\\ S(t+1)\equiv a' \mod m'}}\sum_{j\leqslant \frac{x}{t}} j^s.\end{aligned}$$

Then, we should treat the subcases $s=1$ and $s\neq 1$ separately, using in both cases [5, Theorem 3.2] and (1.2).

○ If $s=1$, then

$$\sum_{n\leqslant x}\hat{\sigma}_1(n) = \frac{1}{2}\ x^2 \sum_{\substack{t\leqslant x\\ t\equiv r \mod h\\ S(t)\equiv a \mod m\\ S(t+1)\equiv a' \mod m'}}\frac{1}{t^2} + O\left(x\sum_{t\leqslant x}\frac{1}{t}\right)$$

$$= \frac{1}{2hmm'}\ x^2\sum_{u=0}^{h-1}\sum_{\ell=0}^{m-1}\sum_{k=0}^{m'-1} e\left(-\frac{ur}{h}-\frac{\ell a}{m}-\frac{ka'}{m'}\right)\sum_{t\leqslant x}\frac{e\left(\frac{u}{h}t+\frac{\ell}{m}S(t)+\frac{k}{m'}S(t+1)\right)}{t^2}$$
$$+ O(x\log x)$$

$$= \frac{1}{2hmm'}\ x^2\left\{-\frac{1}{x}+\zeta(2)+O\left(x^{-2}\right)\right\} + \frac{1}{2hmm'}\ x^2\sum_{u=1}^{h-1} e\left(-\frac{ur}{h}\right)\sum_{t\leqslant x}\frac{e\left(\frac{u}{h}t\right)}{t^2}$$

$$+ \frac{1}{2hmm'}\ x^2\sum_{u=0}^{h-1}\sum_{(\ell,k)\in\mathfrak{F}^*} e\left(-\frac{ur}{h}-\frac{\ell a}{m}-\frac{ka'}{m'}\right)\sum_{t\leqslant x}\frac{e\left(\frac{u}{h}t+\frac{\ell}{m}S(t)+\frac{k}{m'}S(t+1)\right)}{t^2}$$
$$+ O(x\log x).$$

○ If $s\neq 1$, we set $z=\max(1,s)$ so that

$$\begin{aligned}\sum_{n\leqslant x}\hat{\sigma}_s(n) &= \frac{1}{s+1}\ x^{s+1}\sum_{\substack{t\leqslant x\\ t\equiv r \mod h\\ S(t)\equiv a \mod m\\ S(t+1)\equiv a' \mod m'}}\frac{1}{t^{s+1}} + O\left(x^s\sum_{t\leqslant x}\frac{1}{t^s}\right)\\ &= \frac{1}{(s+1)hmm'}\ x^{s+1}\sum_{u=0}^{h-1}\sum_{\ell=0}^{m-1}\sum_{k=0}^{m'-1} e\left(-\frac{ur}{h}-\frac{\ell a}{m}-\frac{ka'}{m'}\right)\end{aligned}$$

$$\begin{aligned}
&\times \sum_{t\leqslant x} \frac{e\left(\frac{u}{h}t+\frac{\ell}{m}S(t)+\frac{k}{m'}S(t+1)\right)}{t^{s+1}} \\
&+O\left(x^{s}\left\{\frac{x^{1-s}}{1-s}+\zeta(s)+O(x^{-s})\right\}\right) \\
= \; & \frac{1}{(s+1)hmm'}\; x^{s+1}\left\{-\frac{x^{-s}}{s}+\zeta(s+1)+O\left(x^{-s-1}\right)\right\} \\
&+\frac{1}{(s+1)hmm'}\; x^{s+1}\sum_{u=1}^{h-1} e\left(-\frac{ur}{h}\right)\sum_{t\leqslant x}\frac{e\left(\frac{u}{h}t\right)}{t^{s+1}} \\
&+\frac{1}{(s+1)hmm'}\; x^{s+1}\sum_{u=0}^{h-1}\sum_{(\ell,k)\in\mathfrak{F}^*} e\left(-\frac{ur}{h}-\frac{\ell a}{m}-\frac{ka'}{m'}\right) \\
&\times \sum_{t\leqslant x} \frac{e\left(\frac{u}{h}t+\frac{\ell}{m}S(t)+\frac{k}{m'}S(t+1)\right)}{t^{s+1}} \\
&+O(x^{z}).
\end{aligned}$$

Thanks to (3.1), we summarize by the following formula

$$\sum_{n\leqslant x}\hat{\sigma}_s(n)=\frac{\zeta(s+1)}{(s+1)hmm'}\; x^{s+1}+B_1+B_2+\begin{cases} O(x\log x) & \text{if } s=1\\ O(x^{z}) & \text{if } s\neq 1,\end{cases} \tag{3.2}$$

where

$$B_1 = \frac{1}{(s+1)hmm'}\; x^{s+1}\sum_{u=1}^{h-1} e\left(-\frac{ur}{h}\right)\sum_{t\leqslant x}\frac{e\left(\frac{u}{h}t\right)}{t^{s+1}},$$

$$\begin{aligned} B_2 = \frac{1}{(s+1)hmm'}\; x^{s+1}\sum_{u=0}^{h-1}\sum_{(\ell,k)\in\mathfrak{F}^*} e\left(-\frac{ur}{h}-\frac{\ell a}{m}-\frac{ka'}{m'}\right)& \\ \times \sum_{t\leqslant x} \frac{e\left(\frac{u}{h}t+\frac{\ell}{m}S(t)+\frac{k}{m'}S(t+1)\right)}{t^{s+1}}&. \end{aligned} \tag{3.3}$$

Applying Abel's summation formula to B_1 implies that for $u\in\{1,\ldots,h-1\}$,

$$\sum_{t\leqslant x}\frac{e\left(\frac{u}{h}t\right)}{t^{s+1}} = \frac{1}{x^{s+1}}\sum_{t\leqslant x} e\left(\frac{u}{h}t\right)+(s+1)\int_1^x\left(\sum_{t\leqslant v} e\left(\frac{u}{h}t\right)\right)\frac{\mathrm{d}v}{v^{s+2}}.$$

But, $\sum_{t\leqslant x} e\left(\frac{u}{h}t\right)=O(x)$ and the integral converges absolutely as done above, so that

$$B_1 = \frac{\beta_1}{hmm'}\; x^{s+1}+O(x), \tag{3.4}$$

where, $\beta_1 = \sum_{u=1}^{h-1} \left(\int_1^{+\infty} \left(\sum_{t \leqslant v} e\left(\frac{u}{h} t\right)\right) \frac{\mathrm{d}v}{v^{s+2}} \right) e\left(-\frac{ur}{h}\right)$.

Next, apply Abel's summation formula again to find

$$\begin{aligned} T_{u,\ell,k}(x) &= \sum_{t \leqslant x} \frac{e\left(\frac{u}{h}t + \frac{\ell}{m}S(t) + \frac{k}{m'}S(t+1)\right)}{t^{s+1}} \\ &= \frac{1}{x^{s+1}} \sum_{t \leqslant x} e\left(\frac{u}{h}t + \frac{\ell}{m}S(t) + \frac{k}{m'}S(t+1)\right) \\ &+ (s+1) \int_1^{x} \left(\sum_{t \leqslant v} e\left(\frac{u}{h}t + \frac{\ell}{m}S(t) + \frac{k}{m'}S(t+1)\right)\right) \frac{\mathrm{d}v}{v^{s+2}}. \end{aligned}$$

Now the assumption $(m,m') = 1$ implies that $(q-1)\left(\frac{\ell}{m} + \frac{k}{m'}\right) \in \mathbb{R} \setminus \mathbb{Z}$ for each couple $(\ell, k) \in \mathfrak{F}^*$ so thanks to Theorem B, the latter integral is again absolutely convergent. Thus, we get

(3.5)

$$\int_1^{x} \left(\sum_{t \leqslant v} e\left(\frac{u}{h}t + \frac{\ell}{m}S(t) + \frac{k}{m'}S(t+1)\right)\right) \frac{\mathrm{d}v}{v^{s+2}} = \beta_{u,\ell,k} + O\left(x^{\lambda-s-1}\log x\right),$$

where

$$\beta_{u,\ell,k} = \int_1^{+\infty} \left(\sum_{t \leqslant v} e\left(\frac{u}{h}t + \frac{\ell}{m}S(t) + \frac{k}{m'}S(t+1)\right)\right) \frac{\mathrm{d}v}{v^{s+2}},$$

for every $u \in \{0, \dots, h-1\}$ and $(\ell, k) \in \mathfrak{F}^*$.
Using Theorem B once again, we get

$$\frac{1}{x^{s+1}} \sum_{t \leqslant x} e\left(\frac{u}{h}t + \frac{\ell}{m}S(t) + \frac{k}{m'}S(t+1)\right) = O\left(x^{\lambda-s-1}\log x\right). \tag{3.6}$$

Considering the identities (3.5) and (3.6) jointly, we find

$$T_{u,\ell,k}(x) = (s+1)\beta_{u,\ell,k} + O\left(x^{\lambda-s-1}\log x\right).$$

Going back to (3.3), we write

$$B_2 = \frac{\beta_2}{hmm'}\, x^{s+1} + O(x^{\lambda}\log x), \tag{3.7}$$

with $\beta_2 = \sum_{u=0}^{h-1} \sum_{(\ell,k) \in \mathfrak{F}^*} e\left(-\frac{ur}{h} - \frac{la}{m} - \frac{ka'}{m'}\right) \beta_{u,\ell,k}$.

Hence, gathering (3.2), (3.4) and (3.7) provides the required result. ☐

In order to find the average order of $\hat{\sigma}_s(n)$ for negative s, we shall set $z = -s$ where $z > 0$.

THEOREM 3.2. *Let q, h, m and m' be integers $\geqslant 2$ such that $(m, m') = (m, q-1) = (m', q-1) = 1$, let $(r, a, a') \in \mathbb{Z}^3$ and $s < 0$ be a real number. Thus*

$$\sum_{n\leqslant x} \hat{\sigma}_s(n) = \frac{\zeta(1-s)}{hmm'}\, x + \left\{ \begin{array}{ll} O(x^{\lambda}\log x) & \text{if } s = \lambda - 1 \\ O(x^{v}) & \text{if } s \neq \lambda - 1 \end{array} \right., \text{ as } x \longrightarrow +\infty,$$

where λ is the constant mentioned in Theorem A and $v = \max(\lambda, 1+s)$.

Proof. In fact, if we set $z = -s > 0$, then

$$\begin{aligned} \sum_{n\leqslant x} \hat{\sigma}_s &= \sum_{n\leqslant x} \sum_{\substack{t\mid n \\ t\equiv r \mod h \\ S(t)\equiv a \mod m \\ S(t+1)\equiv a' \mod m'}} \left(\frac{t}{n}\right)^{z} \\ &= \sum_{j\leqslant x} \frac{1}{j^z} \sum_{\substack{t\leqslant \frac{x}{j} \\ t\equiv r \mod h \\ S(t)\equiv a \mod m \\ S(t+1)\equiv a' \mod m'}} 1 \\ &= \Sigma_1 + \Sigma_2 + \Sigma_3. \end{aligned}$$

where

$$\Sigma_1 = \frac{1}{hmm'} \sum_{j\leqslant x} \frac{1}{j^z} \left\lfloor \frac{x}{j} \right\rfloor,$$

$$\Sigma_2 = \frac{1}{hmm'} \sum_{u=1}^{h-1} e\left(-\frac{ur}{h}\right) \sum_{j\leqslant x} \frac{1}{j^z} \sum_{t\leqslant \frac{x}{j}} e\left(\frac{u}{h} t\right)$$

and

$$\begin{aligned} \Sigma_3 = \frac{1}{hmm'} \sum_{u=0}^{h-1} \sum_{(\ell,k)\in\mathfrak{F}^*} & e\left(-\frac{ur}{h} - \frac{la}{m} - \frac{ka'}{m'}\right) \\ & \times \sum_{j\leqslant x} \frac{1}{j^z} \sum_{t\leqslant \frac{x}{j}} e\left(\frac{u}{h} t + \frac{\ell}{m} S(t) + \frac{k}{m'} S(t+1)\right). \end{aligned}$$

We first apply [5, Theorem 3.2] to get

$$\begin{aligned}
\Sigma_1 &= \frac{1}{hmm'}\, x \sum_{j \leqslant x} \frac{1}{j^{1+z}} + O\left(\sum_{j \leqslant x} \frac{1}{j^z}\right) \\
&= \frac{1}{hmm'}\, x \left\{\zeta(1+z) + O(x^{-z})\right\} \\
&\quad + \left\{ \begin{array}{ll} O(\log x) & \text{if } z = 1 \\ O\left(\left\{\frac{x^{1-z}}{1-z} + \zeta(z) + O(x^{-z})\right\}\right) & \text{else} \end{array} \right. \qquad (3.8) \\
&= \frac{\zeta(1+z)}{hmm'}\, x + \left\{ \begin{array}{ll} O(1) & \text{if } z > 1 \\ O(\log x) & \text{if } z = 1 \\ O\left(x^{1-z}\right) & \text{if } z < 1. \end{array} \right.
\end{aligned}$$

Second, for $u \in \{1, \ldots, h-1\}$ and fixed $j \leqslant x$, we shall use the estimate

$$\sum_{t \leqslant \frac{x}{j}} e\left(\frac{u}{h} t\right) \leqslant \frac{1}{\sin\left(\frac{\pi}{h}\right)}.$$

So that, following (3.8), we obtain

$$\begin{aligned}
\Sigma_2 &\ll \sum_{j \leqslant x} \frac{1}{j^z} \\
&\ll \left\{ \begin{array}{ll} O(1) & \text{if } z > 1 \\ O(\log x) & \text{if } z = 1 \\ O\left(x^{1-z}\right) & \text{if } z < 1. \end{array} \right.
\end{aligned}$$

Third, Theorem B implies that for $u \in \{0, \ldots, h-1\}$ and $(\ell, k) \in \mathfrak{F}^*$ (in which case $(q-1)\left(\frac{\ell}{m} + \frac{k}{m'}\right) \in \mathbb{R} \setminus \mathbb{Z}$),

$$\begin{aligned}
\sum_{j \leqslant x} \frac{1}{j^z} \sum_{t \leqslant \frac{x}{j}} e\left(\frac{u}{h} t + \frac{\ell}{m} S(t) + \frac{k}{m'} S(t+1)\right) &\ll \sum_{j \leqslant x} \frac{1}{j^z} \left(\frac{x}{j}\right)^{\lambda} \\
&\ll x^{\lambda} \sum_{j \leqslant x} \frac{1}{j^{\lambda+z}}.
\end{aligned}$$

Thanks to [5, Theorem 3.2], we find

$$\Sigma_3 = \left\{ \begin{array}{ll} O\left(x^{\lambda} \log x\right) & \text{if } z = 1-\lambda \\ O\left(x^{\lambda}\right) & \text{if } z > 1-\lambda \\ O\left(x^{1-z}\right) & \text{if } z < 1-\lambda. \end{array} \right.$$

Ordering of the real numbers 1, z and $1-\lambda$, we complete the proof. □

4. AVERAGE OF $\hat{\varphi}$

THEOREM 4.1. *Let q, h, m and m' be integers $\geqslant 2$ such that $(m, q-1) = (m', q-1) = 1$, let $(r, a, a') \in \mathbb{Z}^3$. Then*

$$\sum_{n\leqslant x}\hat{\varphi}(n) = \frac{\frac{3}{\pi^2}+\rho}{hmm'}\ x^2 + O\left(x\log x\right),\ \text{ as } x \longrightarrow +\infty,$$

where

$$\begin{aligned}\rho \ &= \sum_{(u,\ell,k)\in\mathfrak{G}^*} e\left(-\frac{ur}{h}-\frac{\ell a}{m}-\frac{ka'}{m'}\right)\\ &\qquad\times\left(\int_1^{+\infty}\left(\sum_{t\leqslant v}\mu(t)e\left(\frac{u}{h}t+\frac{\ell}{m}S(t)+\frac{k}{m'}S(t+1)\right)\right)\frac{\mathrm{d}v}{v^3}\right).\end{aligned}$$

Proof. Let x be large enough, following [5, Theorem 3.2] and (1.2) we find

$$\begin{aligned}\sum_{n\leqslant x}\hat{\varphi}(n) &= \sum_{n\leqslant x}\ \sum_{\substack{t|n\\ t\equiv r \mod h\\ S(t)\equiv a \mod m\\ S(t+1)\equiv a' \mod m'}}\mu(t)\frac{n}{t}\\ &= \sum_{\substack{t\leqslant x\\ t\equiv r \mod h\\ S(t)\equiv a \mod m\\ S(t+1)\equiv a' \mod m'}}\mu(t)\sum_{j\leqslant\frac{x}{t}} j\\ &= \sum_{\substack{t\leqslant x\\ t\equiv r \mod h\\ S(t)\equiv a \mod m\\ S(t+1)\equiv a' \mod m'}}\mu(t)\left(\frac{x^2}{2t^2}+O\left(\frac{x}{t}\right)\right)\\ &= \frac{1}{2hmm'}\ x^2\sum_{u=0}^{h-1}\sum_{\ell=0}^{m-1}\sum_{k=0}^{m'-1} e\left(-\frac{ur}{h}-\frac{\ell a}{m}-\frac{ka'}{m'}\right)\\ &\qquad\times\sum_{t\leqslant x}\mu(t)\frac{e\left(\frac{u}{h}t+\frac{\ell}{m}S(t)+\frac{k}{m'}S(t+1)\right)}{t^2}+O(x\log x)\\ &= \frac{1}{2hmm'}\ x^2\sum_{t=1}^{+\infty}\frac{\mu(t)}{t^2}+O\left(x^2\sum_{t>x}\frac{1}{t^2}\right)\\ &\quad+\frac{1}{2hmm'}\ x^2\sum_{u=1}^{h-1}e\left(-\frac{ur}{h}\right)\sum_{t\leqslant x}\mu(t)\frac{e\left(\frac{u}{h}t\right)}{t^2}\end{aligned}\tag{4.1}$$

$$
\begin{aligned}
&+\frac{1}{2hmm'}\ x^2\sum_{u=0}^{h-1}\sum_{(\ell,k)\in\mathfrak{F}^*} e\left(-\frac{ur}{h}-\frac{\ell a}{m}-\frac{ka'}{m'}\right)\\
&\qquad\times\sum_{t\leqslant x}\mu(t)\frac{e\left(\frac{u}{h}t+\frac{\ell}{m}S(t)+\frac{k}{m'}S(t+1)\right)}{t^2}+O(x\log x)\\
&=\frac{1}{2hmm'}\ x^2\sum_{t=1}^{+\infty}\frac{\mu(t)}{t^2}+C_1+C_2+O(x\log x),
\end{aligned}
$$

where

$$
C_1 \;=\; \frac{1}{2hmm'}\ x^2\sum_{u=1}^{h-1}e\left(-\frac{ur}{h}\right)\sum_{t\leqslant x}\mu(t)\frac{e\left(\frac{u}{h}t\right)}{t^2},
$$

$$
\begin{aligned}
C_2 = \frac{1}{2hmm'}\ x^2\sum_{u=0}^{h-1}\sum_{(\ell,k)\in\mathfrak{F}^*}\left(\sqrt{x}+O(1)\right)e\left(-\frac{ur}{h}-\frac{\ell a}{m}-\frac{ka'}{m'}\right)\\
\times\sum_{t\leqslant x}\mu(t)\frac{e\left(\frac{u}{h}t+\frac{\ell}{m}S(t)+\frac{k}{m'}S(t+1)\right)}{t^2}.
\end{aligned}\tag{4.2}
$$

Möbius inversion formula (see [5, p.32]) gives

$$
\left(\sum_{t=1}^{+\infty}\frac{1}{t^2}\right)\left(\sum_{m=1}^{+\infty}\frac{\mu(m)}{m^2}\right)=\sum_{k=1}^{+\infty}\frac{1}{k^2}\left(\sum_{\substack{t,m\\ tm=k}}\mu(t)\right)=\sum_{k=1}^{+\infty}\frac{1}{k^2}\left(\sum_{m\mid k}\mu(m)\right)=1,
$$

so that

$$
\sum_{t=1}^{+\infty}\frac{\mu(t)}{t^2}=\frac{6}{\pi^2}.\tag{4.3}
$$

Next, We may apply Abel's formula to the first sum, which gives

$$
\sum_{t\leqslant x}\mu(t)\frac{e\left(\frac{u}{h}t\right)}{t^2} \;=\; \frac{1}{x^2}\sum_{t\leqslant x}\mu(t)e\left(\frac{u}{h}t\right)+2\int_1^x\left(\sum_{t\leqslant v}\mu(t)e\left(\frac{u}{h}t\right)\right)\frac{\mathrm{d}v}{v^3}.
$$

But, $\sum_{t\leqslant x}\mu(t)e\left(\frac{u}{h}t\right)=O(x)$ and the integral converges absolutely, so that

$$
C_1 \;=\; \frac{\rho_1}{hmm'}\ x^2+O(x\log x),\tag{4.4}
$$

where, $\rho_1=\sum_{u=1}^{h-1}\left(\int_1^{+\infty}\left(\sum_{t\leqslant v}\mu(t)e\left(\frac{u}{h}t\right)\right)\frac{dv}{v^3}\right)e\left(-\frac{ur}{h}\right).$

Now, we apply Abel's summation formula again to C_2 so to get

$$\begin{aligned} U_{u,\ell,k}(x) &= \sum_{t\leqslant x}\mu(t)\frac{e\left(\frac{u}{h}t+\frac{\ell}{m}S(t)+\frac{k}{m'}S(t+1)\right)}{t^2} \\ &= \frac{1}{x^2}\sum_{t\leqslant x}\mu(t)e\left(\frac{u}{h}t+\frac{\ell}{m}S(t)+\frac{k}{m'}S(t+1)\right) \\ &+ 2\int_1^x\left(\sum_{t\leqslant v}\mu(t)e\left(\frac{u}{h}t+\frac{\ell}{m}S(t)+\frac{k}{m'}S(t+1)\right)\right)\frac{\mathrm{d}v}{v^3}. \end{aligned}$$

The integral is trivially convergent. Consequently, we obtain

$$\int_1^x\left(\sum_{t\leqslant v}\mu(t)e\left(\frac{u}{h}t+\frac{\ell}{m}S(t)+\frac{k}{m'}S(t+1)\right)\right)\frac{dv}{v^3}=\rho_{u,\ell,k}+O\left(x^{-1}\right), \tag{4.5}$$

where

$$\rho_{u,\ell,k}=\int_1^{+\infty}\left(\sum_{t\leqslant v}\mu(t)e\left(\frac{u}{h}t+\frac{\ell}{m}S(t)+\frac{k}{m'}S(t+1)\right)\right)\frac{dv}{v^3},$$

for each $u\in\{0,\ldots,h-1\}$ and $(\ell,k)\in\mathfrak{F}^*$.
Obviously, we have

$$\frac{1}{x^2}\sum_{t\leqslant x}\mu(t)e\left(\frac{u}{h}t+\frac{\ell}{m}S(t)+\frac{k}{m'}S(t+1)\right)=O\left(x^{-1}\right). \tag{4.6}$$

Gathering the identities (4.5) and (4.6) jointly, we get

$$U_{u,\ell,k}(x)=2\rho_{u,\ell,k}+O\left(x^{-1}\right).$$

Going back to (4.2), we write

$$C_2 \;=\; \frac{\rho_2}{hmm'}\,x^2+O(x\log x), \tag{4.7}$$

with $\rho_2=\sum_{u=0}^{h-1}\sum_{(\ell,k)\in\mathfrak{F}^*}e\left(-\frac{ur}{h}-\frac{\ell a}{m}-\frac{ka'}{m'}\right)\rho_{u,\ell,k}$.

Finally, joining (4.1), (4.3), (4.4) and (4.7) altogether provides the required result. ☐

5. AVERAGE OF $\hat{R}$

THEOREM 5.1. *Let q,h,m and m' be integers $\geqslant 2$ such that $(m,m')=(m,q-1)=(m',q-1)=1$, let $(r,a,a')\in\mathbb{Z}^3$. We affirm that*

$$\sum_{n\leqslant x}\hat{r}(n)=\frac{\pi+4\nu}{hmm'}\,x+O\left(x^{\frac{\lambda+1}{2}}\log x\right),\ \text{as } x\longrightarrow+\infty,$$

where λ stands for the constant stated in Theorem A and

$$\begin{aligned}\nu &= \sum_{(u,\ell,k)\in\mathfrak{G}^*} e\left(-\frac{ur}{h}-\frac{\ell a}{m}-\frac{ka'}{m'}\right)\\ &\qquad\times\left(\int_1^{+\infty}\left(\sum_{t\leqslant v}\chi(t)e\left(\frac{u}{h}t+\frac{\ell}{m}S(t)+\frac{k}{m'}S(t+1)\right)\right)\frac{\mathrm{d}v}{v^2}\right).\end{aligned}$$

Proof. Given x large enough, we have

$$\begin{aligned}\sum_{n\leqslant x}\hat{r}(n) &= 4\sum_{n\leqslant x}\sum_{\substack{t\mid n\\ t\equiv r\ \bmod h\\ S(t)\equiv a\ \bmod m\\ S(t+1)\equiv a'\ \bmod m'}}\chi(t)\\ &= 4\sum_{\substack{tj\leqslant x\\ t\equiv r\ \bmod h\\ S(t)\equiv a\ \bmod m\\ S(t+1)\equiv a'\ \bmod m'}}\chi(t)\\ &= 4\sum_{\substack{t\leqslant x\\ t\equiv r\ \bmod h\\ S(t)\equiv a\ \bmod m\\ S(t+1)\equiv a'\ \bmod m'}}\chi(t)\sum_{j\leqslant\frac{x}{t}}1\\ &= 4\sum_{\substack{t\leqslant x\\ t\equiv r\ \bmod h\\ S(t)\equiv a\ \bmod m\\ S(t+1)\equiv a'\ \bmod m'}}\chi(t)\left\lfloor\frac{x}{t}\right\rfloor\\ &= 4\sum_{\substack{t\leqslant \sqrt{x}\\ t\equiv r\ \bmod h\\ S(t)\equiv a\ \bmod m\\ S(t+1)\equiv a'\ \bmod m'}}\chi(t)\left\lfloor\frac{x}{t}\right\rfloor+4\sum_{v\leqslant\sqrt{x}}\sum_{\substack{\sqrt{x}<t\leqslant\frac{x}{v}\\ t\equiv r\ \bmod h\\ S(t)\equiv a\ \bmod m\\ S(t+1)\equiv a'\ \bmod m'}}\chi(t).\end{aligned}$$

The second sum is $O\left(\sqrt{x}\ \right)$, and since

$$\sum_{t=1}^{+\infty}\frac{\chi(t)}{t}=1-\frac{1}{3}+\frac{1}{5}-\ldots=\frac{\pi}{4},$$

then, (1.2) implies

$$\begin{aligned}\sum_{\substack{t\leqslant \sqrt{x}\\ t\equiv r\ \bmod h\\ S(t)\equiv a\ \bmod m\\ S(t+1)\equiv a'\bmod m'}}\chi(t)\left\lfloor\frac{x}{t}\right\rfloor &= \sum_{\substack{t\leqslant \sqrt{x}\\ t\equiv r\ \bmod h\\ S(t)\equiv a\ \bmod m\\ S(t+1)\equiv a'\bmod m'}}\chi(t)\frac{x}{t}+O\left(\sqrt{x}\ \right)\\ &= \frac{1}{hmm'}\ x\sum_{u=0}^{h-1}\sum_{\ell=0}^{m-1}\sum_{k=0}^{m'-1}e\left(-\frac{ur}{h}-\frac{\ell a}{m}-\frac{ka'}{m'}\right)\end{aligned}\tag{5.1}$$

$$\begin{aligned}
&\times \sum_{t\leqslant\sqrt{x}} \chi(t)\frac{e\left(\frac{u}{h}t+\frac{\ell}{m}S(t)+\frac{k}{m'}S(t{+}1)\right)}{t} + O\left(\sqrt{x}\right)\\
&= \frac{1}{hmm'}\ x\left(\sum_{t=1}^{+\infty}\frac{\chi(t)}{t} + O\left(\frac{1}{\sqrt{x}}\right)\right)\\
&+ \frac{1}{hmm'}\ x\sum_{u=1}^{h-1} e\left(-\frac{ur}{h}\right)\sum_{t\leqslant\sqrt{x}}\chi(t)\frac{e\left(\frac{u}{h}t\right)}{t}\\
&+ \frac{1}{hmm'}\ x\sum_{u=0}^{h-1}\sum_{(\ell,k)\in\mathfrak{F}^*} e\left(-\frac{ur}{h}-\frac{\ell a}{m}-\frac{ka'}{m'}\right)\\
&\times \sum_{t\leqslant\sqrt{x}} \chi(t)\frac{e\left(\frac{u}{h}t+\frac{\ell}{m}S(t)+\frac{k}{m'}S(t{+}1)\right)}{t} + O\left(\sqrt{x}\right)\\
&= \frac{\pi}{4hmm'}\ x + D_1 + D_2 + O\left(\sqrt{x}\ \right),
\end{aligned}$$

where

$$D_1 \quad = \quad \frac{1}{hmm'}\ x\sum_{u=1}^{h-1} e\left(-\frac{ur}{h}\right)\sum_{t\leqslant\sqrt{x}}\chi(t)\frac{e\left(\frac{u}{h}t\right)}{t},$$

$$\begin{aligned}
D_2 = \frac{1}{hmm'}\ x\sum_{u=0}^{h-1}\sum_{(\ell,k)\in\mathfrak{F}^*} e\left(-\frac{ur}{h}-\frac{\ell a}{m}-\frac{ka'}{m'}\right)\\
\times \sum_{t\leqslant\sqrt{x}} \chi(t)\frac{e\left(\frac{u}{h}t+\frac{\ell}{m}S(t)+\frac{k}{m'}S(t+1)\right)}{t}.
\end{aligned} \tag{5.2}$$

Abel's summation formula applied to D_1 gives

$$\sum_{t\leqslant\sqrt{x}}\chi(t)\frac{e\left(\frac{u}{h}t\right)}{t} \quad = \quad \frac{1}{\sqrt{x}}\sum_{t\leqslant\sqrt{x}}\chi(t)e\left(\frac{u}{h}t\right) + \int_1^{\sqrt{x}}\left(\sum_{t\leqslant v}\chi(t)e\left(\frac{u}{h}t\right)\right)\frac{\mathrm{d}v}{v^2}.$$

In particular, we recognize the geometric sum

$$\begin{aligned}
\sum_{t\leqslant\sqrt{x}}\chi(t)e\left(\frac{u}{h}t\right) \quad &= \quad e\left(\frac{u}{h}\right) - e\left(\frac{3u}{h}\right) + \dots\\
&+ \quad (-1)^{\left\lfloor\frac{\lfloor\sqrt{x}\rfloor-1}{2}\right\rfloor} e\left(\frac{u}{h}\left(2\left\lfloor\frac{\lfloor\sqrt{x}\rfloor-1}{2}\right\rfloor+1\right)\right)\\
&= \quad O(1).
\end{aligned}$$

So, the integral is absolutely convergent, and then

$$D_1 \quad = \quad \frac{\nu_1}{hmm'}\ x + O\left(\sqrt{x}\ \right), \tag{5.3}$$

where, $\nu_1 = \sum_{u=1}^{h-1} \left(\int_1^{+\infty} \left(\sum_{t \leqslant v} \chi(t) e\left(\frac{u}{h} t\right)\right) \frac{\mathrm{d}v}{v^2} \right) e\left(-\frac{ur}{h}\right)$.

Using Abel's summation formula again, we get

$$\begin{aligned} V_{u,\ell,k}(x) &= \sum_{t \leqslant \sqrt{x}} \chi(t) \frac{e\left(\frac{u}{h}t + \frac{\ell}{m}S(t) + \frac{k}{m'}S(t+1)\right)}{t} \\ &= \frac{1}{\sqrt{x}} \sum_{t \leqslant \sqrt{x}} \chi(t) e\left(\frac{u}{h}t + \frac{\ell}{m}S(t) + \frac{k}{m'}S(t+1)\right) \\ &+ \int_1^{\sqrt{x}} \left(\sum_{t \leqslant v} \chi(t) e\left(\frac{u}{h}t + \frac{\ell}{m}S(t) + \frac{k}{m'}S(t+1)\right)\right) \frac{\mathrm{d}v}{v^2}. \end{aligned}$$

The integral converges absolutely since χ is a character satisfying the identity

$$\chi(t) = \frac{e\left(\frac{t-1}{4}\right) + e\left(-\frac{t-1}{4}\right)}{2},$$

which leads to two sums of the type $\sum_{t \leqslant u} e\left(\alpha t + \frac{\ell}{m}S(t) + \frac{k}{m'}S(t+1)\right)$ (with $\alpha = \frac{u}{h} + \frac{1}{4}$ and $\alpha = \frac{u}{h} - \frac{1}{4}$) that shall be bounded by $u^{\lambda} \log u$ thanks to Theorem B, so that the integral converges absolutely and following Theorem B (which might be used since $(m, m') = 1$ and so $(q-1)\left(\frac{\ell}{m} + \frac{k}{m'}\right) \in \mathbb{R} \setminus \mathbb{Z}$)

$$\int_{\sqrt{x}}^{+\infty} \left| \sum_{t \leqslant v} \chi(t) e\left(\frac{u}{h}t + \frac{\ell}{m}S(t) + \frac{k}{m'}S(t+1)\right)\right| \frac{\mathrm{d}v}{v^2} \quad \ll \quad x^{\frac{\lambda-1}{2}} \log x.$$

Hence, we get

(5.4)

$$\int_1^{\sqrt{x}} \left(\sum_{t \leqslant v} \chi(t) e\left(\frac{u}{h}t + \frac{\ell}{m}S(t) + \frac{k}{m'}S(t+1)\right)\right) \frac{\mathrm{d}v}{v^2} = \nu_{u,\ell,k} + O\left(x^{\frac{\lambda-1}{2}} \log x\right),$$

where

$$\nu_{u,\ell,k} = \int_1^{+\infty} \left(\sum_{t \leqslant v} \chi(t) e\left(\frac{u}{h}t + \frac{\ell}{m}S(t) + \frac{k}{m'}S(t+1)\right)\right) \frac{\mathrm{d}v}{v^2},$$

for every $u \in \{0, \ldots, h-1\}$ and $(\ell, k) \in \mathfrak{F}^*$.

Furthermore, we have

$$(5.5) \quad \frac{1}{\sqrt{x}} \sum_{t \leqslant \sqrt{x}} \chi(t) e\left(\frac{u}{h}t + \frac{\ell}{m}S(t) + \frac{k}{m'}S(t+1)\right) = O\left(x^{\frac{\lambda-1}{2}} \log x\right).$$

Joining the identities (5.4) and (5.5) altogether, we get

$$V_{u,\ell,k}(x) = \nu_{u,\ell,k} + O\left(x^{\frac{\lambda-1}{2}} \log x\right).$$

Going back to (5.2), we write

$$D_2 \;=\; \frac{\nu_2}{hmm'}\, x + O\left(x^{\frac{\lambda-1}{2}} \log x\right), \tag{5.6}$$

with $\nu_2 = \sum_{u=0}^{h-1} \sum_{(\ell,k)\in\mathfrak{F}^*} e\left(-\frac{ur}{h} - \frac{\ell a}{m} - \frac{ka'}{m'}\right) \nu_{u,\ell,k}$.

Finally, combining (5.1), (5.3) and (5.6) provides the result. □

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