ON THE PARTITION DIMENSION OF INFINITE GRAPHS

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Little is known about the partition dimension of infinite graphs. Tomescu studied graphs where the set of vertices is the set of points of the integer lattice. We generalize these graphs and present several exact values, lower bounds and upper bounds on the partition dimension of infinite graphs.

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1. INTRODUCTION

The metric dimension is an invariant with extensive applications in robot navigation [8], pharmaceutical chemistry [4], pattern recognition and image processing [9]. The concept of metric dimension was introduced by Slater [10] and independently by Harary and Melter [6]. Slater referred to a metric dimension of a graph as its location number and motivated the study of this invariant by its application to the placement of minimum number of loran/sonar detecting devices in a network so that the position of every vertex in the network can be uniquely represented in terms of its distances to the devices in the set.

We investigate the partition dimension of graphs, which is an invariant defined more generally than the metric dimension. The partition dimension was introduced by Chartrand, Salehi and Zhang [5], who gave some basic results on the partition dimension of graphs. We consider connected infinite graphs G with the vertex set $V(G)$ and the edge set $E(G)$. The distance $d(u, v)$ between two vertices $u, v \in G$ is the number of edges in a shortest path connecting them. For a vertex v and a set $S \subseteq V(G)$ the distance between v and S is defined as

$$d(v, S) = \min\{d(v, u) \mid u \in S\}.$$

For two sets $S', S \subseteq V(G)$, the distance between S' and S is defined as $d(S', S) = \min\{d(v, u) \mid v \in S', u \in S\}$. Let $\Pi = \{S_1, S_2, \ldots, S_k\}$ be an ordered partition of $V(G)$. The partition representation of a vertex v with respect to Π is the k-tuple

$$r(v|\Pi) = (d(v, S_1), d(v, S_2), \ldots, d(v, S_k)).$$

If for every pair of distinct vertices $u, v \in V(G)$, we have $r(u|\Pi) \neq r(v|\Pi)$, then Π is a resolving partition and the minimum cardinality of a resolving partition of $V(G)$ is called the partition dimension of G and it is denoted by $pd(G)$. It follows that if for every pair of distinct vertices $u, v \in V(G)$, we have $d(u, S_j) \neq d(v, S_j)$ for some set S_j, where $j \in \{1, 2, \ldots, k\}$, then $\Pi = \{S_1, S_2, \ldots, S_k\}$ is a resolving partition of $V(G)$.

We define the ray, P_∞, as the graph with the vertex set $V(P_\infty) = \{v_i \mid i \in \mathbb{N}\}$ and the edge set $E(P_\infty) = \{v_i v_{i+1} \mid i \in \mathbb{N}\}$. The double ray $P_{2\infty}$, is the graph with the vertex set $V(P_{2\infty}) = \{v_i \mid i \in \mathbb{Z}\}$ and the edge set $E(P_{2\infty}) = \{v_i v_{i+1} \mid i \in \mathbb{Z}\}$. Let $P^z_{2\infty}$ be the graph with the vertex set $V(P^z_{2\infty}) = \{v_i \mid i \in \mathbb{Z}\}$ and the edge set $E(P^z_{2\infty}) = \{v_i v_{i+1}, v_i v_{i+2}, \ldots, v_i v_{i+z} \mid i \in \mathbb{Z}\}$. The Cartesian product $G \Box H$ of graphs G and H is a graph having the vertex set $V(G \Box H) = V(G) \times V(H)$ and any two vertices (u, u') and (v, v') are adjacent in $G \Box H$ if and only if either $u = v$ and u' is adjacent to v' in H, or $u' = v'$ and u is adjacent to v in G.

The partition dimension of infinite graphs was investigated in [11] and [12]. Tomescu and Imran [12] studied infinite regular graphs including planar triangular and hexagonal networks. Tomescu [11] studied graphs where the set of vertices is the set of points of the integer lattice. He showed that $pd(P_{2\infty} \Box P_{2\infty}) = 3$. We generalize graphs studied in [11] by considering the graphs $P^z_{2\infty} \Box P_{2\infty}$ for any $z \geq 2$ and $P^z_{2\infty} \Box P^z_{2\infty}$ for $z = 2$.

Note that the partition dimension of graph products was considered also in [1, 2, 13, 14], the partition dimension of disconnected graphs was investigated in [7] and the metric dimension of infinite graphs was studied in [3].

2. RESULTS

First we prove that the ray and the double ray are the only infinite graphs having the partition dimension 2. Chartrand, Salehi and Zhang [5] proved a similar result for finite graphs. Their method can be used to obtain Theorem 2.1. We present an alternative proof.

THEOREM 2.1. *Let G be an infinite graph. We have $pd(G) = 2$ if and only if G is P_∞ or $P_{2\infty}$.*

Proof. Note that the vertex sets and edge sets of P_∞ and $P_{2\infty}$ are defined in the previous section.

Let $S_1 = \{v_1\}$ for P_∞ and $S_1 = \{v_i \mid i \leq 1\}$ for $P_{2\infty}$. Let $S_2 = \{v_i \mid i \geq 2\}$. Clearly, $\Pi = \{S_1, S_2\}$ is a resolving partition, thus $pd(P_\infty) = 2$ and $pd(P_{2\infty}) = 2$.

It remains to show that if G is not P_∞ or $P_{2\infty}$, then $pd(G) > 2$. Assume to the contrary that $pd(G) = 2$ and let $\Pi' = \{S'_1, S'_2\}$ be a resolving partition of G. G contains a vertex, say v, adjacent to at least 3 vertices, say v_1, v_2, v_3 (since every infinite graph except for P_∞ or $P_{2\infty}$ contains a vertex of degree at least 3). We can assume that $v \in S'_1$. Then 2 vertices adjacent to v cannot be in S'_2, otherwise their representation would be $(0, 1)$. So, without loss of generality we can assume that $v_1, v_2 \in S'_1$.

If $v_3 \in S'_2$, then the representations of v, v_1, v_2 would be $(0, 1)$ or $(0, 2)$, so we would have two vertices having the same representations. Thus $v_3 \in S'_1$. Let $V' = \{v, v_1, v_2, v_3\}$ and let $d(V', S_2) = x$. Since the distance between any two vertices in V' is at most 2, the representation of any vertex in V' is $(x, 0), (x + 1, 0)$ or $(x + 2, 0)$. Hence G contains at least 2 vertices with the same representations. A contradiction. $\square$

We use the notation $v_{i,j}$ instead of (v_i, v_j) for the vertices of the Cartesian products of graphs investigated in this paper.

The distance between two vertices $v_{i,j}$ and $v_{k,l}$ in $P^z_{2\infty}\Box P_{2\infty}$, where $i \le k$ is

$$d(v_{i,j}, v_{k,l}) = \left\lceil \frac{k-i}{z} \right\rceil + |l - j|. \tag{1}$$

We present an upper bound on the partition dimension of the graphs $P^z_{2\infty}\Box P_{2\infty}$.

THEOREM 2.2. *$pd(P^z_{2\infty}\Box P_{2\infty}) \le z + 2$ for every $z \ge 2$.*

Proof. Let G be the graph $P^z_{2\infty}\Box P_{2\infty}$, where $z \ge 2$. Let $S_0^+ = \{v_{i,j'} \mid i \le 0,\ j' > 0\}$ and $S_0^- = \{v_{i,j} \mid i \le 0,\ j \le 0\}$. For $p = 1, 2, \ldots, z - 1$ let $S_p = \{v_{p,j} \mid j \in \mathbb{Z}\}$ and $S_z = \{v_{k,l} \mid k \ge z,\ l \in \mathbb{Z}\}$, We show that $\Pi = \{S_0^+, S_0^-, S_1, S_2, \ldots, S_z\}$ is a resolving partition of $V(G)$.

First we show that all vertices in S_p for $p = 1, 2, \ldots, z - 1$ are resolved by S_0^+ and S_0^-. Among the vertices in S_0^+, $v_{0,1}$ is the closest vertex to $v_{p,j} \in S_p$ for any $j \le 0$. From (1) we have $d(v_{0,1}, v_{p,j}) = d(v_{p,j}, S_0^+) = 2 - j$. For $v_{p,j} \in S_p$ and $v_{0,j} \in S_0^-$, where $j \le 0$, we have $d(v_{0,j}, v_{p,j}) = 1$, thus $d(v_{p,j}, S_0^-) = 1$.

For $v_{p,j'} \in S_p$ and $v_{0,j'} \in S_0^+$, where $j' > 0$, we have $d(v_{0,j'}, v_{p,j'}) = 1$, thus $d(v_{p,j'}, S_0^+) = 1$. Among the vertices in S_0^-, $v_{0,0}$ is the closest vertex to $v_{p,j'} \in S_p$ for any $j' > 0$. From (1) we have $d(v_{0,0}, v_{p,j'}) = d(v_{p,j'}, S_0^-) = j' + 1$. So

$$r(v_{p,j}, \{S_0^+, S_0^-\}) = (2 - j, 1) \quad \text{and} \quad r(v_{p,j'}, \{S_0^+, S_0^-\}) = (1, j' + 1).$$

Hence the vertices in S_p for $p = 1, 2, \ldots, z - 1$ are resolved.

Let $S_z^+ = \{v_{k,l'} \mid k \geq z,\, l' > 0\}$ and $S_z^- = \{v_{k,l} \mid k \geq z,\, l \leq 0\}$. Clearly $S_z = S_z^+ \cup S_z^-$. We show that a vertex in S_z^+ and a vertex in S_z^- cannot have the same representations.

Among the vertices in S_0^+, $v_{0,1}$ is the closest vertex to $v_{k,l} \in S_z^-$. From (1) we have $d(v_{0,1}, v_{k,l}) = d(v_{k,l}, S_0^+) = \lceil \frac{k}{z} \rceil + 1 - l \geq \lceil \frac{k}{z} \rceil + 1$ since $l \leq 0$. Among the vertices in S_0^-, $v_{0,l}$ is the closest vertex to $v_{k,l} \in S_z^-$, so $d(v_{0,l}, v_{k,l}) = d(v_{k,l}, S_0^-) = \lceil \frac{k}{z} \rceil$.

Among the vertices in S_0^+, $v_{0,l'}$ is the closest vertex to $v_{k,l'} \in S_z^+$, so $d(v_{0,l'}, v_{k,l'}) = d(v_{k,l'}, S_0^+) = \lceil \frac{k}{z} \rceil$. Among the vertices in S_0^-, $v_{0,0}$ is the closest vertex to $v_{k,l'} \in S_z^+$. From (1) we have $d(v_{0,0}, v_{k,l'}) = d(v_{k,l'}, S_0^-) = \lceil \frac{k}{z} \rceil + l' \geq \lceil \frac{k}{z} \rceil + 1$ since $l' > 0$.

We have

$$r(v_{k,l}, \{S_0^+, S_0^-\}) = \left(\left\lceil \frac{k}{z} \right\rceil + 1 - l, \left\lceil \frac{k}{z} \right\rceil \right)$$

and

$$r(v_{k,l'}, \{S_0^+, S_0^-\}) = \left(\left\lceil \frac{k}{z} \right\rceil, \left\lceil \frac{k}{z} \right\rceil + l' \right).$$

The first coordinate of the representation $r(v_{k,l}, \{S_0^+, S_0^-\})$ is greater than the second coordinate. On the other hand, the first coordinate of $r(v_{k,l'}, \{S_0^+, S_0^-\})$ is smaller than the second coordinate, therefore a vertex in S_z^+ and a vertex in S_z^- cannot have the same representations.

It remains to show that no two vertices in S_0^+, no two vertices in S_0^-, no two vertices in S_z^+ and no two vertices in S_z^- have the same representations with respect to Π. These four cases are very similar, thus we only show that any two different vertices in S_z^+ have different representations.

For $v_{tz+\epsilon,l'} \in S_z^+$ and $v_{0,l'} \in S_0^+$ we have $d(v_{0,l'}, v_{tz+\epsilon,l'}) = d(v_{tz+\epsilon,l'}, S_0^+) = t+1$, where $1 \leq \epsilon \leq z$ and $t \geq 1$, which means that vertices of S_z^+ with the first coordinate $z+1, z+2, \ldots, 2z$ have the same representations with respect to S_0^+, vertices of S_z^+ with the first coordinate $2z+1, 2z+2, \ldots, 3z$ have the same representations with respect to S_0^+, and so on.

For $v_{tz+p+\epsilon,l'} \in S_z^+$ and $v_{p,l'} \in S_p$ where $l' > 0$ we have $d(v_{p,l'}, v_{tz+p+\epsilon,l'}) = d(v_{tz+p+\epsilon,l'}, S_p) = t+1$, where $1 \leq \epsilon \leq z$ and $t \geq 0$. This means that for $p = 1, 2, \ldots, z-1$, vertices of S_z^+ with the first coordinates $1+p, 2+p, \ldots, z+p$ have the same representations with respect to S_p (note that the first coordinate must be at least z for the vertex to be in S_z^+), vertices of S_z^+ with the first coordinates $z+1+p, z+2+p, \ldots, 2z+p$ have the same representations with respect to S_p, vertices with the first coordinates $2z+1+p, 2z+2+p, \ldots, 3z+p$ have the same representations, and so on.

It follows that the vertices $v_{z,1}, v_{z,2}, v_{z,3}, \ldots$ have the same representations with respect to $S_0^+, S_1, S_2, \ldots, S_{z-1}$, the vertices $v_{z+1,1}, v_{z+1,2}, v_{z+1,3}, \ldots$

have the same representations, and so on. So any two vertices of S_z^+ that have the same first coordinate have the same representations with respect to $S_0^+, S_1, S_2, \ldots, S_{z-1}$. We show that these vertices are resolved by S_0^-. Among the vertices in S_0^-, $v_{0,0}$ is the closest vertex to $v_{k,l'} \in S_z^+$. From (1) we have $d(v_{0,0}, v_{k,l'}) = d(v_{k,l'}, S_0^-) = \lceil \frac{k}{z} \rceil + l'$. Hence the vertices $v_{k,1}, v_{k,2}, v_{k,3}, \ldots$ are resolved by S_0^- for any $k \geq z$ since $d(v_{k,1}, S_0^-) = \lceil \frac{k}{z} \rceil + 1$, $d(v_{k,2}, S_0^-) = \lceil \frac{k}{z} \rceil + 2$, $d(v_{k,3}, S_0^-) = \lceil \frac{k}{z} \rceil + 3, \ldots$, which means that any two vertices in S_z^+ have different representations with respect to Π. $\square$

By Theorem 2.1 we have $pd(P_{2\infty}^2 \Box P_{2\infty}) \geq 3$ and from Theorem 2.2 we get $pd(P_{2\infty}^2 \Box P_{2\infty}) \leq 4$, thus we obtain Corollary 2.1.

COROLLARY 2.1. $3 \leq pd(P_{2\infty}^2 \Box P_{2\infty}) \leq 4$.

Let us present a lower bound on the partition dimension of $P_{2\infty}^3 \Box P_{2\infty}$.

THEOREM 2.3. *$pd(P_{2\infty}^3 \Box P_{2\infty}) \geq 4$.*

Proof. Let G be the graph $P_{2\infty}^3 \Box P_{2\infty}$. By Theorem 2.1 we have $pd(G) \geq 3$. We prove Theorem 2.3 by contradiction. Assume that $pd(G) = 3$. Let $\Pi = \{S_1, S_2, S_3\}$ be a resolving partition of $V(G)$. Let $V' = \{v_{p,r},\ v_{p+1,r},\ v_{p+2,r},\ v_{p+3,r},\ v_{p,r+1},\ v_{p+1,r+1},\ v_{p+2,r+1},\ v_{p+3,r+1}\}$ be a set such that $V' \nsubseteq S_i$ for some $i = 1, 2, 3$. Clearly, such a set exists, otherwise we would have $V(G) = S_i$ and $pd(G) = 1$. Note that any two vertices in V' are of distance at most 2, thus if $d(v, V') = x$ for $v \in V(G)$, then for any $v' \in V'$, we have

$$x \leq d(v, v') \leq x + 2. \tag{2}$$

Case 1. $|V' \cap S_i| = \emptyset$ for some $i = 1, 2, 3$.
Say $|V' \cap S_3| = \emptyset$. So $|V' \cap S_1| + |V' \cap S_2| = 8$. Without loss of generality we can assume that $|V' \cap S_1| \geq 4$. Then it is easy to check that there are at least 4 vertices in $|V' \cap S_1|$ having distance 1 from S_2 and these 4 vertices cannot be resolved by S_3, since by (2), $x \leq d(v', S_3) \leq x + 2$ for any $v' \in V'$.

Case 2. $|V' \cap S_i| \geq 1$ for each $i = 1, 2, 3$.
The vertices $v_{p,r'}$, $v_{p+1,r'}$, $v_{p+2,r'}$, $v_{p+3,r'}$, where $r' = r$ or $r + 1$, do not belong to 3 different sets S_1, S_2, S_3, otherwise there is a set, say S_1, containing 2 of these vertices and their representations would be $(0, 1, 1)$.

If $\{v_{p,r},\ v_{p+1,r},\ v_{p+2,r},\ v_{p+3,r}\} \subseteq S_i$ for some $i = 1,\ 2,\ 3$, say $\{v_{p,r},\ v_{p+1,r},\ v_{p+2,r},\ v_{p+3,r}\} \subseteq S_1$, then from the previous sentence $\{v_{p,r+1},\ v_{p+1,r+1},\ v_{p+2,r+1},\ v_{p+3,r+1}\} \subseteq S_2 \cup S_3$. At least 2 of these vertices would belong to one of the sets S_2, S_3, say S_2, and these vertices would have the representations $(1, 0, 1)$.

So the vertices $v_{p,r}$, $v_{p+1,r}$, $v_{p+2,r}$, $v_{p+3,r}$ belong to exactly two sets, say S_1, S_2, and equivalently it can be shown that the vertices $v_{p,r+1}$, $v_{p+1,r+1}$, $v_{p+2,r+1}$, $v_{p+3,r+1}$ belong to two sets, say S_1, S_3.

Clearly, 3 of the vertices $v_{p,r}$, $v_{p+1,r}$, $v_{p+2,r}$, $v_{p+3,r}$ cannot be in the same set, say S_1, because they would be of distance 1 from S_2 and of distance 1 or 2 from S_3 (so there would be 2 vertices having the same representations). Therefore, exactly 2 of the vertices $v_{p,r}$, $v_{p+1,r}$, $v_{p+2,r}$, $v_{p+3,r}$ are in S_1 and the other two vertices are in S_2. Similarly, two of the vertices $v_{p,r+1}$, $v_{p+1,r+1}$, $v_{p+2,r+1}$, $v_{p+3,r+1}$ are in S_1 and the other two vertices are in S_3.

But then we have 4 vertices of S_1 in V' and they have representations $(0,1,1),(0,1,2)$ or $(0,2,1)$, since each vertex in S_1 has distance 1 from S_2 or S_3. This implies that two vertices of S_1 have the same representations. A contradiction. $\square$

From Theorems 2.2 and 2.3 we obtain the following corollary.

Corollary 2.2. $4 \leq pd(P_{2\infty}^3 \Box P_{2\infty}) \leq 5$.

Tomescu [11] proved that $pd(P_{2\infty} \Box P_{2\infty}) = 3$. We show that we can obtain the same result for the graphs $P_m \Box P_{2\infty}$ and $P_\infty \Box P_{2\infty}$.

Let G be the graph $P_m \Box P_{2\infty}$ or $P_\infty \Box P_{2\infty}$. Let $0 \leq i \leq k$ and let j, l be any integers. The distance between the vertices $v_{i,j}$ and $v_{k,l}$ in G is $d(v_{i,j}, v_{k,l}) = k - i + |l - j|$. If $i = 0$, we obtain

$$d(v_{0,j}, v_{k,j+t}) = k + |t|. \tag{3}$$

Theorem 2.4. *$pd(P_m \Box P_{2\infty}) \leq 3$ and $pd(P_\infty \Box P_{2\infty}) \leq 3$.*

Proof. Let G be the graph $P_m \Box P_{2\infty}$. Let $S_1 = \{v_{0,t} \mid t \geq 1, t \in \mathbb{Z}\}$, $S_2 = \{v_{0,j} \mid j \leq 0, j \in \mathbb{Z}\}$ and $S_3 = \{v_{k,l} \mid 1 \leq k \leq m-1, l \in \mathbb{Z}\}$. We show that $\Pi = \{S_1, S_2, S_3\}$ is a resolving partition of $V(G)$.

First we show that all vertices in S_1 are resolved by S_2. Since among the vertices in S_2, $v_{0,0}$ is the closest vertex to $v_{0,t} \in S_1$ for any $t \geq 1$, from (3) we have $d(v_{0,t}, S_2) = t$. Thus all vertices in S_1 have unique partition representations with respect to Π.

Similarly, among the vertices in S_1, $v_{0,1}$ is the closest vertex to $v_{0,j} \in S_2$ for any $j \leq 0$, therefore by (3), $d(v_{0,j}, S_1) = 1 - j$, so no two vertices in S_2 have the same partition representations.

It remains to prove that all vertices of S_3 are resolved by S_1 and S_2. It suffices to show that the vertices in S_3 of distance p from S_2 (where $p \geq 1$) are resolved by S_1. Let $S_3 = V' \cup V''$ where $V' = \{v_{k,l'} \mid 1 \leq k \leq m-1, l' \geq 1\}$ and $V'' = \{v_{k,l} \mid 1 \leq k \leq m-1, l \leq 0\}$. For any $v_{p,l} \in V''$ and $v_{0,l} \in S_2$ $(1 \leq p \leq m-1, l \leq 0)$, by (3) we have $d(v_{0,l}, v_{p,l}) = p$, thus $d(v_{p,l}, S_2) = p$.

We find all vertices in V' of distance p from S_2. Note that among the vertices in S_2, $v_{0,0}$ is the closest vertex to any $v_{k,l'} \in V'$ $(1 \leq k \leq m-1, l' \geq 1)$, therefore $d(v_{k,l'}, S_2) = d(v_{k,l'}, v_{0,0}) = k + l'$. It follows that the vertices $v_{k,l'}$ of distance p from S_2 satisfy $k + l' = p$, which implies that $v_{k,p-k}$ for $1 \le k < p$ are the vertices of V' at distance p from S_2.

We show that S_1 resolves the set $V_p = \{v_{p,l} \mid l \le 0, p \le m-1\} \cup \{v_{k,p-k} \mid 1 \le k < p\}$, where p is any positive integer. Among the vertices in S_1, $v_{0,1}$ is the closest vertex to $v_{p,l}$, therefore

$$d(v_{p,l}, v_{0,1}) = d(v_{p,l}, S_1) = p + |l-1| = p + 1 - l$$

where $l \le 0$, and $v_{0,p-k} \in S_1$ is the closest vertex to $v_{k,p-k}$, thus

$$d(v_{k,p-k}, v_{0,p-k}) = d(v_{k,p-k}, S_1) = k = p - l'$$

where $l' \ge 1$. It is easy to see that no two vertices in V_p are of the same distance from S_1, hence Π is a resolving partition of $V(G)$.

If G is the graph $P_\infty \square P_{2\infty}$, the only modification of the proof is to remove the upper bound $m-1$ which is included in several sets considered in the proof of $pd(P_m \square P_{2\infty}) \le 3$. $\square$

Theorems 2.1 and 2.4 yield Corollary 2.3.

COROLLARY 2.3. *$pd(P_m \square P_{2\infty}) = 3$ and $pd(P_\infty \square P_{2\infty}) = 3$.*

Finally, we consider the graphs $P^z_{2\infty} \square P^z_{2\infty}$ for $z = 2$. We present a lower bound on the partition dimension of the graph $P^2_{2\infty} \square P^2_{2\infty}$. The distance between two vertices $v_{i,j}$ and $v_{k,l}$ in $P^2_{2\infty} \square P^2_{2\infty}$ is

$$d(v_{i,j}, v_{k,l}) = \left\lceil \frac{|k-i|}{2} \right\rceil + \left\lceil \frac{|l-j|}{2} \right\rceil.$$

THEOREM 2.5. *$pd(P^2_{2\infty} \square P^2_{2\infty}) \ge 4$.*

Proof. Let $G = P^2_{2\infty} \square P^2_{2\infty}$. By Theorem 2.1 we have $pd(G) \ge 3$. We prove Theorem 2.5 by contradiction. Assume that $pd(G) = 3$. Let $\Pi = \{S_1, S_2, S_3\}$ be a resolving partition of $V(G)$. Let $V' = \{v_{p,r},\ v_{p,r+1},\ v_{p,r+2},\ v_{p+1,r},\ v_{p+1,r+1},\ v_{p+1,r+2},\ v_{p+2,r},\ v_{p+2,r+1},\ v_{p+2,r+2}\}$ where $p, r \in \mathbb{Z}$. Note that any two vertices in V' are of distance at most 2, thus if $d(v, V') = x$ for $v \in V(G)$, then for any $v' \in V'$, we have $x \le d(v, v') \le x + 2$.

Without loss of generality we can assume that $|V' \cap S_1| \ge |V' \cap S_2| \ge |V' \cap S_3|$. We distinguish a few cases.

Case 1. $V' \subseteq S_1$.

Let $d(V', S_2) = x$. Then $x \le d(v', S_2) = x + 2$ for any $v' \in V'$ and $y \le d(v', S_3) = y + 2$ for some positive integer y and any $v' \in V'$.

If there are at most 2 vertices of V' at distance x from S_2, then we have 4 vertices of V' at distance $x+1$ from S_2 or 4 vertices of V' are distance $x+2$ from S_2. These 4 vertices cannot be resolved by S_3 since for any $v' \in V'$,

$y \le d(v', S_3) = y + 2$ (2 vertices in V' have the same representations with respect to S_2 and S_3).

If there are at least 3 vertices of V' at distance x from S_2, it is easy to check that there is at most one vertex of V' at distance $x+2$ from S_2. Thus we have 4 vertices of V' at distance x from S_2 or 4 vertices of V' are of distance $x+1$ from S_2, which again cannot be resolved by S_3.

Case 2. $7 \le |V' \cap S_1| \le 8$.
Then $|V' \cap S_2| \ge 1$. Since $|V' \cap S_1| \ge 7$, there are 4 vertices in $V' \cap S_1$ at distance 1 from S_2 or 4 vertices in $V' \cap S_1$ at distance 2 from S_2. These 4 vertices cannot be resolved by S_3, since for any $v' \in V'$, $y \le d(v', S_3) = y + 2$ for some positive integer y.

Case 3. $|V' \cap S_1| = 6$.
We can assume that $2 \le |V' \cap S_2| \le 3$. Let $u, u' \in V' \cap S_2$. Then there are at least 4 vertices in $V' \cap S_1$ at distance 1 from S_2 (the vertices of S_1 that have the first or second coordinate same as u or u'). These 4 vertices cannot be resolved by S_3.

Case 4. $|V' \cap S_1| = 5$ and $|V' \cap S_2| = 4$.
Every vertex u in $V' \cap S_2$ is at distance 2 from exactly 4 vertices in V'. Since $|V' \cap S_1| > 4$, there exists a vertex in $V' \cap S_1$ at distance 1 from u. Thus $d(u, S_1) = 1$ for every u in $V' \cap S_2$, and 4 vertices of $V' \cap S_2$ cannot be resolved by S_3.

Case 5. $|V' \cap S_1| = 5$, $V' \cap S_2 \ne \emptyset$ and $V' \cap S_3 \ne \emptyset$.
Since every vertex in $V' \cap S_1$ is at distance 1 or 2 from S_2, we either have (at least) 3 vertices in $V' \cap S_1$ at distance 1 from S_2 or 3 vertices in $V' \cap S_1$ at distance 2 from S_2. Those 3 vertices cannot be resolved by S_3 (since every vertex in $V' \cap S_1$ is at distance 1 or 2 from S_3).

Case 6. $|V' \cap S_1| = 4$, $V' \cap S_2 \ne \emptyset$ and $V' \cap S_3 \ne \emptyset$.
The partition representation of any vertex in $V' \cap S_1$ is $(0, x, y)$ where $x, y \le 2$, which implies that 4 vertices in $V' \cap S_1$ have representations $(0, 1, 1)$, $(0, 1, 2)$, $(0, 2, 1)$, $(0, 2, 2)$. The vertex, say u, with the representation $(0, 2, 2)$ must be at distance 2 from all 5 vertices in $(V' \cap S_2) \cup (V' \cap S_3)$, which is impossible, since there at exactly 4 vertices in V' at distance 2 from u.

Case 7. $|V' \cap S_1| = |V' \cap S_2| = |V' \cap S_3| = 3$.
A vertex $u \in V'$ cannot have the representation $(0, 2, 2)$, because V' contains only 4 vertices at distance 2 from u (and $|V' \cap S_2| + |V' \cap S_3| = 6$). Similarly, no vertex of V' has the representation $(2, 0, 2)$ or $(2, 2, 0)$. So the vertices in V' have the representations $(0, 1, 1)$, $(0, 1, 2)$, $(0, 2, 1)$, $(1, 0, 1)$, $(1, 0, 2)$, $(2, 0, 1)$, $(1, 1, 0)$, $(1, 2, 0)$ and $(2, 1, 0)$.

Without loss of generality we can assume that $v_{p,r}$ has the representation $(0, 1, 2)$. Since, $d(v_{p,r}, S_3) = 2$, no vertex of V' which is in S_3 has the first coordinate p or the second coordinate r. So $V' \cap S_3 \subseteq \{v_{p+1,r+1},\ v_{p+1,r+2},\ v_{p+2,r+1},\ v_{p+2,r+2}\}$. Then $v_{p,r}$ is the only vertex in V' having distance 2 from S_3 and no vertex can have the representation $(1,0,2)$. A contradiction. $\square$

3. CONCLUSION

We proved that

$$3 \leq pd(P_{2\infty}^2 \Box P_{2\infty}) \leq 4 \leq pd(P_{2\infty}^3 \Box P_{2\infty}) \leq 5 \quad \text{and} \quad pd(P_{3\infty}^2 \Box P_{2\infty}) \leq z+2$$

for any $z \geq 4$. To find exact values of the partition dimension for the graphs $P_{2\infty}^z \Box P_{2\infty}$ is an open problem.

Problem 3.1. Find exact values of $pd(P_{2\infty}^z \Box P_{2\infty})$ for $z \geq 2$.

We showed that $pd(P_{2\infty}^2 \Box P_{2\infty}^2) \geq 4$. We believe that the partition dimension of $P_{2\infty}^2 \Box P_{2\infty}^2$ is 4, therefore we state the following conjecture.

Conjecture 3.1. *$pd(P_{2\infty}^2 \Box P_{2\infty}^2) = 4$.*

It would be interesting to study the partition dimension also for the graphs $P_{2\infty}^z \Box P_{2\infty}^z$ where $z \geq 3$. Thus we introduce Problem 3.2.

Problem 3.2. Give lower and upper bounds on $pd(P_{2\infty}^z \Box P_{2\infty}^z)$ for $z \geq 3$.

These problems remain open for future research.

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REFERENCES

[1] R. Alfarisi, Darmaji, and Dafik, *On the star partition dimension of comb product of cycle and complete graph.* J Physics: Conference Series **855** (2017), 012005.

[2] E. T. Baskoro and Darmaji, *The partition dimension of the corona product of two graphs.* Far East J. Math. Sci. (FJMS) **66** (2012). 181–196.

[3] J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo, and M. L. Puertas, *On the metric dimension of infinite graphs.* Discrete Appl. Math. **160** (2012), 2618–2626.

[4] G. Chartrand, L. Eroh, M. A. Johnson, and O. R. Oellermann, *Resolvability in graphs and the metric dimension of a graph.* Discrete Appl. Math. **105** (2000), 99–113.

[5] G. Chartrand, E. Salehi, and P. Zhang, *The partition dimension of a graph.* Aequationes Math. **59** (2000), 45–54.

[6] F. Harary and R. A. Melter, *On the metric dimension of a graph*. Ars Comb. **2** (1976), 191–195.

[7] D. O. Haryeni, E. T. Baskoro, S. W. Saputro, M. Bača, and A. Semaničová-Feňovčíková, *On the partition dimension of two-component graphs*. Proc. Indian Acad. Sci. (Math. Sci.) **127** (2017), 755–767.

[8] S. Khuller, B. Raghavachari, and A. Rosenfeld, *Landmarks in graphs*. Discrete Appl. Math. **70** (1996), 217–229.

[9] R. A. Melter and I. Tomescu, *Metric bases in digital geometry*. Comput. Vision Graphics Image Process. **25** (1984), 113–121.

[10] P. J. Slater, *Leaves of trees*. Congress. Numer. **14** (1975), 549–559.

[11] I. Tomescu, *Discrepancies between metric dimension and partition dimension of a connected graph*. Discrete Math. **308** (2008), 5026–5031.

[12] I. Tomescu and M. Imran, *On metric and partition dimensions of some infinite regular graphs*. Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **52** (2009), 461–472.

[13] I. G. Yero, M. Jakovac, D. Kuziak, and A. Taranenko, *The partition dimension of strong product graphs and Cartesian product graphs*. Discrete Math. **331** (2014), 43–52.

[14] G. I. Yero and J. A. Rodríguez-Velázquez, *A note on the partition dimension of Cartesian product graphs*. Appl. Math. Comput. **217** (2010), 3571–3574.

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