Dedicated to my father, Professor Iuliu Crivei,
on the occasion of his 80th birthday

ON ISOMORPHIC INJECTIVE OBJECTS IN CATEGORIES

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We show that if X and Y are quasi-injective objects of a Grothendieck category such that there exist monomorphisms $f : X \to Y$ and $g : Y \to X$, then X and Y are isomorphic. We also prove that if X and Y are pure-injective objects of a finitely accessible additive category such that there exist pure monomorphisms $f : X \to Y$ and $g : Y \to X$, then X and Y are isomorphic. In particular, if $\mathcal{C}$ is either a Grothendieck category or a finitely accessible additive category, and X and Y are objects of $\mathcal{C}$ such that there exist monomorphisms $f : X \to Y$ and $g : Y \to X$, then X and Y are isomorphic.

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1. INTRODUCTION

The classical Cantor-Bernstein-Schröder Theorem states that if A and B are two sets such that there exist injective functions $f : A \to B$ and $g : B \to A$, then there exists a bijection between A and B. This can be rewritten in the terminology of category theory as follows: if A and B are two injective objects of the category **Set** of sets such that there exist monomorphisms $f : A \to B$ and $g : B \to A$, then A and B are isomorphic. Note that every object of **Set** which is not initial is injective. Two objects A and B of a category with the property that there exist monomorphisms $f : A \to B$ and $g : B \to A$ are called *mono-equivalent* by R. Wisbauer [15], and are said to *belong to the same monogeny class* by A. Facchini [3]. Using the former terminology, the Cantor-Bernstein-Schröder Theorem can be reformulated as follows: injective mono-equivalent objects of **Set** are isomorphic.

A natural question is whether one can obtain similar results by replacing **Set** by other categories, arbitrary monomorphisms by particular ones, and injective objects by some generalizations. A significant progress in module categories was made by R. T. Bumby, who proved that quasi-injective mono-equivalent modules are isomorphic [2, Corollary 2]. Recently, J. E. Macías-Díaz and S. Macías showed that $\mathcal{H}$-injective mono-equivalent modules with respect to a so-called algebraic class $\mathcal{H}$ of monomorphisms are isomorphic [10, Theorem 4.1]. If $\mathcal{C}$ is a category admitting a notion of purity, then two objects A and B of $\mathcal{C}$ with the property that there exist pure monomorphisms $f : A \to B$ and $g : B \to A$ are called *pure-mono-equivalent* [15]. Using different approaches, R. Wisbauer [15, 4.3] and J. E. Macías-Díaz [10, Theorem 4] proved that pure-injective pure-mono-equivalent modules are isomorphic. Also, L. M. Gurrola-Ramos, S. Macías and J. E. Macías-Díaz showed that injective mono-equivalent objects of a Grothendieck category are isomorphic [8, Theorem 12]. Recently, Guil Asensio, Kaleboğaz and Srivastava have obtained some new results on the (Cantor-)Bernstein-Schröder problem for modules [7].

In the present note we generalize such results from module categories to Grothendieck categories and finitely accessible additive categories. By using functorial techniques, we are able to give shorter and more conceptual proofs. We show that if $\mathcal{C}$ is a Grothendieck category, then quasi-injective mono-equivalent objects of $\mathcal{C}$ are isomorphic. We also prove that if $\mathcal{C}$ is a finitely accessible additive category, then pure-injective pure-mono-equivalent objects of $\mathcal{C}$ are isomorphic. Moreover, the same is true for definable full subcategories of a finitely accessible additive category with products. In particular, if $\mathcal{C}$ is either a Grothendieck category or a finitely accessible additive category, then injective mono-equivalent objects of $\mathcal{C}$ are isomorphic.

2. GROTHENDIECK CATEGORIES

We begin with a property on the transfer of quasi-injectivity between categories.

PROPOSITION 2.1. *Let (L, R) be an adjoint pair of covariant functors $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$ between arbitrary categories $\mathcal{A}$ and $\mathcal{B}$ such that L preserves monomorphisms and R is fully faithful. Then an object M of $\mathcal{B}$ is quasi-injective if and only if $R(M)$ is quasi-injective in $\mathcal{A}$.*

Proof. Let $\varepsilon : LR \to 1_{\mathcal{B}}$ and $\eta : 1_{\mathcal{A}} \to RL$ be the counit and the unit of adjunction respectively. Since R is fully faithful, $\varepsilon_B : LR(B) \to B$ is an isomorphism for every object B of $\mathcal{B}$.

Assume that M is quasi-injective in $\mathcal{B}$. Let $\alpha : A \to R(M)$ be a monomorphism and let $\beta : A \to R(M)$ be a morphism in $\mathcal{A}$. Since L preserves monomorphisms, $\varepsilon_M L(\alpha) : L(A) \to M$ is a monomorphism. By the quasi-injectivity of M, there exists a morphism $h : M \to M$ such that $h\varepsilon_M L(\alpha) = \varepsilon_M L(\beta)$.

Let $\gamma = R(h) : R(M) \to R(M)$. Since (L, R) is an adjoint pair, we have $R(\varepsilon_M)\eta_{R(M)} = 1_{R(M)}$ and η is a natural transformation. It follows that:

$$\begin{aligned}\gamma\alpha &= R(h)\alpha = R(h)R(\varepsilon_M)\eta_{R(M)}\alpha = R(h)R(\varepsilon_M)RL(\alpha)\eta_A \\ &= R(h\varepsilon_M L(\alpha))\eta_A = R(\varepsilon_M L(\beta))\eta_A = R(\varepsilon_M)RL(\beta)\eta_A \\ &= R(\varepsilon_M)\eta_{R(M)}\beta = \beta.\end{aligned}$$

This shows that $R(M)$ is quasi-injective in $\mathcal{A}$.

Conversely, assume that $R(M)$ is quasi-injective in $\mathcal{A}$. Let $\alpha : B \to M$ be a monomorphism and let $\beta : B \to M$ be a morphism in $\mathcal{B}$. Since R preserves monomorphisms as a right adjoint, $R(\alpha) : R(B) \to R(M)$ is a monomorphism. By the quasi-injectivity of $R(M)$, there exists a morphism $h : R(M) \to R(M)$ such that $hR(\alpha) = R(\beta)$. Let $\gamma = \varepsilon_M L(h)\varepsilon_M^{-1} : M \to M$. Since ε is a natural transformation, it follows that:

$$\begin{aligned}\gamma\alpha &= \gamma\alpha\varepsilon_B\varepsilon_B^{-1} = \varepsilon_M L(h)\varepsilon_M^{-1}\alpha\varepsilon_B\varepsilon_B^{-1} = \varepsilon_M L(h)\varepsilon_M^{-1}\varepsilon_M LR(\alpha)\varepsilon_B^{-1} \\ &= \varepsilon_M L(hR(\alpha))\varepsilon_B^{-1} = \varepsilon_M LR(\beta)\varepsilon_B^{-1} = \beta\varepsilon_B\varepsilon_B^{-1} = \beta.\end{aligned}$$

This shows that M is quasi-injective in $\mathcal{B}$. □

A *Grothendieck* category is an abelian category with coproducts, exact direct limits and a generator. Examples of such categories include module categories and categories $\sigma[M]$ of modules subgenerated by a module M.

Grothendieck categories are related to module categories by the following famous theorem.

THEOREM 2.2 (Gabriel-Popescu [5]). *Let $\mathcal{C}$ be a Grothendieck category with a generator U, let $R = \mathrm{End}_{\mathcal{C}}(U)$ be the endomorphism ring of U and let $\mathrm{Mod}(R)$ be the category of unitary right R-modules. Then the functor $H = \mathrm{Hom}_{\mathcal{C}}(U, -) : \mathcal{C} \to \mathrm{Mod}(R)$ is fully faithful and has an exact left adjoint.*

The following theorem generalizes [8, Theorem 12] from injectivity to quasi-injectivity.

THEOREM 2.3. *Let $\mathcal{C}$ be a Grothendieck category. Then quasi-injective mono-equivalent objects of $\mathcal{C}$ are isomorphic.*

Proof. Let X and Y be quasi-injective mono-equivalent objects of $\mathcal{C}$. We use the notation and the conclusion of the Gabriel-Popescu Theorem 2.2. The functor H preserves quasi-injective objects by Proposition 2.1. Since H is left exact, it follows that $H(X)$ and $H(Y)$ are quasi-injective mono-equivalent right R-modules. Then $H(X)$ and $H(Y)$ are isomorphic by [2, Corollary 2]. Since H is fully faithful, it follows that X and Y are isomorphic. □

Since every object of a Grothendieck category has an injective envelope and a quasi-injective envelope, we immediately have the following corollary.

COROLLARY 2.4. *Let $\mathcal{C}$ be a Grothendieck category. Then mono-equivalent objects of $\mathcal{C}$ have isomorphic injective envelopes and isomorphic quasi-injective envelopes.*

3. FINITELY ACCESSIBLE ADDITIVE CATEGORIES

We recall, mainly from [12], some terminology on finitely accessible additive categories, which are suitable frameworks for defining purity. An additive category $\mathcal{C}$ is called *finitely accessible* if it has direct limits, the class of finitely presented objects is skeletally small, and every object is a direct limit of finitely presented objects. Some typical examples of finitely accessible additive categories are the category of modules over a ring with enough idempotents and the category of torsion-free abelian groups. The former is a Grothendieck category, while in general the latter is not even abelian [6]. Also, note that there are Grothendieck categories which are not finitely accessible, e.g. a category of the form $\sigma[M]$ having no non-zero finitely presented object [13, Example 1.7].

Let $\mathcal{C}$ be a finitely accessible additive category. By a *sequence*

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

in $\mathcal{C}$ we mean a pair of composable morphisms $f : X \to Y$ and $g : Y \to Z$ such that $gf = 0$. The above sequence in $\mathcal{C}$ is called *pure exact* if it induces an exact sequence of abelian groups

$$0 \to \mathrm{Hom}_{\mathcal{C}}(P, X) \to \mathrm{Hom}_{\mathcal{C}}(P, Y) \to \mathrm{Hom}_{\mathcal{C}}(P, Z) \to 0$$

for every finitely presented object P of $\mathcal{C}$. Then f and g form a kernel-cokernel pair, f is called a *pure monomorphism* and g a *pure epimorphism*. The *pure-injective* objects of $\mathcal{C}$ are those objects which are injective with respect to every pure exact sequence in $\mathcal{C}$.

To every finitely accessible additive category $\mathcal{C}$ one may associate a ring R with enough idempotents, called the *functor ring* of $\mathcal{C}$ [4]. Denote by $\mathrm{Mod}(R)$ the category of unitary right R-modules. Note that $\mathrm{Mod}(R)$ is equivalent to the functor category $(\mathrm{fp}(\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$ of all contravariant additive functors from the full subcategory $\mathrm{fp}(\mathcal{C})$ of finitely presented objects of $\mathcal{C}$ to the category Ab of abelian groups.

Recall that a module C is called *cotorsion* if $\mathrm{Ext}^1_R(F, C) = 0$ for every flat module F [14, Definition 3.1.1]. A homomorphism $f : M \to C$ from a module M to a cotorsion module C is called a *cotorsion envelope* if for every cotorsion module C' the induced homomorphism $\mathrm{Hom}_R(C, C') \to \mathrm{Hom}_R(M, C')$ is an epimorphism, and if $g : C \to C$ is an endomorphism such that $gf = f$, then g is an automorphism [14, Definition 1.2.1]. Every module M has a cotorsion envelope $M \to C(M)$, which is a pure monomorphism [1].

Theorem 3.1 ([12, Theorem 3.4], [9, Lemma 3]). *Let $\mathcal{C}$ be a finitely accessible additive category, let $(U_i)_{i\in I}$ be a representative set of finitely presented objects of $\mathcal{C}$ and let R be the functor ring of $\mathcal{C}$. Then the (Yoneda) functor $H : \mathcal{C} \to \mathrm{Mod}(R)$, given on objects by $H(X) = \bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{C}}(U_i, X)$, is fully faithful.*

It preserves pure exact sequences, and induces an equivalence between $\mathcal{C}$ and the category of flat unitary right R-modules. The equivalence restricts to one between pure-injective objects of $\mathcal{C}$ and flat cotorsion unitary right R-modules.

In view of Theorem 3.1, in order to obtain properties on pure-injective objects of finitely accessible additive categories we first need to give some results on flat cotorsion modules over a ring with enough idempotents, which will be denoted by R in the following results.

Lemma 3.2. *Let M be a flat cotorsion right R-module and let K be a pure submodule of M. Then $C(K)$ is isomorphic to a direct summand of M.*

Proof. Let $i : K \to C(K)$ and $k : K \to M$ be inclusion homomorphisms. Since M/K is flat, we have $\mathrm{Ext}^1_R(M/K, C(K)) = 0$. Hence there exists a homomorphism $\alpha : M \to C(K)$ such that $\alpha k = i$. Since $i : K \to C(K)$ is a cotorsion envelope, there exists a homomorphism $\beta : C(K) \to M$ such that $\beta i = k$. Then $\alpha\beta i = i$, which implies that $\alpha\beta$ is an automorphism by the envelope property. Hence $\beta : C(K) \to M$ is a split monomorphism. □

Lemma 3.3 ([11, Lemma 4.1]). *Let M and N be right R-modules which are isomorphic to direct summands of each other. Then there exist sequences $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$, $(M_n)_{n\in\mathbb{N}^*}$ and $(N_n)_{n\in\mathbb{N}^*}$ such that for every $n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$ we have:*

1. $M_n \cong M$, $N_n \cong N$, $A_n \cong A_0$ *and* $B_n \cong B_0$;
2. $M_n = A_n \oplus N_{n+1}$ *and* $N_n = B_{n-1} \oplus M_n$;
3. $M = \left(\bigoplus_{m=0}^{n-1} A_m\right) \oplus \left(\bigoplus_{m=0}^{n-1} B_m\right) \oplus M_n$;
4. $N_1 = (\bigoplus_{m=1}^{n} A_m) \oplus \left(\bigoplus_{m=0}^{n-1} B_m\right) \oplus N_{n+1}$.

Lemma 3.4. *Let M and N be flat cotorsion pure-mono-equivalent right R-modules. Then there exist sequences $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$, $(M_n)_{n\in\mathbb{N}^*}$ and $(N_n)_{n\in\mathbb{N}^*}$ of flat cotorsion right R-modules such that the conditions (1)-(4) from Lemma 3.3 hold for every $n \in \mathbb{N}^*$.*

Proof. By Lemma 3.2, it follows that M and N are isomorphic to flat cotorsion direct summands of each other. The existence of the required sequences follows by Lemma 3.3. All their terms are direct summands of M and $N_1 \cong N$, and so they are flat cotorsion. □

The next result has also appeared as [7, Corollary 2.5], where it has a different proof.

Proposition 3.5. *Flat cotorsion pure-mono-equivalent right R-modules are isomorphic.*

Proof. Let M and N be flat cotorsion pure-mono-equivalent right R-modules. We follow the idea of proof of [10, Theorem 4]. By Lemma 3.4, there exist sequences $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$, $(M_n)_{n\in\mathbb{N}^*}$ and $(N_n)_{n\in\mathbb{N}^*}$ of flat cotorsion right R-modules such that for every $n \in \mathbb{N}^*$ the conditions (1)-(4) from Lemma 3.3 hold. For every $n \in \mathbb{N}^*$, let

$$C_n = \left(\bigoplus_{m=1}^{n} A_m\right) \oplus \left(\bigoplus_{m=0}^{n-1} B_m\right).$$

Then $(C_n)_{n\in\mathbb{N}^*}$ is an increasing sequence of pure submodules of M and N_1. Then $U = \bigcup_{n\in\mathbb{N}^*} C_n$ is a pure submodule of M and N_1, and we have:

$$U \cong \left(\bigoplus_{m\in\mathbb{N}^*} A_m\right) \oplus \left(\bigoplus_{m\in\mathbb{N}} B_m\right) \cong \left(\bigoplus_{m\in\mathbb{N}} A_m\right) \oplus \left(\bigoplus_{m\in\mathbb{N}} B_m\right) \cong A_0 \oplus U.$$

Since M and $N_1 \cong N$ are flat cotorsion, Lemma 3.2 implies that $M = C \oplus A$ and $N_1 = D \oplus B$ for some submodules A and $C \cong C(U)$ of M, and B and $D \cong C(U)$ of N_1. Since A_0 is cotorsion, we have $C(U) \cong C(A_0) \oplus C(U) \cong A_0 \oplus C(U)$. It follows that:

$$M \cong M_1 = A_0 \oplus N_1 \cong A_0 \oplus C(U) \oplus B \cong C(U) \oplus B \cong N_1 \cong N,$$

which finishes the proof. □

Now we may use the results on flat cotorsion modules in order to deduce the following theorem.

Theorem 3.6. *Let $\mathcal{C}$ be a finitely accessible additive category. Then pure-injective pure-mono-equivalent objects of $\mathcal{C}$ are isomorphic.*

Proof. We use the notation and the conclusion of Theorem 3.1. Let X and Y be pure-injective pure-mono-equivalent objects of $\mathcal{C}$. Then $H(X)$ and $H(Y)$ are flat cotorsion pure-mono-equivalent right R-modules. Then $H(X)$ and $H(Y)$ are isomorphic by Proposition 3.5. Since H is fully faithful, it follows that X and Y are isomorphic. □

Corollary 3.7. *Let $\mathcal{C}$ be a finitely accessible additive category. Then injective mono-equivalent objects of $\mathcal{C}$ are isomorphic.*

Since every object of a finitely accessible additive category has a pure-injective envelope, which is a pure monomorphism [9, Theorem 6], we immediately have the following corollary.

Corollary 3.8. *Let $\mathcal{C}$ be a finitely accessible additive category. Then pure-mono-equivalent objects of $\mathcal{C}$ have isomorphic pure-injective envelopes.*

Following [12], a full subcategory $\mathcal{D}$ of a finitely accessible additive category with products is called *definable* if $\mathcal{D}$ is closed under products, direct limits and pure subobjects. Note that definable subcategories of finitely accessible additive categories need not be finitely accessible. For instance, the category of divisible abelian groups is a definable subcategory of the category of abelian groups, but it is not finitely accessible [12, Example 10.3]. Since purity and pure-injectivity in a definable subcategory are just the restriction of purity and pure-injectivity in the larger finitely accessible additive category (e.g., see [12, Section 5]), we immediately deduce the following corollary.

Corollary 3.9. *Let $\mathcal{D}$ be a definable subcategory of a finitely accessible additive category with products. Then pure-injective pure-mono-equivalent objects of $\mathcal{D}$ are isomorphic.*

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