SOME CHARACTERIZATIONS OF g-UNCONDITIONAL BASES

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Due to their potential in pure and applied mathematics, the notions of basis and frame have various generalizations. Recently, g-frame and g-basis were introduced and studied by some mathematicians. It is well known that a frame-based series expansion of a vector is unconditionally convergent, while a basis-based one need not be. In applications unconditionality is more favourable than conditionality. In this paper, we introduce the notion of g-unconditional basis which leads to unconditional convergence, and establish a characterization of g-unconditional bases.

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1. INTRODUCTION

As one of fundamental problems in functional analysis, the theory of bases in Banach spaces has interested numerous mathematicians for the last more than ninety years. It is well known that an arbitrary infinite-dimensional Banach space X admits a Hamel basis – a lineary independent subset of X spanning X, when viewed as a vector space. But such bases are almost useless since they cannot in general be constructed, their very existence depending on the axiom of choice. Of far more importance and applicability in analysis is the notion of a basis first introduced by Schauder in [40, 41]. In the literatures or books, "basis" is also called "Schauder basis".

Definition 1.1. A sequence $\{x_j\}_{j\in\mathbb{N}}$ in an infinite-dimensional Banach space X is said to be a (Schauder) basis for X if to every $x \in X$ there corresponds a unique scalar sequence $\{c_j\}_{j\in\mathbb{N}}$ such that

$$x = \sum_{j=1}^{\infty} c_j x_j, \tag{1.1}$$

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i.e.

$$\lim_{J\to\infty} \|x - \sum_{j=1}^{J} c_j x_j\| = 0.$$

It is easy to check that a Banach space with a basis must be separable. The outstanding "basis problem" – whether or not every separable Banach space has a basis – was raised by Banach in [5] in 1955. In 1973, Enflo in [16] gave a negative answer to the basis problem by constructing a separable Banach space having no basis. For fundamentals of bases in Banach spaces, we refer to [42, 43]. Recall that the series in (1.1) of Definition 1.1 need not be unconditionally convergent.

Let $\{x_j\}_{j\in\mathbb{N}}$ be a sequence in a Banach space X, where $\mathbb{N}$ denotes the set of positive integers. If $\sum\limits_{j=1}^{\infty} x_{\sigma(j)}$ is convergent for each permutation σ of $\mathbb{N}$, we say that $\sum\limits_{j=1}^{\infty} x_j$ is unconditionally convergent. In that case, the limit is the same regardless of the order of summation, and so we may write $\sum\limits_{j\in\mathbb{N}} x_j$ for $\sum\limits_{j=1}^{\infty} x_{\sigma(j)}$ for each permutation σ. Hereafter, we say $\sum\limits_{j\in\mathbb{N}} x_j$ is well defined means that $\sum\limits_{j=1}^{\infty} x_j$ is unconditionally convergent. We write

$$\lim_F \sum_{j\in F} x_j = x$$

if for every $\epsilon > 0$, there exists a finite subset F_ϵ of $\mathbb{N}$ such that

$$\|x - \sum_{j\in F} x_j\| < \epsilon$$

for all finite subsets F of $\mathbb{N}$ with $F_\epsilon \subset F$.

Collecting [27, Theorem 3.10, Corollary 3.11] and [46, Theorem 7.2], we have the following characterization of unconditional convergence.

PROPOSITION 1.1. *Let $\{x_j\}_{j\in\mathbb{N}}$ be a sequence in X. Then the following are equivalent:*

(i) $\sum\limits_{j\in\mathbb{N}} x_j$ *is well defined.*

(ii) $\lim\limits_F \sum\limits_{j\in F} x_j$ *exists, where F is finite subset of* $\mathbb{N}$.

(iii) $\sum\limits_{j=1}^{\infty} \mu_j x_j$ *is well defined for each sequence $\mu = \{\mu_j\}_{j\in\mathbb{N}}$ with $|\mu_j| \le 1$ for $j \in \mathbb{N}$.*

(iv) $\sum_{j=1}^{\infty} \mu_j x_j$ *is well defined for each bounded sequence* $\mu = \{\mu_j\}_{j\in\mathbb{N}}$.

In case the equivalent conditions are satisfied, $\sum_{j\in\mathbb{N}} x_j = \lim_F \sum_{j\in F} x_j$.

Definition 1.2. A sequence $\{x_j\}_{j\in\mathbb{N}}$ in a infinite-dimensional Banach space X is said to be a unconditional basis for X if to every $x \in X$ there corresponds a unique scalar sequence $\{c_j\}_{j\in\mathbb{N}}$ such that

$$x = \sum_{j\in\mathbb{N}} c_j x_j,$$

i.e., $\sum_{j=1}^{\infty} c_j x_j$ unconditionally converges to x.

Unconditionality is an important property, and in practice we prefer "unconditional basis" over "conditional basis". For unconditional bases on some concrete spaces, we refer to [1, 2, 22, 29, 30, 34, 48] and references therein. For unconditional bases on abstract spaces, we refer to [21, 7, 17, 31, 36, 39] and the references therein. It is worth noting that Gowers and Maurey in [21] constructed a hereditarily indecomposable separable Banach space which contains no infinite unconditional basic sequence, where a Banach space is said to be indecomposable if it is not isomorphic to a direct sum of two infinite-dimensional Banach spaces, and an infinite-dimensional Banach space is called hereditarily indecomposable if all of its infinite-dimensional closed subspaces are indecomposable. This paper addresses a class of "g-unconditional bases" with respect to operators between Hilbert spaces.

Another notion closely related to bases is "frame" in a Hilbert space which is a generalization of "orthonormal basis" and may be redundant. It was first formally introduced in [12] by Duffin and Schaeffer in 1952 to study nonharmonic Fourier series. However, it had not attracted enough attention from mathematics until the advent of wavelet analysis. The development of wavelet analysis has brought new vitality to the frame theory.

During the last more than thirty years, the frame theory has seen great achievements in pure and applied mathematics ([8, 11, 13, 14, 26, 20, 32, 28]). For the fundamentals of the frame theory, see the references ([10, 12, 25-28]) and therein. In 2006, Sun in [44] proposed the concept of g-frame, which covers the existing frames: bounded quasi-projectors in [18, 19], frames of subspaces in [4, 6], pseudo-frames in [33], oblique frames in [9, 15], and outer frames in [3]. Casazza, Han and Larson in [8] proved that every separable Banach space has a Banach frame, and that a Banach space having an atomic decomposition is equivalent to it having the bounded approximation property. Kaftal, Larson and Zhang in [32] developed operator-valued frame theory, extended many results in [26]. Găvruta and Găvruta in [20] presented a new formula for operator-valued frames in finite-dimensional Hilbert spaces. For basic results on g-frames, we refer to [35, 37, 45, 47, 49] and the references therein.

Motivated by the above works, Guo in [23] first introduced the concept of g-basis, and characterized g-bases, g-orthonormal bases and the equivalence between them. In [24], he obtained a perturbation result and some new properties for g-bases.

Let $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ be complex separable Hilbert spaces, and $\{V_j : j \in \mathbb{N}\}$ a sequence of closed subspaces of $\mathcal{V}$. We denote by $\mathcal{B}(\mathcal{U}, \mathcal{W})$ the set of bounded linear operators from $\mathcal{U}$ to $\mathcal{W}$; by $\mathcal{M}(\mathcal{U}, \mathcal{W})$ the set of all linear mappings from $\mathcal{U}$ to $\mathcal{W}$; and by $l(\{V_j\}_{j\in\mathbb{N}})$ the vector space:

$$l(\{V_j\}_{j\in\mathbb{N}}) = \{\mathbf{g} = \{g_j\}_{j\in\mathbb{N}} : g_j \in V_j \text{ for } j \in \mathbb{N}\}$$

with the usual scalar multiplication and coordinate-wise addition.

Definition 1.3. Let $\{\Lambda_j : j \in \mathbb{N}\}$ be a sequence with each $\Lambda_j \in \mathcal{B}(\mathcal{U}, V_j)$.

(i) $\{\Lambda_j : j \in \mathbb{N}\}$ is said to be g-complete if $\{f : \Lambda_j f = 0 \text{ for } j \in \mathbb{N}\} = \{0\}$.

(ii) $\{\Lambda_j : j \in \mathbb{N}\}$ is called a g-basis for $\mathcal{U}$ if to every $x \in \mathcal{U}$ there corresponds a unique $\mathbf{g} \in l(\{V_j\}_{j\in\mathbb{N}})$ such that

$$x = \sum_{j=1}^{\infty} \Lambda_j^* g_j. \tag{1.2}$$

Define Γ_j: $\mathcal{U} \to V_j$ by $\Gamma_j x = g_j$ for each $j \in \mathbb{N}$. Then $\Gamma_j \in \mathcal{M}(\mathcal{U}, V_j)$ for each $j \in \mathbb{N}$. We call $\{\Gamma_j : j \in \mathbb{N}\}$ the g-dual sequence of $\{\Lambda_j : j \in \mathbb{N}\}$.

(iii) $\{\Lambda_j : j \in \mathbb{N}\}$ is said to be g-linearly independent if $\sum\limits_{j=1}^{\infty} \Lambda_j^* g_j = 0$ for some $\mathbf{g} \in l(\{V_j\}_{j\in\mathbb{N}})$ implies $\mathbf{g} = 0$.

Remark 1.1. (i) By [38, Proposition 2.11], $\{\Lambda_j : j \in \mathbb{N}\}$ is g-complete if and only if $\overline{\text{span}}\{\Lambda_j^*(V_j)\}_{j\in\mathbb{N}} = \mathcal{U}$.

(ii) By a standard argument, the g-dual sequence $\{\Gamma_j : j \in \mathbb{N}\}$ is the unique solution to

$$x = \sum_{j=1}^{\infty} \Lambda_j^* \Gamma_j x \text{ for all } x \in \mathcal{U} \tag{1.3}$$

with $\Gamma_j \in \mathcal{M}(\mathcal{U}, V_j)$ for each $j \in \mathbb{N}$, and it is a g-biorthogonal sequence of $\{\Lambda_j : j \in \mathbb{N}\}$ in the sense that

$$\Gamma_j \Lambda_j^* = I_{V_j} \text{ for } j \in \mathbb{N} \text{ and } \Gamma_{j_1} \Lambda_{j_2}^* = 0 \text{ for } j_1 \neq j_2 \text{ in } \mathbb{N}, \tag{1.4}$$

where I_{V_j} denotes the identity operator on V_j for each $j \in \mathbb{N}$. And by [24, Theorems 13], $\Gamma_j \in \mathcal{B}(\mathcal{U}, V_j)$ for $j \in \mathbb{N}$ if each Λ_j is surjective in addition.

Guo in [24, Theorem 17] obtained the following characterization of g-bases.

PROPOSITION 1.2. *Let* $\{\Lambda_j \in \mathcal{B}(\mathcal{U}, V_j) : j \in \mathbb{N}\}$ *be a* g*-complete sequence with each* Λ_j *being onto. Then it is a* g*-basis for* $\mathcal{U}$ *if and only if there exists a constant* C *such that*

$$\Big\| \sum_{j=1}^{M} \Lambda_j^* g_j \Big\| \le C \Big\| \sum_{j=1}^{N} \Lambda_j^* g_j \Big\| \tag{1.5}$$

for $M, N \in \mathbb{N}$ *with* $M \le N$ *and* $g_j \in V_j$ *with* $1 \le j \le N$.

As mentioned before, unconditionality is more favourable than conditionality in applications. It is natural to ask what is "g-unconditional basis" and how we characterize it. In this paper, we introduce the notion of g-unconditional basis, and characterize g-unconditional bases.

Definition 1.4. Let $\{\Lambda_j : j \in \mathbb{N}\}$ be a sequence with each $\Lambda_j \in \mathcal{B}(\mathcal{U}, V_j)$.

(i) $\{\Lambda_j : j \in \mathbb{N}\}$ is called a g-unconditional basis for $\mathcal{U}$ if to every $x \in \mathcal{U}$ there corresponds a unique $\mathbf{g} \in l(\{V_j\}_{j\in\mathbb{N}})$ such that

$$x = \sum_{j\in\mathbb{N}} \Lambda_j^* g_j, \tag{1.6}$$

i.e., the series $\sum_{j=1}^{\infty} \Lambda_j^* g_j$ is unconditionally convergent to x. Define $\Gamma_j \colon \mathcal{U} \to V_j$ by $\Gamma_j x = g_j$ for each $j \in \mathbb{N}$. Similarly to Definition 1.3, $\Gamma_j \in \mathcal{M}(\mathcal{U}, V_j)$ for each $j \in \mathbb{N}$, and $\{\Gamma_j : j \in \mathbb{N}\}$ is called the g-dual sequence of $\{\Lambda_j : j \in \mathbb{N}\}$.

(ii) $\{\Lambda_j : j \in \mathbb{N}\}$ is said to be g-unconditionally linearly independent if $\sum_{j\in\mathbb{N}} \Lambda_j^* g_j = 0$ for some $\mathbf{g} \in l(\{V_j\}_{j\in\mathbb{N}})$ implies $\mathbf{g} = 0$.

Remark 1.2. (i) Clearly, a g-unconditional basis must be a g-basis. So the g-dual sequence $\{\Gamma_j : j \in \mathbb{N}\}$ is the unique solution to (1.6) with $\Gamma_j \in \mathcal{M}(\mathcal{U}, V_j)$ and $\Gamma_j x = g_j$ for each $j \in \mathbb{N}$, and a g-biorthogonal sequence of $\{\Lambda_j : j \in \mathbb{N}\}$ in the sense of (1.4) by Remark 1.1 (ii). And by Remark 1.1 (ii), $\Gamma_j \in \mathcal{B}(\mathcal{U}, V_j)$ for $j \in \mathbb{N}$ if each Λ_j is surjective in addition.

(ii) Observe that, if $\sum_{j=1}^{\infty} \Lambda_j^* g_j$ is unconditionally convergent, then the sum of $\sum_{j=1}^{\infty} \Lambda_{\sigma(j)}^* g_{\sigma(j)}$ is the same regardless of the choice of the permutation σ of $\mathbb{N}$. It follows that $\{\Lambda_j : j \in \mathbb{N}\}$ is a g-unconditional basis for $\mathcal{U}$ if and only if to every $x \in \mathcal{U}$ there corresponds a unique $\mathbf{g} \in l(\{V_j\}_{j\in\mathbb{N}})$ such that

$$x = \sum_{j=1}^{\infty} \Lambda_j^* g_j,$$

and $\sum_{j=1}^{\infty} \Lambda_j^* g_j$ is unconditionally convergent (i.e., $\sum_{j=1}^{\infty} \Lambda_{\sigma(j)}^* g_{\sigma(j)}$ is convergent for each permutation σ of $\mathbb{N}$).

To state our results, we first give the following definition and notations.

Definition 1.5. Let $\{\Lambda_j : j \in \mathbb{N}\}$ be a sequence with each $\Lambda_j \in \mathcal{B}(\mathcal{U}, V_j)$.

(i) Given a permutation σ of $\mathbb{N}$, $\{\Lambda_j : j \in \mathbb{N}\}$ is called a σ permuted g-basis for $\mathcal{U}$ if $\{\Lambda_{\sigma(j)} : j \in \mathbb{N}\}$ is a g-basis. And it is said to be σ permuted g-linearly independent if $\sum_{j=1}^{\infty} \Lambda_{\sigma(j)}^* g_{\sigma(j)} = 0$ for some $\mathbf{g} \in l(\{V_j\}_{j\in\mathbb{N}})$ implies that $\mathbf{g} = 0$.

(ii) $\{\Lambda_j : j \in \mathbb{N}\}$ is called an arbitrarily permuted g-basis for $\mathcal{U}$ if, for every permutation σ of $\mathbb{N}$, $\{\Lambda_j : j \in \mathbb{N}\}$ is a σ permuted g-basis for $\mathcal{U}$. And it is said to be arbitrarily permuted g-linearly independent if, for every permutation σ of $\mathbb{N}$, $\{\Lambda_j : j \in \mathbb{N}\}$ is σ permuted g-linearly independent.

Let $\{\Lambda_j \in \mathcal{B}(\mathcal{U}, V_j) : j \in \mathbb{N}\}$ be a g-unconditional basis for $\mathcal{U}$ with the g-dual sequence $\{\Gamma_j : j \in \mathbb{N}\}$, and let F be a finite subset of $\mathbb{N}$, $\epsilon = \{\epsilon_j\}_{j\in F}$ with $\epsilon_j = \pm 1$, and $\lambda = \{\lambda_j\}_{j\in F}$ with $|\lambda_j| \leq 1$. Define $S_F : \mathcal{U} \to \mathcal{U}$ by

$$S_F(x) = \sum_{j\in F} \Lambda_j^* \Gamma_j x, \tag{1.7}$$

$S_{F,\epsilon} : \mathcal{U} \to \mathcal{U}$ by

$$S_{F,\epsilon}(x) = \sum_{j\in F} \epsilon_j \Lambda_j^* \Gamma_j x, \tag{1.8}$$

and $S_{F,\lambda} : \mathcal{U} \to \mathcal{U}$ by

$$S_{F,\lambda}(x) = \sum_{j\in F} \lambda_j \Lambda_j^* \Gamma_j x \tag{1.9}$$

for $x \in \mathcal{U}$, respectively. In particular, we also write S_N, $S_{N,\epsilon}$ and $S_{N,\lambda}$ in (1.7)-(1.9) for S_F, $S_{F,\epsilon}$ and $S_{F,\lambda}$ if $F = \{1, 2, \cdots, N\}$ with $N \in \mathbb{N}$. For $x \in \mathcal{U}$, we write

$$\begin{aligned} |||x||| &= \sup\{\|S_F(x)\| : F \text{ is a finite subset of } \mathbb{N}\}, \\ |||x|||_\epsilon &= \sup\{\|S_{F,\epsilon}(x)\| : F \text{ is a finite subset of } \mathbb{N}, \\ &\qquad \text{and } \epsilon = \{\epsilon_j\}_{j\in F} \text{ with } \epsilon_j = \pm 1\}, \\ |||x|||_\lambda &= \sup\{\|S_{F,\lambda}(x)\| : F \text{ is a finite subset of } \mathbb{N}, \\ &\qquad \text{and } \lambda = \{\lambda_j\}_{j\in F} \text{ with } |\lambda_j| \leq 1\}, \end{aligned} \tag{1.10}$$

and

$$\mathcal{C} = \sup\{\|S_F\| : F \text{ is a finite subset of } \mathbb{N}\}, \tag{1.11}$$

(1.12) $\mathcal{C}_\epsilon = \sup\{\|S_{F,\epsilon}\| : F \text{ is a finite subset of } \mathbb{N}, \epsilon = \{\epsilon_j\}_{j\in F} \text{ with } \epsilon_j = \pm 1\}$,

(1.13)
$$\mathcal{C}_\lambda = \sup\{\|S_{F,\lambda}\| : F \text{ is a finite subset of } \mathbb{N}, \lambda = \{\lambda_j\}_{j\in F} \text{ with } |\lambda_j| \leq 1\}.$$

The main result of this paper is as follows.

THEOREM 1.1. *Let $\{\Lambda_j \in \mathcal{B}(\mathcal{U}, V_j) : j \in \mathbb{N}\}$ be a g-complete sequence with each Λ_j being onto. Then the following are equivalent:*

(i) *$\{\Lambda_j : j \in \mathbb{N}\}$ is a g-unconditional basis for $\mathcal{U}$.*

(ii) *$\{\Lambda_j : j \in \mathbb{N}\}$ is arbitrarily permuted g-basis for $\mathcal{U}$.*

(iii) *There exists a constant C such that*

$$\Big\|\sum_{j=1}^{N} \epsilon_j \Lambda_j^* g_j\Big\| \leq C\Big\|\sum_{j=1}^{N} \Lambda_j^* g_j\Big\| \tag{1.14}$$

for $N \in \mathbb{N}$, $\epsilon_j = \pm 1$ and $g_j \in V_j$ with $1 \leq j \leq N$.

(iv) *There exists a constant C such that*

$$\Big\|\sum_{j=1}^{N} \mu_j \Lambda_j^* g_j\Big\| \leq C\Big\|\sum_{j=1}^{N} \Lambda_j^* g_j\Big\|$$

for $N \in \mathbb{N}$, $g_j \in V_j$ and $|\mu_j| \leq 1$ with $1 \leq j \leq N$.

(v) *$\{\Lambda_j : j \in \mathbb{N}\}$ is a g-basis for $\mathcal{U}$, and for each bounded sequence $\mu = \{\mu_j\}_{j\in\mathbb{N}}$ there exists an operator $T_\mu \in \mathcal{B}(\mathcal{U})$ such that*

$$T_\mu \Lambda_j^* g_j = \mu_j \Lambda_j^* g_j$$

for $g_j \in V_j$ with $j \in \mathbb{N}$.

(vi) *There exists a constant C such that*

$$\Big\|\sum_{j\in F_1} \Lambda_j^* g_j\Big\| \leq C\Big\|\sum_{j\in F_2} \Lambda_j^* g_j\Big\| \tag{1.15}$$

for finite sets $F_1, F_2 \subset \mathbb{N}$ with $F_1 \subset F_2$ and $g_j \in V_j$ with $j \in F_2$.

2. SOME AUXILIARY LEMMAS

This section focuses on some auxiliary lemmas for later use. The first lemma is repeated from [27, Theorem 3.13] and [42].

LEMMA 2.1. *Suppose $\{\lambda_n\}_{n=1}^N$ is a sequence of real numbers with $|\lambda_n| \leq 1$ for $1 \leq n \leq N$. Then there exist sequences $\{\epsilon_k^n\}_{k=1}^N$ with $\epsilon_k^n = \pm 1$ and real numbers $c_k \geq 0$ for $k = 1, 2, \cdots, N+1$ and $n = 1, 2, \cdots, N$ such that*

$$\sum_{k=1}^{N+1} c_k = 1 \text{ and } \sum_{k=1}^{N+1} \epsilon_k^n c_k = \lambda_n$$

for $1 \leq n \leq N$.

A slight modification of the proof of [24, Theorem 11] leads to the following lemma.

LEMMA 2.2. *Let* $\{\Lambda_j \in \mathcal{B}(\mathcal{U}, V_j) : j \in \mathbb{N}\}$ *be a sequence with each* Λ_j *being onto. Define*

$$\mathcal{G} = \Big\{\mathbf{g} \in l(\{V_j\}_{j\in\mathbb{N}}) : \sum_{j\in\mathbb{N}} \Lambda_j^* g_j \text{ is well defined } \Big\},$$

and

$$\|\mathbf{g}\|_{\mathcal{G}} = \sup\Big\{\Big\|\sum_{j\in F} \Lambda_j^* g_j\Big\| : F \text{ is a finite subset of } \mathbb{N}\Big\} \text{ for } \mathbf{g} \in l(\{V_j\}_{j\in\mathbb{N}}).$$

Then

(i) $\mathcal{G}$ *is a Banach space.*

(ii) *The operator* $T : \mathcal{G} \to \mathcal{U}$ *defined by*

$$T\mathbf{g} = \sum_{j\in\mathbb{N}} \Lambda_j^* g_j$$

is a bounded and invertible operator from $\mathcal{G}$ *onto* $\mathcal{U}$ *provided that* $\{\Lambda_j : j \in \mathbb{N}\}$ *is a g-unconditional basis for* $\mathcal{U}$.

LEMMA 2.3. *Let* $\{\Lambda_j \in \mathcal{B}(\mathcal{U}, V_j) : j \in \mathbb{N}\}$ *be a g-unconditional basis for* $\mathcal{U}$ *with the g-dual sequence* $\{\Gamma_j : j \in \mathbb{N}\}$, *and each* Λ_j *be onto. Then the numbers* $\mathcal{C}$, $\mathcal{C}_\epsilon$ *and* $\mathcal{C}_\lambda$ *in (1.11)-(1.13) are finite, and* $|||\cdot|||$, $|||\cdot|||_\epsilon$ *and* $|||\cdot|||_\lambda$ *in (1.10) form norms defined on* $\mathcal{U}$, *each equivalent to the initial norm* $\|\cdot\|$ *with*

$$\|\cdot\| \le |||\cdot||| \le \mathcal{C}\|\cdot\|, \tag{2.1}$$

$$\|\cdot\| \le |||\cdot|||_\epsilon \le \mathcal{C}_\epsilon\|\cdot\|, \tag{2.2}$$

$$\|\cdot\| \le |||\cdot|||_\lambda \le \mathcal{C}_\lambda\|\cdot\|. \tag{2.3}$$

Proof. We first deal with $|||\cdot|||$-case. Since $\{\Lambda_j : j \in \mathbb{N}\}$ is a g-unconditional basis for $\mathcal{U}$ with the g-dual sequence $\{\Gamma_j : j \in \mathbb{N}\}$, we have

$$x = \sum_{j\in\mathbb{N}} \Lambda_j^* \Gamma_j x, \tag{2.4}$$

equivalently,

$$x = \lim_F S_F(x) \tag{2.5}$$

for $x \in \mathcal{U}$ by Proposition 1.1. This implies that

$$|||x||| < \infty \text{ for } x \in \mathcal{U}. \tag{2.6}$$

So $|||\cdot|||$ is well defined on $\mathcal{U}$ by the arbitrariness of x. Also it is easy to check that $|||\cdot|||$ is a semi-norm. Now suppose $|||x||| = 0$. Then $\sum_{j\in F} \Lambda_j^* \Gamma_j x = 0$ for each finite set F. This implies that $x = 0$ by (2.5), and thus $|||\cdot|||$ is a norm defined on $\mathcal{U}$. Now let $\mathcal{G}$ and the operator T be defined as in Lemma 2.2. Then for each $x \in \mathcal{U}$, we have $\{\Gamma_j x\}_{j\in\mathbb{N}} \in \mathcal{G}$ by (2.4). It follows that

$$\begin{aligned} \|S_F(x)\| &= \|\sum_{j\in F} \Lambda_j^* \Gamma_j x\| \\ &\leq \|\{\Gamma_j x\}_{j\in\mathbb{N}}\|_{\mathcal{G}} \\ &= \|T^{-1}(x)\| \\ &\leq \|T^{-1}\|\|x\| \end{aligned}$$

for $x \in \mathcal{U}$ and arbitrary finite set $F \subset \mathbb{N}$ by Lemma 2.2. So $\|S_F\| \leq \|T^{-1}\|$ for each finite set $F \subset \mathbb{N}$, and thus

$$\begin{aligned} \mathcal{C} &= \sup\{\|S_F\| : \text{ for all finite set } F \subset \mathbb{N}\} \\ &\leq \|T^{-1}\| \\ &< \infty. \end{aligned}$$

This implies that

$$\|S_F(x)\| \leq \mathcal{C}\|x\| \tag{2.7}$$

for $x \in \mathcal{U}$ and each finite set $F \subset \mathbb{N}$, and thus

$$|||x||| \leq \mathcal{C}\|x\| \tag{2.8}$$

for $x \in \mathcal{U}$. On the other hand,

$$\|S_F(x)\| \leq |||x|||$$

for $x \in \mathcal{U}$ and each finite set $F \subset \mathbb{N}$ by the definition of $|||\cdot|||$, which implies that

$$\|x\| \leq |||x||| \tag{2.9}$$

for $x \in \mathcal{U}$ by (2.5). Collecting (2.8) and (2.9) gives (2.1).

Next we deal with (2.2). Arbitrarily fix a finite set $F \subset \mathbb{N}$ and a sequence $\epsilon = \{\epsilon_j\}_{j\in F}$ with $\epsilon_j = \pm 1$. Let $F_+ = \{j \in F : \epsilon_j = 1\}$ and $F_- = \{j \in F : \epsilon_j = -1\}$. Then

$$\begin{aligned} \|S_{F,\epsilon}(x)\| &= \|\sum_{j\in F} \epsilon_j \Lambda_j^* \Gamma_j x\| \\ &= \|\sum_{j\in F_+} \Lambda_j^* \Gamma_j x - \sum_{j\in F_-} \Lambda_j^* \Gamma_j x\| \\ &\leq \|S_{F_+}(x)\| + \|S_{F_-}(x)\| \\ &\leq 2\mathcal{C}\|x\| \qquad (2.10) \end{aligned}$$

for $x \in \mathcal{U}$ by (2.7). This implies that $\|S_{F,\epsilon}\| \leq 2\mathcal{C}$, and

$$\mathcal{C}_\epsilon \leq 2\mathcal{C} < \infty.$$

By the definition of $\mathcal{C}_\epsilon$, we have

$$\|S_{F,\epsilon}(x)\| \leq \mathcal{C}_\epsilon \|x\|, \tag{2.11}$$

and thus

$$|||x|||_\epsilon \leq \mathcal{C}_\epsilon \|x\| \tag{2.12}$$

for $x \in \mathcal{U}$. On the other hand, it is easy to check that $|||\cdot|||_\epsilon$ is a semi-norm defined on $\mathcal{U}$, and

$$\|x\| \leq |||x|||_\epsilon \tag{2.13}$$

for $x \in \mathcal{U}$. This implies that $x = 0$ if $|||x|||_\epsilon = 0$ due to $\|\cdot\|$ being a norm on $\mathcal{U}$. So $|||\cdot|||_\epsilon$ is a norm defined on $\mathcal{U}$. Collecting (2.12) and (2.13) gives (2.2).

Now we turn to (2.3). Arbitrarily fix a finite set $F \subset \mathbb{N}$ and a sequence $\lambda = \{\lambda_j\}_{j \in F}$ with $|\lambda_j| \leq 1$. Then

$$\begin{aligned} \|S_{F,\lambda}(x)\| &= \|\sum_{j\in F} \lambda_j \Lambda_j^* \Gamma_j x\| \\ &= \|\sum_{j\in F} (\mathrm{Re}(\lambda_j) + i\mathrm{Im}(\lambda_j))\, \Lambda_j^* \Gamma_j x\| \\ &\leq \|\sum_{j\in F} \mathrm{Re}(\lambda_j)\Lambda_j^* \Gamma_j x\| + \|\sum_{j\in F} \mathrm{Im}(\lambda_j)\Lambda_j^* \Gamma_j x\| \end{aligned} \tag{2.14}$$

for $x \in \mathcal{U}$, where $\mathrm{Re}(\lambda_j)$ and $\mathrm{Im}(\lambda_j)$ denotes the real part and imaginary part of λ_j with $j \in F$, respectively. Observe that $\mathrm{Re}(\lambda_j)$ belongs to $\mathbb{R}$ and $|\mathrm{Re}(\lambda_j)| \leq 1$ for $j \in F$. Then by letting $N = \mathrm{card}(F)$, we can find a scalar sequence $\alpha = \{\alpha_l\}_{l=1}^{N+1}$ with $\alpha_l \geq 0$ and sequences $\epsilon^j = \{\epsilon_l^j\}_{l=1}^{N+1}$ for each $j \in F$ with $\epsilon_l^j = \pm 1$ for $1 \leq l \leq N+1$ such that

$$\sum_{l=1}^{N+1} \alpha_l = 1 \text{ and } \mathrm{Re}(\lambda_j) = \sum_{l=1}^{N+1} \epsilon_l^j \alpha_l \text{ for } j \in F \tag{2.15}$$

by Lemma 2.1. This implies that

$$\begin{aligned} \|\sum_{j\in F} \mathrm{Re}(\lambda_j)\Lambda_j^* \Gamma_j x\| &= \|\sum_{j\in F} \sum_{l=1}^{N+1} \epsilon_l^j \alpha_l \Lambda_j^* \Gamma_j x\| \\ &= \|\sum_{l=1}^{N+1} \alpha_l \sum_{j\in F} \epsilon_l^j \Lambda_j^* \Gamma_j x\| \end{aligned}$$

$$\leq \sum_{l=1}^{N+1} \alpha_l \Big\| \sum_{j\in F} \epsilon_l^j \Lambda_j^* \Gamma_j x \Big\|$$

$$\leq 2\mathcal{C}\Big(\sum_{l=1}^{N+1} \alpha_l \Big) \|x\|$$

$$= 2\mathcal{C}\|x\| \tag{2.16}$$

by (2.10). Similarly,

$$\Big\| \sum_{j\in F} \mathrm{Im}(\lambda_j)\Lambda_j^*\Gamma_j x \Big\| \leq 2\mathcal{C}\|x\|. \tag{2.17}$$

So

$$\|S_{F,\lambda}(x)\| \leq 4\mathcal{C}\|x\| \tag{2.18}$$

for $x \in \mathcal{U}$ by (2.14). It follows that $\|S_{F,\lambda}\| \leq 4\mathcal{C}$, and

$$\mathcal{C}_\lambda \leq 4\mathcal{C} < \infty.$$

By the definition of $\mathcal{C}_\lambda$,

$$|||x|||_\lambda \leq \mathcal{C}_\lambda \|x\| \tag{2.19}$$

for $x \in \mathcal{U}$, and $|||\cdot|||_\lambda$ is well defined by the arbitrariness of x. On the other hand,

$$\|x\| \leq |||x|||_\lambda \tag{2.20}$$

for $x \in \mathcal{U}$ by (2.5). By a standard argument, we can prove that $|||\cdot|||_\lambda$ is a norm defined on $\mathcal{U}$. The proof is completed. □

3. PROOF OF THEOREM 1.1

We prove the theorem by showing that

$$\text{(i)} \Rightarrow \text{(iii)} \Rightarrow \text{(ii)} \Rightarrow \text{(i)} \Rightarrow \text{(iv)} \Rightarrow \text{(ii) and (v)} \Leftrightarrow \text{(i)} \Leftrightarrow \text{(vi)}.$$

(i) $\Longrightarrow$ (iii). Suppose $\{\Lambda_j : j \in \mathbb{N}\}$ is a g-unconditional basis for $\mathcal{U}$ with the g-dual sequence $\{\Gamma_j : j \in \mathbb{N}\}$. Arbitrarily fix $N \in \mathbb{N}$, $\epsilon_j \in \{\pm 1\}$ and $g_j \in V_j$ with $1 \leq j \leq N$. Take

$$x = \sum_{j=1}^{N} \Lambda_j^* g_j.$$

Then

$$\Gamma_j x = \begin{cases} g_j, & 1 \leq j \leq N; \\ 0, & \text{otherwise} \end{cases}$$

by Remark 1.1. It follows that

$$\|\sum_{j=1}^{N}\epsilon_j\Lambda_j^*g_j\| \;=\; \|\sum_{j=1}^{N}\epsilon_j\Lambda_j^*\Gamma_j x\| = \|S_{N,\epsilon}(x)\| \le \||x\||_\epsilon \le \mathcal{C}_\epsilon\|\sum_{j=1}^{N}\Lambda_j^*g_j\|.$$

(iii) $\Longrightarrow$ (ii). Suppose (iii) holds. Arbitrarily fix a permutation σ of $\mathbb{N}$. Observe that $\{\Lambda_j : j\in\mathbb{N}\}$ is g-complete and each Λ_j is onto. To demonstrate that $\{\Lambda_j : j\in\mathbb{N}\}$ is a σ permuted g-basis, we only need to prove that there exists a constant C_σ such that

$$\|\sum_{j=1}^{M}\Lambda_{\sigma(j)}^*g_{\sigma(j)}\| \le C_\sigma\|\sum_{j=1}^{N}\Lambda_{\sigma(j)}^*g_{\sigma(j)}\| \tag{3.1}$$

for $M,N\in\mathbb{N}$ with $M\le N$ and $\mathbf{g}\in l(\{V_j\}_{j\in\mathbb{N}})$ by Proposition 1.2. Arbitrarily fix $M,N\in\mathbb{N}$ with $M\le N$ and $\mathbf{g}\in l(\{V_j\}_{j\in\mathbb{N}})$. Let $L=\max\{\sigma(j): 1\le j\le N\}$. Define

$$\widetilde{g}_j = \begin{cases} g_j, & j\in\{\sigma(l): 1\le l\le N\};\\ 0, & \text{otherwise,}\end{cases}$$

and

$$\epsilon_j = 1 \text{ and } \gamma_j = \begin{cases} 1, & j\in\{\sigma(l): 1\le l\le M\};\\ -1, & \text{otherwise}\end{cases}$$

for $j\in\mathbb{N}$, respectively. Then

$$\begin{aligned}\|\sum_{j=1}^{M}\Lambda_{\sigma(j)}^*g_{\sigma(j)}\| &= \|\sum_{j=1}^{L}\frac{\epsilon_j+\gamma_j}{2}\Lambda_j^*\widetilde{g}_j\|\\ &\le \frac{1}{2}\|\sum_{j=1}^{L}\epsilon_j\Lambda_j^*\widetilde{g}_j\| + \frac{1}{2}\|\sum_{j=1}^{L}\gamma_j\Lambda_j^*\widetilde{g}_j\|\\ &\le \frac{C}{2}\|\sum_{j=1}^{L}\Lambda_j^*\widetilde{g}_j\| + \frac{C}{2}\|\sum_{j=1}^{L}\Lambda_j^*\widetilde{g}_j\|\\ &= C\|\sum_{j=1}^{N}\Lambda_{\sigma(j)}^*g_{\sigma(j)}\|\end{aligned}$$

by (iii), and thus (3.1) holds with $C_\sigma = C$.

(ii) $\Longrightarrow$ (i). Suppose $\{\Lambda_j : j\in\mathbb{N}\}$ is an arbitrarily permuted g-basis for $\mathcal{U}$. Arbitrarily fix a permutation σ of $\mathbb{N}$. Then $\{\Lambda_j : j\in\mathbb{N}\}$ is both a g-basis and σ permutated g-basis. Let $\{\Gamma_j : j\in\mathbb{N}\}$ and $\{\Gamma'_{\sigma(j)} : j\in\mathbb{N}\}$ be the g-dual sequences of $\{\Lambda_j : j\in\mathbb{N}\}$ and $\{\Lambda_{\sigma(j)} : j\in\mathbb{N}\}$, respectively. Then

$$x=\sum_{j=1}^{\infty}\Lambda_j^*\Gamma_j x, \tag{3.2}$$

and

$$x=\sum_{j=1}^{\infty}\Lambda^*_{\sigma(j)}\Gamma'_{\sigma(j)}x \tag{3.3}$$

for $x\in\mathcal{U}$. It follows that

$$\Gamma_j\Lambda^*_j=I_{V_j} \text{ for } j \text{ in } \mathbb{N},\quad \Gamma_j\Lambda^*_l=0 \text{ for } j\neq l \text{ in } \mathbb{N},$$

and

$$\Gamma^{'}_j\Lambda^*_j=I_{V_j} \text{ for } j \text{ in } \mathbb{N},\quad \Gamma^{'}_j\Lambda^*_l=0 \text{ for } j\neq l \text{ in } \mathbb{N} \tag{3.4}$$

by Remark 1.1 (ii), where we used the fact that σ is a permutation of $\mathbb{N}$ in (3.4). It follows that

$$\Gamma_j\Lambda^*_l=\Gamma'_j\Lambda^*_l \text{ for } j,l\in\mathbb{N}. \tag{3.5}$$

Since $\{\Lambda_j : j\in\mathbb{N}\}$ is g-complete and each Λ_j is onto, $\overline{\text{span}}\{\Lambda^*_j(V_j) : j\in\mathbb{N}\})=\mathcal{U}$ and $\Gamma_j,\ \Gamma^{'}_j\in\mathcal{B}(\mathcal{U},\ V_j)$ for $j\in\mathbb{N}$ by Remark 1.1. This, together with (3.5), implies that

$$\Gamma_j=\Gamma'_j \text{ for } j\in\mathbb{N}.$$

So collecting (3.2) and (3.3) leads to

$$x=\sum_{j\in\mathbb{N}}\Lambda^*_j\Gamma_jx \tag{3.6}$$

for $x\in\mathcal{U}$ by the arbitrariness of σ. Thus $\{\Lambda_j : j\in\mathbb{N}\}$ is a g-unconditional basis by its definition.

(i) $\Longrightarrow$ (iv). Suppose $\{\Lambda_j : j\in\mathbb{N}\}$ is a g-unconditional basis for $\mathcal{U}$ with the g-dual sequence $\{\Gamma_j : j\in\mathbb{N}\}$. Arbitrarily fix $N\in\mathbb{N}$, $g_j\in V_j$ and $\{\mu_j\}_{j=1}^N$ with $|\mu_j|\leq 1$ for $1\leq j\leq N$. Let $x=\sum\limits_{j=1}^{N}\Lambda^*_jg_j$. Then

$$\begin{aligned}\|\sum_{j=1}^{N}\mu_j\Lambda^*_jg_j\| &= \|\sum_{j=1}^{N}\mu_j\Lambda^*_j\Gamma_jx\|\\ &\leq |||x|||_\lambda\\ &\leq \mathcal{C}_\lambda\|x\|\\ &= \mathcal{C}_\lambda\|\sum_{j=1}^{N}\Lambda^*_jg_j\|\end{aligned}$$

by Lemma 2.3.

(iv) $\Longrightarrow$ (ii). Suppose (iv) holds. Arbitrarily fix a permutation σ. It is enough to demonstrate that $\{\Lambda_j : j\in\mathbb{N}\}$ is a σ permutated g-basis. Observe that $\{\Lambda_j : j \in \mathbb{N}\}$ is g-complete and each Λ_j is onto. So, we only need to prove that there exists a constant C_σ such that

$$\|\sum_{j=1}^{M} \Lambda^*_{\sigma(j)} g_{\sigma(j)}\| \leq C_\sigma \|\sum_{j=1}^{N} \Lambda^*_{\sigma(j)} g_{\sigma(j)}\| \tag{3.7}$$

for $M, N \in \mathbb{N}$ with $M \leq N$ and $\mathbf{g} \in l(\{V_j\}_{j\in\mathbb{N}})$ by Proposition 1.2. Now arbitrarily fix $M, N \in \mathbb{N}$ with $M \leq N$ and $\mathbf{g} \in l(\{V_j\}_{j\in\mathbb{N}})$. Let $L = \max\{\sigma(j) : 1 \leq j \leq N\}$ and define

$$\widetilde{g}_j = \begin{cases} g_j, & j \in \{\sigma(l) : 1 \leq l \leq N\}; \\ 0, & \text{otherwise,} \end{cases} \quad \text{and } \mu_j = \begin{cases} 1, & j \in \{\sigma(l) : 1 \leq l \leq M\}; \\ 0, & \text{otherwise} \end{cases}$$

for $j \in \mathbb{N}$. Then

$$\begin{aligned} \|\sum_{j=1}^{M} \Lambda^*_{\sigma(j)} g_{\sigma(j)}\| &= \|\sum_{j=1}^{L} \mu_j \Lambda^*_j \widetilde{g}_j\| \\ &\leq C\|\sum_{j=1}^{L} \Lambda^*_j \widetilde{g}_j\| \\ &= C\|\sum_{j=1}^{N} \Lambda^*_{\sigma(j)} g_{\sigma(j)}\| \end{aligned}$$

by (iv), and thus (3.7) holds with $C_\sigma = C$.

(i) $\Rightarrow$ (v). Suppose $\{\Lambda_j : j \in \mathbb{N}\}$ is a g-unconditional basis for $\mathcal{U}$ with $\{\Gamma_j : j \in \mathbb{N}\}$ its g-dual sequence. Then it must be a g-basis for $\mathcal{U}$, and

$$x = \sum_{j\in\mathbb{N}} \Lambda^*_j \Gamma_j x \tag{3.8}$$

for $x \in \mathcal{U}$, which implies that

$$\sum_{j=1}^{\infty} \lambda_j \Lambda^*_j \Gamma_j x \tag{3.9}$$

is well defined for $x \in \mathcal{U}$ and each sequence $\lambda = \{\lambda_j\}_{j\in\mathbb{N}}$ with each $|\lambda_j| \leq 1$ by Proposition 1.1. Arbitrarily fix a bounded sequence $\mu = \{\mu_j\}_{j\in\mathbb{N}}$ with $|\mu_j| < M < \infty$ for $j \in \mathbb{N}$. Then the mapping $T_\mu : \mathcal{U} \to \mathcal{U}$ defined by

$$T_\mu x = \sum_{j=1}^{\infty} \mu_j \Lambda^*_j \Gamma_j x \tag{3.10}$$

for $x \in \mathcal{U}$ is well defined by (3.9). Since $\sum_{j=1}^{\infty} \mu_j \Lambda_j^* \Gamma_j x = M \sum_{j=1}^{\infty} \frac{\mu_j}{M} \Lambda_j^* \Gamma_j x$, we see that

$$\begin{aligned} \|T_\mu x\| &= M \| \sum_{j=1}^{\infty} \frac{\mu_j}{M} \Lambda_j^* \Gamma_j x \| \\ &\leq M \mathcal{C}_\lambda \|x\| \end{aligned}$$

for $x \in \mathcal{U}$ by (3.9) and Lemma 2.3. Also observe that

$$T_\mu(\Lambda_j^* g_j) = \sum_{j'=1}^{\infty} \mu_{j'} \Lambda_{j'}^* \Gamma_{j'} \Lambda_j^* g_j = \mu_j \Lambda_j^* g_j.$$

T_μ is as desired.

(v) $\Rightarrow$ (i). Suppose (v) holds. Let $\{\Lambda_j : j \in \mathbb{N}\}$ be a g-basis with the g-dual sequence $\{\Gamma_j : j \in \mathbb{N}\}$. Then

$$x = \sum_{j=1}^{\infty} \Lambda_j^* \Gamma_j x \tag{3.11}$$

for $x \in \mathcal{U}$. By Remark 1.1 (ii), to obtain (i), we only need to prove that, for every $x \in \mathcal{U}$, the series $\sum_{j=1}^{\infty} \Lambda_j^* \Gamma_j x$ is unconditionally convergent for every $x \in \mathcal{U}$, equivalently, $\sum_{j=1}^{\infty} \mu_j \Lambda_j^* \Gamma_j x$ is well defined for each bounded sequence $\mu = \{\mu_j\}_{j \in \mathbb{N}}$ by Proposition 1.1. Let μ be such a sequence. Then there exists an operator $T_\mu \in \mathcal{B}(\mathcal{U})$ defined by

$$T_\mu \Lambda_j^* g_j = \mu_j \Lambda_j^* g_j$$

for $g_j \in V_j$ with $j \in \mathbb{N}$ by (v). It follows that

$$T_\mu\left(\sum_{j=1}^{\infty} \Lambda_j^* \Gamma_j x\right) = \sum_{j=1}^{\infty} T_\mu(\Lambda_j^* \Gamma_j x) = \sum_{j=1}^{\infty} \mu_j \Lambda_j^* \Gamma_j x \tag{3.12}$$

for $x \in \mathcal{U}$, and thus $\sum_{j=1}^{\infty} \mu_j \Lambda_j^* \Gamma_j x$ is well defined for $x \in \mathcal{U}$ by (3.11).

(i) $\Rightarrow$ (vi). Suppose $\{\Lambda_j : j \in \mathbb{N}\}$ is a g-unconditional basis for $\mathcal{U}$ with the g-dual sequence $\{\Gamma_j : j \in \mathbb{N}\}$. Then we have

$$\|x\| \leq |||x||| \leq \mathcal{C}\|x\| \tag{3.13}$$

for $x \in \mathcal{U}$ by Lemma 2.3. Arbitrarily fix $\mathbf{g} \in l(\{V_j\}_{j \in \mathbb{N}})$ and a finite subset

$F_0 \subset \mathbb{N}$. Take $x = \sum_{j\in F_0} \Lambda_j^* g_j$. Then

$$\Gamma_j x = \begin{cases} g_j, & \text{if } j \in F_0; \\ 0, & \text{otherwise,} \end{cases} \tag{3.14}$$

and thus

$$\begin{aligned} |||x||| &= |||\sum_{j\in F_0} \Lambda_j^* \Gamma_j x||| \\ &\leq \mathcal{C}\|\sum_{j\in F_0} \Lambda_j^* \Gamma_j x\| \\ &= \mathcal{C}\|\sum_{j\in F_0} \Lambda_j^* g_j\| \end{aligned} \tag{3.15}$$

by (3.13). Also observe that

$$\begin{aligned} S_F(x) &= \sum_{j\in F} \Lambda_j^* \Gamma_j (\sum_{j'\in F_0} \Lambda_{j'}^* g_{j'}) \\ &= \sum_{j\in F}\sum_{j'\in F_0} \Lambda_j^* \Gamma_j \Lambda_{j'}^* g_{j'} \\ &= \sum_{j\in F\cap F_0} \Lambda_j^* g_j \end{aligned} \tag{3.16}$$

for each finite set $F \subset \mathbb{N}$ by Remark 1.1 (ii). It follows that

$$\|\sum_{j\in F\cap F_0} \Lambda_j^* g_j\| \leq |||x||| \leq \mathcal{C}\|\sum_{j\in F_0} \Lambda_j^* g_j\| \tag{3.17}$$

for each finite set $F \subset \mathbb{N}$. In particular,

$$\|\sum_{j\in F} \Lambda_j^* g_j\| \leq \mathcal{C}\|\sum_{j\in F_0} \Lambda_j^* g_j\| \tag{3.18}$$

for $F \subset F_0$. This leads to (1.15) by the arbitrariness of F_0.

(vi) ⇒ (i). Suppose (vi) holds. Then (1.5) holds, and thus $\{\Lambda_j : j \in \mathbb{N}\}$ is a g-basis by Proposition 1.2, that is, to every $x \in \mathcal{U}$ there corresponds a unique $\mathbf{g} \in l(\{V_j\}_{j\in\mathbb{N}})$ such that

$$x = \sum_{j=1}^{\infty} \Lambda_j^* g_j. \tag{3.19}$$

By Remark 1.2 (ii), to obtain (i), we only need to prove that the series $\sum_{j=1}^{\infty} \Lambda_j^* g_j$ is unconditionally convergent, equivalently,

$$\sum_{j=1}^{\infty} \Lambda_{\sigma(j)}^* g_{\sigma(j)}$$

is convergent for an arbitrary permutation σ of $\mathbb{N}$. Arbitrarily fix an arbitrary permutation σ. Next, we prove the series $\sum_{j=1}^{\infty} \Lambda^*_{\sigma(j)} g_{\sigma(j)}$ is convergent. By (3.19), to every $\epsilon > 0$ there corresponds $N \in \mathbb{N}$ such that

$$\Big\| \sum_{j=m}^{n} \Lambda^*_j g_j \Big\| < \epsilon \text{ for } n \geq m > N. \tag{3.20}$$

Take

$$K = \min \big\{ k \in \mathbb{N} : \{1, 2, \cdots, N\} \subset \{\sigma(1), \sigma(2), \cdots, \sigma(k)\} \big\}.$$

Then $\sigma(j) > N$ for $j > K$, this implies that

$$\{\sigma(j) : k \leq j \leq l\} \subset \{j \in \mathbb{N} : N + 1 \leq j \leq M\}$$

for $l \geq k > K$, where $M = \max\{\sigma(j) : k \leq j \leq l\}$. It follows that

$$\Big\| \sum_{j=k}^{l} \Lambda^*_{\sigma(j)} g_{\sigma(j)} \Big\| \leq C \Big\| \sum_{j=N+1}^{M} \Lambda^*_j g_j \Big\| < C\epsilon$$

for $l \geq k > K$ by (vi) and (3.20). Therefore, $\sum_{j=1}^{\infty} \Lambda^*_{\sigma(j)} g_{\sigma(j)}$ converges. The proof is completed.

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