GENERALIZED WEIGHTED COMPOSITION OPERATORS BETWEEN THE WEIGHTED BERGMAN SPACES

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Let φ and u be analytic maps on the open unit disc $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $H(\mathbb{D})$ be the spaces of analytic functions on $\mathbb{D}$. For a non-negative integer n, the generalized weighted composition operator $D^n_{\varphi,u}$ is defined by $D^n_{\varphi,u} f = u \cdot \left(f^{(n)} \circ \varphi\right), f \in H(\mathbb{D})$. In this paper, we characterize the boundedness of the generalized weighted composition operator $D^n_{\varphi,u}$ between the weighted Bergman spaces. We also compute the upper and lower bounds for the essential norm of this operator.

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1. INTRODUCTION

Let $\mathbb{D}$ be the open unit disc of the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the class of all analytic functions on $\mathbb{D}$. Let dA denotes the normalized Lebesgue area measure on $\mathbb{D}$. For $0 < p < +\infty$ and $-1 < \alpha < \infty$, the weighted Bergman space A^p_α is the space of all analytic functions f on $\mathbb{D}$ such that

$$\|f\|^p_{A^p_\alpha} = \int_{\mathbb{D}} |f(z)|^p \mathrm{d}A_\alpha < \infty,$$

where $\mathrm{d}A_\alpha = (\alpha+1)\left(1-|z|^2\right)^\alpha \mathrm{d}A$. When $\alpha = 0, A^p = A^p_0$ is the classic Bergman space (see[6]).

For $a, z \in \mathbb{D}$, let $\beta_a(z) = \frac{a-z}{1-\bar{a}z}$ be a Mobius transformation on $\mathbb{D}$.

For a fixed $u \in H(\mathbb{D})$ and a nonconstant analytic self-map of $\mathbb{D}$, we define a linear operator uC_φ on $H(\mathbb{D})$, called the weighted composition operator, as follows:

$$uC_\varphi f = u(f \circ \varphi), \qquad f \in H(\mathbb{D}).$$

Putting $u = 1$, the operator C_φ is called the composition operator. The weighted composition operator is a generalization of the composition operator and the multiplication operator, which is defined by $M_u f = uf$, where

$f \in H(\mathbb{D})$. It is a well-known consequence of Littlewood's subordination principle that the composition operator C_φ is bounded on the classical Hardy space and the Bergman space. It is interesting to give a function theoretic characterization of φ when φ induces a bounded or compact composition operator on various spaces. For (weighted) composition operators on various spaces of analytic functions, we refer for example to [2, 3, 4, 9, 16, 19] and references therein.

Let D be the differentiation operator and n be a nonnegative integer, i.e., we have

$$Df = f', \qquad D^n f = f^{(n)}, \qquad f \in H(\mathbb{D}).$$

Now, we define a new operator $D^n_{\varphi,u}$ as follows:

$$D^n_{\varphi,u} f = u \cdot f^{(n)} \circ \varphi, \qquad f \in H(\mathbb{D}).$$

This operator is called the generalized weighted composition operator which includes many known operators (see [20]). If $n = 0$, then we get the weighted composition operator uC_φ. If $n = 0$ and $u(z) \equiv 1$, then we get the composition operator C_φ. If $n = 1$, $u(z) = \varphi'(z)$, then $D^n_{\varphi,u} = DC_\varphi$, which was studied in[7, 8]. When $n = 1$ and $u(z) = 1$, then $D^n_{\varphi,u} = C_\varphi D$ which was studied in[7, 11]. If we put $n = 1$ and $\varphi(z) = z$, then $D^n_{\varphi,u} = M_u D$, i.e., the product of differentiation operator and multiplication operator. For some other results on the generalized weighted composition operator on various spaces of holomorphic functions, see, for example, [12, 13, 14, 18, 21].

In [15], Vaezi studied the nearly open weighted composition operator on weighted spaces of continuous functions. Symmetric lifting operator acting on some spaces of analytic functions was studied by Vaezi and Nasresfahani in [17]. In this article, first we characterize the boundedness of the generalized weighted composition operator $D^n_{\varphi,u}$ between the weighted Bergman spaces. Then we obtain the upper and lower bounds for the essential norm of this operator.

2. BOUNDEDNESS OF $D^n_{\varphi,u} : \mathcal{A}^p_\alpha \to \mathcal{A}^q_\beta$ FOR $p \leq q$

In what follows, we make extensive use of Carleson measure techniques, so we give a short introduction to Carleson sets and Carleson measure first. Let μ be a positive Borel measure on $\mathbb{D}$ and X be a Banach space of analytic functions on $\mathbb{D}$. Let $q > 0$. We say that μ is an (X, q)-Carleson measure if there is a constant $C > 0$ such that, for any $f \in X$,

$$\int_D |f^{(n)}(z)|^q \mathrm{d}\mu(z) \leq C\|f\|^q_X.$$

Let I be an arc in the unit circle $\partial\mathbb{D}$ and $S(I)$ be the Carleson square defined by

$$S(I)=\{z\in\mathbb{D}:1-|I|\le|z|<1, z/|z|\in I\}.$$

THEOREM 2.1. *Let μ be a positive Borel measure on $\mathbb{D}$. Let $0<p<q\le\infty$ and $-1<\alpha<\infty$. Then the following statements are equivalent:*

(i) *There is a constant $C_1>0$ such that, for any $f\in A^q_\alpha$,*

$$\int_D |f^{(n)}(z)|^q \mathrm{d}\mu(z)\le C_1\|f\|^q_{A^p_\alpha}.$$

(ii) *There is a constant $C_2>0$ such that, for any arc $I\in\partial\mathbb{D}$,*

$$\mu\left(S\left(I\right)\right)\le C_2|I|^{\frac{(2+\alpha)q}{p}(n+1)}.$$

(iii) *There is a constant $C_3>0$ sch that, for every $a\in\mathbb{D}$,*

$$\int_D |\beta_a^{(n+1)}(z)|^{\frac{(2+\alpha)q}{p}}\mathrm{d}\mu(z)\le C_3.$$

The result was proved by several authors. The equivalence of (i) and (ii) can be found in[5] and [10]. The equivalence of (ii) and (iii) can be proved by similar arguments as in [1].

A positive Borel measure μ which satisfies the above equivalent conditions is called a $\frac{(2+\alpha)q}{p}(n+1)$-Carleson measure for the weighted Bergman space. If we define

$$\|\mu\|=\sup_{I\subset\partial\mathbb{D}}\frac{\mu\left(S(I)\right)}{|I|^{\frac{(2+\alpha)q}{p}(n+1)}},$$

then, $\|\mu\|$ and the above constant are comparable.
A positive Borel measure μ on $\mathbb{D}$ is called a vanishing $\frac{(2+\alpha)q}{p}(n+1)$-Carleson measure if

$$\lim_{|I|\to0}\sup_{I\subset\partial\mathbb{D}}\frac{\mu\left(S(I)\right)}{|I|^{\frac{(2+\alpha)q}{p}(n+1)}}=0.$$

The following result completely characterizes the bounded generalized integration operator $D^n_{\varphi,u}$ on the weighted Bergman space.

THEOREM 2.2. *Let φ be an analytic self-map of $\mathbb{D}$ and n be a nonnegative integer. Assume that $0<p\le q<\infty$ and $\alpha,\beta>-1$. Then the following statements are equivalent.*

(a) *The operator $D^n_{\varphi,u}:A^p_\alpha\to A^q_\beta$ is bounded.*

(b) *The pull-back measure $\mu_{\varphi,u}=\nu_u\circ\varphi^{-1}$ of ν_u induced by φ is a $\frac{(2+\alpha)q}{p}(n+1)$ -Carleson measure. Here $\mathrm{d}\nu_u=|u|^q\mathrm{d}A_\beta$.*

(c) $\sup_{z\in\mathbb{D}}\int_{\mathbb{D}}\frac{\left(1-|z|^2\right)^{\frac{q(2+\alpha)}{p}}}{|1-\bar{z}\varphi(w)|^{\frac{q(2+\alpha)}{p}(n+2)}}|u\left(w\right)|^q\mathrm{d}A_\beta\left(w\right)<\infty.$

Proof. (a) $\iff$ (b). By defintion, $D^n_{\varphi,u}$ is bounded from A^p_α into A^q_β if and only if there exists a constant $C>0$ such that for any $f\in A^p_\alpha$,

$$\|D^n_{\varphi,u}f\|^q_{A^q_\beta}\le C\|f\|^q_{A^p_\alpha}.$$

That is,

$$\int_{\mathbb{D}}|f^{(n)}\left(\varphi\left(z\right)\right)|^q|u\left(z\right)|^q\mathrm{d}A_\beta\left(z\right)\le C\|f\|^q_{A^p_\alpha}.$$

Let $d\nu_u(z)=|u\left(z\right)|^q\mathrm{d}A_\beta\left(z\right)$ and $\mu_{\varphi,u}=\nu_u\circ\varphi^{-1}$ be the pull-back measure of ν_u induced by φ. By change of variable $w=\varphi\left(z\right)$, we get

$$\begin{aligned}\int_{\mathbb{D}}|f^{(n)}\left(\varphi\left(z\right)\right)|^q|u\left(z\right)|^q\mathrm{d}A_\beta\left(z\right)&=\int_{\mathbb{D}}|f^{(n)}\left(\varphi\left(z\right)\right)|^q\mathrm{d}\nu_u\left(z\right)\\&=\int_{\mathbb{D}}|f^{(n)}\left(w\right)|^q\mathrm{d}\mu_{\varphi,u}\left(w\right).\end{aligned}$$

Thus,

$$\int_{\mathbb{D}}|f^{(n)}\left(w\right)|^q\mathrm{d}\mu_{\varphi,u}\left(w\right)\le C\|f\|^q_{A^p_\alpha}.\tag{1}$$

By Theorem 2.1, the relation (1) is equivalent with the fact that $\mu_{\varphi,u}$ is a $\frac{(2+\alpha)q}{p}\left(n+1\right)-$Carleson measure. So, (a) and (b) are equivalent.

By Theorem 2.1, the condition that $\mu_{\varphi,u}$ is a $\frac{(2+\alpha)q}{p}\left(n+1\right)-$Carleson measure is equivalent to

$$\int_{\mathbb{D}}|\beta^{(n+1)}_z\left(w\right)|^{\frac{(2+\alpha)q}{p}}\mathrm{d}\mu_{\varphi,u}\left(w\right)\le C.$$

Changing the variable, we get

$$\sup_{z\in\mathbb{D}}\int_{\mathbb{D}}\frac{\left(1-|z|^2\right)^{\frac{(2+\alpha)q}{p}}|z|^{\frac{(2+\alpha)q}{p}n}}{|1-\bar{z}\varphi\left(w\right)|^{\frac{(2+\alpha)q}{p}(n+2)}}|u\left(w\right)|^q\mathrm{d}A_\beta\left(w\right)<\infty.\tag{2}$$

Thus (a), (b) and (2) are equivalent. Clearly, (c) $\Rightarrow$ (2). Suppose that (2) holds. Then for any $0<r_0<1$, we have

$$\sup_{r_0<|z|<1}\int_{\mathbb{D}}\frac{\left(1-|z|^2\right)^{\frac{(2+\alpha)q}{p}}}{|1-\bar{z}\varphi\left(w\right)|^{\frac{(2+\alpha)q}{p}(n+2)}}|u\left(w\right)|^q\mathrm{d}A_\beta\left(w\right)<\infty.\tag{3}$$

By (a), $D^n_{\varphi,u}$ is bounded from A^p_α into A^q_β. Thus, by taking $f\left(z\right)=\frac{z^n}{n!}$ in A^p_α, we get

$$\int_{\mathbb{D}}|u\left(z\right)|^q\mathrm{d}A_\beta\left(z\right)\le\frac{C}{(n!)^q},$$

where C is a constant. So,

$$(4)\qquad \begin{aligned}&\sup_{0\le|z|\le r_0}\int_{\mathbb{D}}\frac{\left(1-|z|^2\right)^{\frac{(2+\alpha)q}{p}}}{|1-\bar{z}\varphi(w)|^{\frac{(2+\alpha)q}{p}(n+2)}}|u(w)|^q\mathrm{d}A_\beta(w)\\&\qquad\le\frac{1}{(1-r_0)^{\frac{(2+\alpha)q}{p}(n+2)}}\int_{\mathbb{D}}|u(w)|^q\mathrm{d}A_\beta(w)<\infty.\end{aligned}$$

Combining (3) and (4), we have (2) $\Rightarrow$ (c). $\square$

3. ESSENTIAL NORM

In this section, we obtain the upper and lower bounds for the essential norm of the generalized weighted composition operator $D^n_{\varphi,u}$ between the weighted Bergman spaces.

Theorem 3.1. *Let u be an analytic function on $\mathbb{D}$ and φ be an analytic self-map of $\mathbb{D}$. Let $1<p\le q<\infty$, $\alpha,\beta>-1$ and $D^n_{\varphi,u}$ be bounded from A^p_α into A^q_β. Then there are positive constants C_1 and C_2 such that*

$$C_1\lim_{|a|\to1}\sup I_{\varphi,\alpha,\beta}(u)(a)\le\|D^n_{\varphi,u}\|^q_e\le C_2\lim_{|a|\to1}\sup I_{\varphi,\alpha,\beta}(u)(a),$$

where

$$I_{\varphi,\alpha,\beta}(u)(a)=\int_{\mathbb{D}}\frac{\left(1-|a|^2\right)^{\frac{q(2+\alpha)}{p}}}{|1-\bar{a}\varphi(z)|^{\frac{q(2+\alpha)}{p}(n+2)}}|u(z)|^q\mathrm{d}A_\beta(z).$$

In order to prove Theorem 3.1, we need several lemmas.

Lemma 3.2. *Let $0<r<1$ and μ be a positive Borel measure on $\mathbb{D}$. Let*

$$N^*_r=\sup_{|a|\ge r}\int_{\mathbb{D}}|\beta^{(n+1)}_a(z)|^{\frac{(2+\alpha)q}{p}}\mathrm{d}\mu(z).$$

If μ is an $\frac{(2+\alpha)q}{p}(n+1)-$ Carleson measure for $0<p\le q<0$, then so is $\mu_r=\mu|_{\mathbb{D}\setminus D_r}$, where $D_r=\{z\in\mathbb{D}:|z|<r\}$. Moreover, $\|\mu_r\|\le MN^*_r$, where M is an absolute constant.

Proof. Let

$$N_r=\sup_{|I|\le1-r}\frac{\mu(S(I))}{|I|^{\frac{(2+\alpha)q}{p}(1+n)}}.$$

Take any arc $I\subset\partial D$. Suppose that $|I|=\gamma(1-r)$ for some constant $\gamma>0$. If $0<\gamma\le1$, then obviously $S(I)\subset D\setminus D_r$ and so

$$\mu_r(S(I))=\mu(S(I))\le N_r|I|^{\frac{(2+\alpha)q}{p}(1+n)}.$$

Now consider the case $\gamma > 1$. Let $[\gamma]$ be the greatest integer that is less than or equal to γ. Then $[\gamma]+1 > \gamma$ and $([\gamma]+1)/\gamma \leq 2$. Let $n = [\gamma]+1$. In this case, it is possible to cover I by n smaller arcs $I_1, I_2, ..., I_n$ such that $|I_k| = 1-r, k = 1,2,...,n$. Thus

$$\begin{aligned}
\mu_r(S(I)) &= \mu(S(I) \cap (D \setminus D_r)) \\
&\leq \sum_{k=1}^{n} \mu(S(I_k)) \\
&\leq \sum_{k=1}^{n} N_r |I_k|^{\frac{(2+\alpha)q}{p}(1+n)} \\
&\leq N_r \left(\sum_{k=1}^{n} |I_k| \right)^{\frac{(2+\alpha)q}{p}(1+n)} \\
&= N_r (n(1-r))^{\frac{(2+\alpha)q}{p}(1+n)} \\
&= N_r ([\gamma]+1)^{\frac{(2+\alpha)q}{p}(1+n)} (1-r)^{\frac{(2+\alpha)q}{p}(1+n)} \\
&\leq N_r (2\gamma)^{\frac{(2+\alpha)q}{p}(1+n)} (1-r)^{\frac{(2+\alpha)q}{p}(1+n)} \\
&= N_r 2^{\frac{(2+\alpha)q}{p}(1+n)} |I|^{\frac{(2+\alpha)q}{p}(1+n)}.
\end{aligned}$$

This implies that $\|\mu_r\| \leq 2^{\frac{(2+\alpha)q}{p}(1+n)} N_r$. Thus, to complete the proof, we just need to prove that $N_r \leq M N_r^*$, where $M > 0$ is a constant depending upon n only. Take any arc $I \subset \partial D$ with $|I| \leq 1-r$. Let $a = (1-|I|)e^{i\theta}$, where $e^{i\theta}$ is the center of I. Then $|a| = 1-|I| \geq r$. Again

$$\begin{aligned}
\frac{\left(1-|a|^2\right)|a|^n}{|1-\bar{a}z|^{2+n}} &\geq \Re\left(\frac{\left(1-|a|^2\right)|a|^n}{(1-\bar{a}z)^{2+n}} \right) \\
&= \frac{\left(1-|a|^2\right)|a|^n}{(1-|a|)^{2+n}} \Re\left(\frac{1-|a|}{1-\bar{a}z} \right)^{2+n} \\
&= \frac{\left(1-|a|^2\right)|a|^n}{(1-|a|)^{2+n}} \Re\left(1 + \frac{|a|\left(1-z\bar{\zeta}\right)}{(1-|a|)} \right)^{-(2+n)} \\
&> \frac{1}{(2)^{\frac{(2+n)}{2}}} \frac{\left(1-|a|^2\right)|a|^n}{(1-|a|)^{2+n}} \\
&\geq \frac{1}{2^{\frac{3n+2}{2}}|I|^{1+n}}, \qquad \left(\zeta = \frac{a}{|a|}\right)
\end{aligned}$$

if $z \in S(I)$. Thus,

$$\frac{\mu(S(I))}{|I|^{\frac{(2+\alpha)q}{p}(1+n)}} \leq 2^{\frac{(2+\alpha)q}{p}\frac{(3n+2)}{2}} \int_{S(I)} |\beta_a^{(n+1)}(z)|^{\frac{(2+\alpha)q}{p}} \mathrm{d}\mu(z)$$
$$\leq 2^{\frac{(2+\alpha)q}{p}\frac{(3n+2)}{2}} \int_{D} |\beta_a^{(n+1)}(z)|^{\frac{(2+\alpha)q}{p}} \mathrm{d}\mu(z)$$
$$\leq 2^{\frac{(2+\alpha)q}{p}\frac{(3n+2)}{2}} N_r^*.$$

Taking the supremum over the arcs I with $|I| \leq 1-r$, we get

$$N_r \leq 2^{\frac{(2+\alpha)q}{p}\frac{(3n+2)}{2}} C N_r^*.$$

□

For $f(z) = \sum_{k=0}^{\infty} a_k z^k$ analytic on $\mathbb{D}$, let $K_n f(z) = \sum_{k=0}^{n} a_k z^k$ and $R_n = I - K_n$, where $If = f$ is the identity map. Hence $R_n f(z) = \sum_{k=n+1}^{\infty} a_k z^k$.

LEMMA 3.3. *For any $\varepsilon > 0$ and $f \in A_\alpha^p$, there exists n large enough such that*

$$|(R_n f)^n(w)| \leq \varepsilon \|f\|_{A_\alpha^p}.$$

Proof. Let n be a non-negative integer. Then for a fixed $0 < r < 1$ and any $z \in \mathbb{D}$ with $|z| \leq 1$, from the Taylor expansion $K_z(w) = \sum_{k=0}^{\infty} (k+1) \bar{z}^k w^k$, we have

$$K_z^{(n)}(w) = \sum_{k=n}^{\infty} (k+1) \frac{k!}{(k-n)!} \bar{z}^k w^{k-n} = \sum_{k=0}^{\infty} \frac{(n+k+1)(n+k)!}{k!} \bar{z}^{n+k} w^k.$$

Hence,

$$R_n K_z^{(n)}(w) = \sum_{k=n+1}^{\infty} \frac{(n+k+1)(n+k)!}{k!} \bar{z}^{n+k} w^k.$$

So,

$$|R_n K_z^{(n)}(w)| \leq \sum_{k=n+1}^{\infty} \frac{(n+k+1)(n+k)!}{k!} r^{n+k}.$$

Hence, for any $\varepsilon > 0$, there exists a large enough n such that

$$\int_{\mathbb{D}} |R_n K_z^{(n)}(w)|^q \left(1-|z|^2\right)^\beta \mathrm{d}A(z) < \varepsilon^q.$$

Therefore,

$$|(R_n f)^{(n)}(z)| = |\left\langle R_n f, K_z^{(n)} \right\rangle| = |\left\langle f, R_n K_z^{(n)} \right\rangle|$$
$$\leq \|f\|_{A_\alpha^p} \|R_n K_z^{(n)}\|_{A_\beta^q} \leq \varepsilon \|f\|_{A_\alpha^p}.$$

□

Finally, we need the following lemma to estimate the essential norm of $D^n_{\varphi,u}$.

LEMMA 3.4. *If $D^n_{\varphi,u}$ is bounded from A^p_α into A^q_β for $0<p\le q<\infty$, then*

$$\|D^n_{\varphi,u}\|_e \le \liminf_{n\to\infty} \|D^n_{\varphi,u}R_n\|.$$

Proof. since $(R_n+K_n)f=f$ and K_n is compact, so, for each n,

$$\|D^n_{\varphi,u}\|_e \le \|D^n_{\varphi,u}R_n + D^n_{\varphi,u}K_n\|_e \le \|D^n_{\varphi,u}R_n\|_e \le \|D^n_{\varphi,u}R_n\|.$$

Therefore, $\|D^n_{\varphi,u}\|_e \le \liminf_{n\to\infty}\|D^n_{\varphi,u}R_n\|$. □

Proof of the Theorem 3.1. First we prove the upper estimate. By Lemma 3.4,

$$\|D^n_{\varphi,u}\|_e \le \liminf_{n\to\infty}\|D^n_{\varphi,u}R_n\| \le \liminf_{n\to\infty} \sup_{\|f\|_{A^p_\alpha}\le 1} \|\left(D^n_{\varphi,u}R_n\right)f\|_{A^q_\beta}.$$

However, for any fixed $0<r<1$,

$$\begin{aligned}
\|\left(D^n_{\varphi,u}R_n\right)f\|^q_{A^q_\beta} &= \int_{\mathbb{D}} |u(z)|^q|\left(R_nf\right)^n(\varphi(z))|^q \mathrm{d}A_\beta(z)\\
&= \int_{\mathbb{D}} |\left(R_nf\right)^{(n)}(w)|^q \mathrm{d}\mu_{\varphi,u}(w)\\
&= \int_{\mathbb{D}\setminus D_r} |\left(R_nf\right)^{(n)}(w)|^q \mathrm{d}\mu_{\varphi,u}(w)\\
&+ \int_{D_r} |\left(R_nf\right)^{(n)}(w)|^q \mathrm{d}\mu_{\varphi,u}(w)\\
&= I_1+I_2,
\end{aligned} \tag{5}$$

where $\mu_{\varphi,u}$ is the pull-back measure induced by φ defined in Section 2. Since $D^n_{\varphi,u}$ is bounded from A^p_α into A^q_β, $\mu_{\varphi,u}$ is an $\frac{(2+\alpha)q}{p}(n+1)$ -Carleson measure. So,

$$\begin{aligned}
I_2 &\le \sup_{|z|<r} |\left(R_nf\right)^{(n)}(z)|^q \int_{D_r} \mathrm{d}\mu_{\varphi,u}(z)\\
&\le \varepsilon^q \|f\|^q_{A^p_\alpha}\mu_{\varphi,u}(D_r)\\
&\le \varepsilon^q \|f\|^q_{A^p_\alpha}\|u\|^q_{A^q_\beta}.
\end{aligned}$$

Hence, for a fixed r,

$$\sup_{\|f\|_{A^p_\alpha}\le 1} I_2 \to 0 \ as\ n\to\infty.$$

On the other hand, if we set $\mu_{\varphi,ur} = \mu_{\varphi,u}\mid_{D\setminus D_r}$, then, by Theorem 2.1 and Lemma 3.2,

$$I_1 = \int_{D\setminus D_r} |(R_n f)^{(n)}(w)|^q \mathrm{d}\mu_{\varphi,u}(w) \leq K\|\mu_{\varphi,ur}\|\|(R_n f)^{(n)}\|^q_{A^p_\alpha} \leq KCMN^*_r\|f\|^q_{A^p_\alpha},$$

where K and C are constants independent of u and r, and M and N^*_r are defined as in Lemma 3.2. Taking the supremum in (5) over analytic functions f in the unit ball of A^p_α, and letting $n \to \infty$, we get

$$\liminf_{n\to\infty} \sup_{\|f\|_{A^p_\alpha}\leq 1} \|(D^n_{\varphi,u}R_n)f\|^q_{A^q_\beta} \leq KCMN^*_r.$$

Thus, $\|D^n_{\varphi,u}\|^q_e \leq KCMN^*_r$. Letting $r \to 1$ we get

$$\begin{aligned}\|D^n_{\varphi,u}\|^q_e &\leq KCM \lim_{r\to 1} N^*_r\\ &= KCM \limsup_{|a|\to 1} \int_{\mathbb{D}} |\beta^{(n+1)}_a(w)|^{(2+\alpha)q/p}\mathrm{d}\mu_{\varphi,u}(w)\\ &= KCM \limsup_{|a|\to 1} \int_{\mathbb{D}} |\beta^{(n+1)}_a(\varphi(z))|^{(2+\alpha)q/p}|u(z)|^q\mathrm{d}A_\beta(z)\\ &= KCM\left((n+1)!\right)^{\frac{(2+\alpha)q}{p}} \limsup_{|a|\to 1} I_{\varphi,\alpha,\beta}(u)(a),\end{aligned}$$

which gives us the desired upper bound.

Lower bound. Consider the normalized kernel function

$$k_a(z) = (1-|a|^2)/(1-\bar{a}z)^2.$$

Let $f_a = k_a^{(2+\alpha)/p}$. Then $\|f_a\|_{A^p_\alpha} = 1$, and $f_a \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1$. Fix a compact operator K from A^p_α into A^q_β. Then $\|\mathrm{K}f_a\|_{A^q_\beta \to 0}$ as $|a| \to 1$. Therefore,

$$\begin{aligned}\|D^n_{\varphi,u} - \mathrm{K}\| &\geq \limsup_{|a|\to 1} \|(D^n_{\varphi,u} - \mathrm{K})f_a\|_{A^q_\beta}\\ &\geq \limsup_{|a|\to 1} \left(\|D^n_{\varphi,u}f_a\|_{A^q_\beta} - \|\mathrm{K}f_a\|_{A^q_\beta}\right)\\ &= \limsup_{|a|\to 1} \|D^n_{\varphi,u}f_a\|_{A^q_\beta}.\end{aligned}$$

Thus, there exists a constant C such that

$$\|D^n_{\varphi,u}\|^q_e \geq \|D^n_{\varphi,u} - \mathrm{K}\|^q \geq \limsup_{|a|\to 1} \|D^n_{\varphi,u}f_a\|^q_{A^q_\beta}$$

$$\geq C \limsup_{|a|\to 1} \int_{\mathbb{D}} \frac{\left(1-|a|^2\right)^{\frac{q(2+\alpha)}{p}} |u(z)|^q}{|1-\bar{a}\varphi(z)|^{\frac{q(2+\alpha)}{p}(n+2)}} \mathrm{d}A_\beta(z)$$
$$= C \limsup I_{\varphi,\alpha,\beta}(u)(a).$$

The theorem is proved. □

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