THE FULL AUTOMORPHISM GROUP OF THE NON-CYCLIC GRAPH OF A FINITE GROUP

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For a finite non-cyclic group G, let $\mathrm{Cyc}(G)$ be the set of elements a of G such that $\langle a, b\rangle$ is cyclic for each b of G. The non-cyclic graph of G is the graph with vertex set $G \setminus \mathrm{Cyc}(G)$, two distinct vertices x and y are adjacent if $\langle x, y\rangle$ is not cyclic. In this paper, we characterize the full automorphism group of the non-cyclic graph of a finite group. As applications, we compute the full automorphism group of the non-cyclic graph of an elementary abelian group, a dihedral group, a semi-dihedral group, and a generalized quaternion group.

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1. INTRODUCTION

In this paper, G always denotes a finite non-cyclic group. The *cyclicizer* $\mathrm{Cyc}(G)$ of G is the set of elements a of G such that $\langle a, b\rangle$ is cyclic for each b of G. It has been proved in [12], that $\mathrm{Cyc}(G)$ is a normal cyclic subgroup of G. The non-cyclic graph Γ_G of G is the graph whose vertex set is $G \setminus \mathrm{Cyc}(G)$, and two distinct vertices are adjacent if they do not generate a cyclic subgroup.

Graphs associated with groups and other algebraic structures have been actively investigated, since they have valuable applications (cf. [9]) and are related to automata theory (cf. [7]). In 2007, Abdollahi and Hassanabadi [2] introduced the concept of a non-cyclic graph and established basic graph theoretical properties. In [3], Abdollahi and Hassanabadi investigated the clique number of a non-cyclic graph. Recently, Costa *et al.* [6] studied the Eulerian properties of non-cyclic graphs of finite groups. Finite groups whose non-cyclic graphs have genus one were classified by Selvakumar and Subajini [13] and, independently, by Ma and Su [10]. Moreover, star-free non-cyclic graphs were studied in [11]. In 2017, Aalipour *et al.* [1] investigated the relationship

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between the complement graph of a non-cyclic graph and two well-studied graphs–power graphs (cf. [5, 8]) and commuting graphs (cf. [4]).

In this paper, we characterize the full automorphism group $\mathrm{Aut}(\Gamma_G)$ of the non-cyclic graph of G. As applications, we compute $\mathrm{Aut}(\Gamma_G)$ if G is elementary abelian, dihedral, semi-dihedral and generalized quaternion.

2. MAIN RESULT

Recall that G is a finite non-cyclic group. Denote by $\mathcal{C}_G$ the set of all cyclic subgroups C of G such that $C \nsubseteq \mathrm{Cyc}(G)$. A cyclic subgroup $\langle x\rangle$ is said to be *maximal* in G if $\langle x\rangle \subseteq \langle y\rangle$ implies $\langle x\rangle = \langle y\rangle$. The set of all maximal cyclic subgroups of G is denoted by $\mathcal{M}_G$. It was noted in [12] that

$$\mathrm{Cyc}(G) = \bigcap_{C\in\mathcal{M}_G} C.$$

Assume now that G has precisely k maximal cyclic subgroups, that is,

$$\mathcal{M}_G = \{C_1, C_2, \ldots, C_k\}.$$

Suppose that for any $1 \le i \le k$, C_i has s_i generators. The set of all generators of C_i is denoted by $[C_i]$. Write

$$[C_i] = \{[C_i]_1, [C_i]_2, \ldots, [C_i]_{s_i}\},\ \mathcal{C}_G \setminus \mathcal{M}_G = \{C_{k+1}, C_{k+2}, \ldots, C_m\}$$

Then it is easy to see that

$$V(\Gamma_G) = \bigcup_{i=1}^{m} [C_i].$$

Now we define $P_{\mathcal{C}_G}$ as the set of all permutations α on $\mathcal{C}_G$ satisfying:
(*a*) $|C_i^\alpha| = |C_i|$ for all $C_i \in \mathcal{C}_G$;
(*b*) $C_i \subseteq C_j$ if and only if $C_i^\alpha \subseteq C_j^\alpha$.

It is easy to check that $P_{\mathcal{C}_G}$ is a permutation group on $\mathcal{C}_G$. Denote by S_Ω the symmetric group on a set Ω. Suppose that $[C_i]_j$ is an arbitrary vertex of $V(\Gamma_G)$. Let

$$f:\ P_{\mathcal{C}_G} \longrightarrow S_{V(\Gamma_G)}$$

be a mapping such that $[C_i]_j^{\alpha^f} = [C_i^\alpha]_j$ for all $\alpha \in P_{\mathcal{C}_G}$, where α^f is the value of α under f. Note that f is a monomorphism. So $P_{\mathcal{C}_G}$ is isomorphic to a subgroup of $S_{V(\Gamma_G)}$.

In the non-cyclic graph Γ_G, the neighborhood of a vertex a, denoted by $N(a)$, is the set of all vertices that are adjacent to a. Now we define an equivalence relation $\equiv$ on $V(\Gamma_G)$ by the rule that for $x, y \in V(\Gamma_G)$, $x \equiv y$ if $N(x) = N(y)$. Let $\overline{w}$ denote the $\equiv$-class containing vertex w. Write

$$\mathcal{W}_G = \{\overline{w} : w \in V(\Gamma_G)\} = \{\overline{w_1}, \overline{w_2}, \ldots, \overline{w_t}\}.$$

Note that the $\overline{w_2}, \ldots, \overline{w_t}$ are distinct. Let

$$f' : \prod_{i=1}^{t} S_{\overline{w_i}} \longrightarrow S_{V(\Gamma_G)}$$

be a mapping satisfying $[C_i]_j^{(\beta_1,\beta_2,\ldots,\beta_t)^{f'}} = ([C_i]_j)^{\beta_l}$ for all $(\beta_1, \beta_2, \ldots, \beta_t) \in \prod_{i=1}^{t} S_{\overline{w_i}}$, where $[C_i]_j \in \overline{w_l}$ for some $l \in \{1, 2, \ldots, t\}$. It is easy to check that f' is a monomorphism. Thus, we see that $\prod_{i=1}^{t} S_{\overline{w_i}}$ is isomorphic to a subgroup of $S_{V(\Gamma_G)}$.

Now, let $P_{\mathcal{C}_G}$ and $\prod_{i=1}^{t} S_{\overline{w_i}}$ act on $V(\Gamma_G)$ as following:

$$[C_i]_j^{\alpha} = [C_i^{\alpha}]_j, \ [C_i]_j^{(\beta_1,\beta_2,\ldots,\beta_t)} = ([C_i]_j)^{\beta_l}, \tag{1}$$

where $\alpha \in P_{\mathcal{C}_G}$, $(\beta_1, \beta_2, \ldots, \beta_t) \in \prod_{i=1}^{t} S_{\overline{w_i}}$, and $[C_i]_j \in \overline{w_l}$ for some $l \in \{1, 2, \ldots, t\}$.

The goal of the paper is to characterize the full automorphism group of the non-cyclic graph of a finite non-cyclic group. Our main result is the following theorem.

THEOREM 2.1. *Let G be a finite non-cyclic group. Then*

$$\mathrm{Aut}(\Gamma_G) = P_{\mathcal{C}_G} \ltimes \prod_{i=1}^{t} S_{\overline{w_i}},$$

where $P_{\mathcal{C}_G}$ and $\prod_{i=1}^{t} S_{\overline{w_i}}$ act on $V(\Gamma_G)$ as in (1).

3. PROOF OF THEOREM 2.1

For $x \in V(\Gamma_G)$, denote by $\mathcal{M}_G(x)$ the set of all maximal cyclic subgroups of G containing x. Then $N(x) = G \setminus \bigcup_{C \in \mathcal{M}_G(x)} C$. Note that x and y are adjacent in Γ_G if and only if $\mathcal{M}_G(x) \cap \mathcal{M}_G(y) = \emptyset$.

FACT 1. *Let $x, y \in V(\Gamma_G)$. Then $x \equiv y$ if and only if $\mathcal{M}_G(x) = \mathcal{M}_G(y)$.*

Note that every automorphism of a graph maps edges to edges and non-edges to non-edges. The proof of the following result is straightforward.

FACT 2. *Let $x \in V(\Gamma_G)$ and $\pi \in \mathrm{Aut}(\Gamma_G)$. Then $\overline{x}^{\pi} = \overline{x^{\pi}}$.*

For a subset $\mathcal{F}$ of $\mathcal{M}_G$, write

$$S_{\mathcal{F}} = \cap_{C \in \mathcal{F}} C \setminus \cup_{C' \in \mathcal{M}_G \setminus \mathcal{F}} C'.$$

LEMMA 3.1. *Let S be a non-empty subset of $V(\Gamma_G)$. Then $S \in \mathcal{W}_G$ if and only if there exists a subset $\mathcal{F}$ of $\mathcal{M}_G$ such that $S = S_{\mathcal{F}}$.*

Proof. "⇒" Write $S = \overline{x}$ and $\mathcal{F} = \mathcal{M}_G(x)$. Then we have that $x \in S_{\mathcal{F}}$. Take an arbitrary element y in $\overline{x}$. Then by Fact 1 one has that $\mathcal{M}_G(x) = \mathcal{M}_G(y)$. This implies that $y \in S_{\mathcal{F}}$ and so $S \subseteq S_{\mathcal{F}}$. Since for any $z \in S_{\mathcal{F}}$, one has that $\mathcal{M}_G(z) = \mathcal{F}$ which implies that $z \equiv x$. Thus, we have that $S_{\mathcal{F}} \subseteq S$, and hence $S = S_{\mathcal{F}}$, as desired.

"⇐" Note that $\mathcal{F}$ is a non-empty proper subset of $\mathcal{M}_G$. Let x be an element of S. Then $\mathcal{M}_G(x) = \mathcal{F}$. It follows that every element of S is equivalent to x. On the other hand, we can see that $\overline{x} \subseteq S$, since $\mathcal{M}_G(y) = \mathcal{F} = \mathcal{M}_G(x)$ for each $y \in \overline{x}$. Therefore, one has that $S = \overline{x}$, as required. □

An *independent set* is a set of vertices in a graph, no two of which are adjacent; that is, a set whose induced subgraph is null. By Lemma 3.1, one has that for any $\overline{x} \in \mathcal{W}_G$, there exists a subset $\mathcal{F}$ of $\mathcal{M}_G$ such that $\overline{x} = S_{\mathcal{F}}$. Let $\bigcap_{C\in\mathcal{F}} C = T$. Then $\overline{x} = T \setminus \bigcup_{C\in\mathcal{M}_G\setminus\mathcal{F}}(C \cap T)$. This means that $\overline{x}$ equals the set obtained by deleting some cyclic subgroups of T from T. Set

$$|T| = n,\ \mathcal{M}_G \setminus \mathcal{F} = \{C_1, C_2, \ldots, C_{|\mathcal{M}_G|-|\mathcal{F}|}\},\ |C_i \cap T| \geq |C_{i+1} \cap T|.$$

The equivalence class $\overline{x}$ is said to be of *type* $(n; n_1, n_2, \ldots, n_{|\mathcal{M}_G|-|\mathcal{F}|})$, where $|C_i \cap T| = n_i$ for each i in $\{1, 2, \ldots, |\mathcal{M}_G| - |\mathcal{F}|\}$.

LEMMA 3.2. *Let* $\pi \in \mathrm{Aut}(\Gamma_G)$ *and* $\overline{x} \in \mathcal{W}_G$. *Then* $\overline{x}$ *and* $\overline{x}^\pi$ *are of the same type.*

Proof. By Lemma 3.1, we may assume that $\overline{x} = S_{\mathcal{F}}$ for some subset $\mathcal{F}$ of $\mathcal{M}_G$. Let $\bigcap_{C\in\mathcal{F}} C = T$. Then it follows from Fact 2 that

$$\overline{x^\pi} = \overline{x}^\pi = S_{\mathcal{F}}^\pi = \big(T \setminus \cup_{C'\in\mathcal{M}_G\setminus\mathcal{F}}(T \cap C')\big)^\pi.$$

We now extend π to G by defining $h^\pi = h$ for all $h \in \mathrm{Cyc}(G)$. Let $M \in \mathcal{M}_G$. We claim that $M^\pi \in \mathcal{M}_G$. If $x, y, z \in G$, and any two of $\{x, y, z\}$ can generate a cyclic subgroup, then $\langle x, y, z\rangle$ is cyclic (cf. [1, Lemma 35]). Now by an induction, the union of a maximal independent set in Γ_G and $\mathrm{Cyc}(G)$ is a maximal cyclic subgroup of G. Clearly, for any $C \in \mathcal{M}_G$, $C \setminus \mathrm{Cyc}(G)$ is a maximal independent set of Γ_G. Since an automorphism maps maximal independent sets to maximal independent sets, it follows that the union of $(M \setminus \mathrm{Cyc}(G))^\pi$ and $\mathrm{Cyc}(G)$ is a maximal cyclic subgroup of G. This implies that M^π is a maximal cyclic subgroup, and the claim follows.

Now note that $(\bigcap_{C\in\mathcal{F}} C)^\pi = \bigcap_{C\in\mathcal{F}} C^\pi$ and $\{C^\pi : C \in \mathcal{F}\} \subseteq \mathcal{M}_G$. For any $C' \in \mathcal{M}_G \setminus \mathcal{F}$, we have

$$\Big|(\bigcap_{C\in\mathcal{F}} C) \cap C'\Big| = \Big|((\bigcap_{C\in\mathcal{F}} C) \cap C')^\pi\Big| = \Big|(\bigcap_{C\in\mathcal{F}} C^\pi) \cap C'^\pi\Big|,$$

and so $\overline{x}$ and $\overline{x}^\pi$ are of the same type. □

Proof of Theorem 2.1. We first prove that $P_{\mathcal{C}_G} \subseteq \mathrm{Aut}(\Gamma_G)$. Let $\{x, y\} \in E(\Gamma_G)$. Then $\mathcal{M}_G(x) \cap \mathcal{M}_G(y) = \emptyset$. Pick any σ in $P_{\mathcal{C}_G}$. One has that $\mathcal{M}_G(x^\sigma) \cap \mathcal{M}_G(y^\sigma) = \emptyset$. It follows that $\{x^\sigma, y^\sigma\} \in E(\Gamma_G)$. Hence, $\sigma \in \mathrm{Aut}(\Gamma_G)$, as desired.

It is similar to the above proof, we can obtain that $\prod_{i=1}^t S_{\overline{w_i}} \subseteq \mathrm{Aut}(\Gamma_G)$. Thus, now we have that both $P_{\mathcal{C}_G}$ and $\prod_{i=1}^t S_{\overline{w_i}}$ are subgroups of $\mathrm{Aut}(\Gamma_G)$.

We now claim that $P_{\mathcal{C}_G} \cap \prod_{i=1}^t S_{\overline{w_i}} = 1$. To see that, pick any π in $P_{\mathcal{C}_G} \cap \prod_{i=1}^t S_{\overline{w_i}}$ and let x in $V(\Gamma_G)$. Then the order of x equals the order of x^π by $\pi \in P_{\mathcal{C}_G}$. Since $\pi \in \prod_{i=1}^t S_{\overline{w_i}}$, x and x^π lie in the same equivalence class, and so x is not adjacent to x^π in Γ_G. It follows that $\langle x, x^\pi \rangle$ is a cyclic subgroup. Then $\langle x \rangle = \langle x^\pi \rangle$. Let $x = [\langle x \rangle]_j$. By (1) one has that

$$x^\pi = ([\langle x \rangle]_j)^\pi = [\langle x \rangle^\pi]_j = [\langle x \rangle]_j = x,$$

and thus $\pi = 1$. So our claim is valid.

Next we prove that $\mathrm{Aut}(\Gamma_G) = P_{\mathcal{C}_G} \prod_{i=1}^t S_{\overline{w_i}}$. Let π be an arbitrary element of $\mathrm{Aut}(\Gamma_G)$. Pick any $x \in V(\Gamma_G)$. Then $\overline{x}$ and $\overline{x}^\pi$ are two equivalence classes of the same type by Lemma 3.2. By Lemma 3.1 we may assume that

$$\overline{x} = \{[C_1]_1, [C_1]_2, \ldots, [C_1]_{k_1}, \ldots, [C_m]_1, [C_m]_2, \ldots, [C_m]_{k_m}\}$$

and

$$\overline{x}^\pi = \{[C_1']_1, [C_1']_2, \ldots, [C_1']_{k_1}, \ldots, [C_m']_1, [C_m']_2, \ldots, [C_m']_{k_m}\},$$

where C_i and C_i' are cyclic groups and $|C_i| = |C_i'|$ for all i. Let $x = [C_i]_p$. Now by (1) we may choose an element τ in $S_{\overline{x}^\pi}$ so that $((([C_i]_p)^\pi)^\tau = [C_i']_p$. Then, we have that $\tau \in \prod_{i=1}^t S_{\overline{w_i}}$ and $\pi\tau \in P_{\mathcal{C}_G}$. It follows that $\pi \in P_{\mathcal{C}_G} \prod_{i=1}^t S_{\overline{w_i}}$. Namely, $\mathrm{Aut}(\Gamma_G) = P_{\mathcal{C}_G} \prod_{i=1}^t S_{\overline{w_i}}$.

In order to complete the proof, now, it suffices to prove that $\prod_{i=1}^t S_{\overline{w_i}}$ is normal in $\mathrm{Aut}(\Gamma_G)$. For any $\sigma \in P_{\mathcal{C}_G}$, $\rho \in \prod_{i=1}^t S_{\overline{w_i}}$ and $\overline{w} \in \mathcal{W}_G$. By Fact 2 we get

$$(\overline{w})^{\sigma^{-1}\rho\sigma} = (\overline{w^{\sigma^{-1}}})^{\rho\sigma} = (\overline{w^{\sigma^{-1}}})^\sigma = \overline{w}.$$

It follows that $\sigma^{-1}\rho\sigma \in \prod_{i=1}^t S_{\overline{w_i}}$. So $P_{\mathcal{C}_G}$ is a subgroup of the normalizer of $\prod_{i=1}^t S_{\overline{w_i}}$ in $\mathrm{Aut}(\Gamma_G)$, and hence $\prod_{i=1}^t S_{\overline{w_i}}$ is normal in $\mathrm{Aut}(\Gamma_G)$. $\square$

4. **EXAMPLES**

In this section, we compute $\mathrm{Aut}(\Gamma_G)$ if G is elementary abelian, dihedral, semi-dihedral or generalized quaternion.

Let H be a group and K be a permutation group on a set Y. The *wreath product* $H \wr K$ is the semidirect product $N \rtimes K$, where N is the direct product of $|Y|$ copies of H (indexed by Y), and K acts on N by permuting the factors in the same way as it permutes elements of Y.

Let $\mathbb{Z}_p^n$ be the elementary abelian p-group for some prime number p. Then $\mathrm{Cyc}(\mathbb{Z}_p^n) = 1$. Let $\mathcal{C}_{\mathbb{Z}_p^n} = \{\langle g_i \rangle : i = 1, 2, \ldots, m\}$, where $m = \frac{p^n-1}{p-1}$ and $|g_i| = p$ for all i. Note that

$$\mathcal{W}_{\mathbb{Z}_p^n} = \{\{g_1, g_1^2, \ldots, g_1^{p-1}\}, \{g_2, g_2^2, \ldots, g_2^{p-1}\}, \ldots, \{g_m, g_m^2, \ldots, g_m^{p-1}\}\}.$$

Thus, we have that $P_{\mathcal{C}_{\mathbb{Z}_p^n}} \cong S_m$ and $\prod_{i=1}^t S_{\overline{w_i}} \cong \prod_{i=1}^m S_{p-1}$. In view of Theorem 2.1, the following result is straightforward.

Example 4.1. $\mathrm{Aut}(\Gamma_{\mathbb{Z}_p^n}) \cong S_{p-1} \wr S_m$, *where* $m = \frac{p^n-1}{p-1}$.

Example 4.2. *For* $n \geq 3$, *let* D_{2n} *denote the dihedral group of order* $2n$. *Then* $\mathrm{Aut}(\Gamma_{D_{2n}}) \cong S_n \times S_{n-1}$.

Proof. Let $G = D_{2n} = \langle a, b : a^n = b^2 = e, b^{-1}ab = a^{-1} \rangle$. Then $\mathrm{Cyc}(G) = \{e\}$,

$$\mathcal{C}_G = \{\langle a^i b \rangle : i = 1, 2, \ldots, n\} \cup \{\langle g \rangle : \{e\} \neq \langle g \rangle \subseteq \langle a \rangle\},$$

and

$$\mathcal{W}_G = \{\{ab\}, \{a^2b\}, \ldots, \{a^nb\}, \{a, a^2, \ldots, a^{n-1}\}\}.$$

Note that $|a^ib| = 2$ for all $i = 1, 2, \ldots, n-1$. We have that

$$P_{\mathcal{C}_G} = S_{\{\langle a^ib \rangle : 1 \leq i \leq n\}} \cong S_n, \ \prod_{i=1}^t S_{\overline{w_i}} = S_{\{a, a^2, \ldots, a^{n-1}\}} \cong S_{n-1}.$$

For any $\sigma \in P_{\mathcal{C}_G}$, $\alpha \in \prod_{i=1}^t S_{\overline{w_i}}$ and $1 \neq C \subseteq \langle a \rangle$, one has that

$$C^{\alpha^{-1}\sigma\alpha} = (C^{\alpha^{-1}})^{\alpha} = C, \ \langle a^ib \rangle^{\alpha^{-1}\sigma\alpha} = \langle a^ib \rangle^{\sigma\alpha} = \langle a^ib \rangle^{\sigma}.$$

It follows that $\alpha^{-1}\sigma\alpha \in P_{\mathcal{C}_G}$ and so $P_{\mathcal{C}_G}$ is normalized by $\prod_{i=1}^t S_{\overline{w_i}}$. By Theorem 2.1, we know that $\mathrm{Aut}(\Gamma_{D_{2n}}) \cong S_n \times S_{n-1}$. □

Let n be a natural number greater than or equal to 4. The semi-dihedral group of order 2^n, denoted by SD_{2^n}, is defined by the following presentation:

$$SD_{2^n} = \langle a, x : a^{2^{n-1}} = x^2 = e, xax = a^{2^{n-2}-1} \rangle.$$

The set of elements of SD_{2^n} is $\{a^ix : 1 \leq i \leq 2^{n-1}\} \cup \langle a \rangle$. Note that $|a^ix| = 2$ for each even number i, and $|a^jx| = 4$ and $(a^jx)^2 = a^{2^{n-2}}$ for each odd number j.

Example 4.3. $\mathrm{Aut}(\Gamma_{SD_{2^n}}) \cong (S_2 \wr S_{2^{n-3}}) \times S_{2^{n-2}} \times S_{2^{n-1}-1}$.

Proof. Let $G = SD_{2^n}$. Then $\mathrm{Cyc}(G) = \{e\}$ and

$$\mathcal{C}_G = \{\langle a^ix \rangle : 1 \leq i \leq 2^{n-1}\} \cup \{\langle g \rangle : \{e\} \neq \langle g \rangle \subseteq \langle a \rangle\}.$$

Therefore, $\mathcal{W}_G = \{\{a^ix\} : 1 \leq i \leq 2^{n-1}, i \text{ is even}\} \cup \{\{a^jx, (a^jx)^{-1}\} : 1 \leq i \leq 2^{n-1}, j \text{ is odd}\} \cup \{a^{2^{n-2}}\} \cup (\langle a \rangle \setminus \{e, a^{2^{n-2}}\})$, where $(a^jx)^{-1} = a^{(2^{n-2}-1)(2^{n-1}-j)}x$. It follows that

$$P_{\mathcal{C}_G} = S_{\{\langle a^jx \rangle : 1 \leq i \leq 2^{n-1}, \ j \text{ is odd}\}} \times S_{\{a^ix : 1 \leq i \leq 2^{n-1}, \ i \text{ is even}\}} \cong S_{2^{n-3}} \times S_{2^{n-2}}$$

and

$$\prod_{i=1}^{t} S_{\overline{w_i}} = \prod_{i=1}^{2^{n-3}} S_{\{a^{j_i}x,(a^{j_i}x)^{-1}\}} \times S_{\langle a\rangle\setminus\{e,a^{2^{n-2}}\}} \cong S_2^{2^{n-3}} \times S_{2^{n-1}-1},$$

where $1 \leq j_i \leq 2^{n-1}$, j_i is odd and

$$\{a^{j_i}x, (a^{j_i}x)^{-1} : 1 \leq i \leq 2^{n-3}\} = \{a^k x : 1 \leq k \leq 2^{n-1},\ k \text{ is odd}\}.$$

We note that any automorphism of Γ_G fixes the vertex $a^{2^{n-2}}$. Now it is easy to check that $S_{\langle a\rangle\setminus\{e,a^{2^{n-2}}\}}$ and $P_{\mathcal{C}_G}$ commute, and $S_{\{a^i x:1\leq i\leq 2^{n-1},\ i \text{ is even}\}}$ and $\prod_{i=1}^{t} S_{\overline{w_i}}$ commute. By Theorem 2.1, $\mathrm{Aut}(\Gamma_{SD_{2^n}})$ is obtained. □

Let $n \geq 2$, denote by Q_{4n} the generalized quaternion group of order $4n$, that is,

$$Q_{4n} = \langle a, b : a^{2n} = e, a^n = b^2, b^{-1}ab = a^{-1}\rangle.$$

EXAMPLE 4.4. $\mathrm{Aut}(\Gamma_{Q_{4n}}) \cong \begin{cases} S_2 \wr S_3, & \text{if } n = 2, \\ (S_2 \wr S_n) \times S_{2n-2}, & \text{otherwise.} \end{cases}$

Proof. Note that $\mathrm{Cyc}(Q_{4n}) = \{e, a^n\}$ and $|a^i b| = 4$ for all $i = 1, 2, \ldots, 2n$. Let $G = Q_{4n}$. It is clear that

$$\mathcal{C}_G = \{\langle a^i b\rangle : i = 1, 2, \ldots, n\} \cup \{\langle g\rangle : \{e\} \neq \langle g\rangle \subseteq \langle a\rangle, g \neq a^n\},$$

and

$$\mathcal{W}_G = \{\langle a^i b\rangle \setminus \langle b\rangle : i = 1, 2, \ldots, n\} \cup \{\langle a\rangle \setminus \langle a^n\rangle\}.$$

Thus, if $n = 2$, then $\prod_{i=1}^{t} S_{\overline{w_i}} \cong S_2^3$ and $P_{\mathcal{C}_G} \cong S_3$, so it follows from Theorem 2.1 that $\mathrm{Aut}(\Gamma_{Q_{4n}}) \cong S_2 \wr S_3$, as desired. Now we suppose that $n \geq 3$. Then

$$P_{\mathcal{C}_G} = S_{\{\langle a^i b\rangle:1\leq i\leq n\}} \cong S_n$$

and

$$\prod_{i=1}^{t} S_{\overline{w_i}} = \prod_{i=1}^{n} S_{\langle a^i b\rangle\setminus\langle b\rangle} \times S_{\{a,a^2,\ldots,a^{n-1},a^{n+1},\ldots,a^{2n-1}\}} \cong S_2^n \times S_{2n-2}.$$

Note that it is easy to check that $S_{\{a,a^2,\ldots,a^{n-1},a^{n+1},\ldots,a^{2n-1}\}}$ and $P_{\mathcal{C}_G}$ commute in $\mathrm{Aut}(\Gamma_{Q_{4n}})$. Now the required result follows from Theorem 2.1. □

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