A NOTE ON UNIFORM STATISTICAL LIMIT POINTS

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Communicated by Lucian Beznea

In this paper, we study the concept of a uniform statistical limit point of a sequence. We present some results about the set of uniform statistical limit points of a sequence and about constructing a (sub)sequence with a prescribed set of uniform statistical limit points. Also, we study the relationship between the set of uniform statistical limit points of a sequence and its subsequences. We give some analogous results concerning (non-uniform) statistical limit points of a sequence and connect to some earlier known results about the same.

AMS 2020 Subject Classification: 40G99, 28A12.

Key words: uniform statistical convergence, subsequences, uniform statistical limit points.

1. INTRODUCTION AND PRELIMINARIES

The convergence of sequences has many generalizations with the aim of providing deeper insights into summability theory. One of the most important generalizations is uniform statistical convergence. It has been introduced by Brown and Freedman [3] with the use of uniform density. This concept of convergence has been studied by many authors in different directions ([2], [13], [14], [19], [20]) and it is quite effective, especially when the classical limit does not exist since it is broader than ordinary convergence. The relationship between the convergence of a given sequence and the summability of its subsequences has been given by Buck [5]. By changing the concept of convergence, Agnew [1], Buck [6], Buck and Pollard [7], Miller and Orhan [16], Zeager [23] have studied this relation. Instead of subsequences, Dawson [8] and Fridy [10] have used stretching and rearrangements, respectively, and have also obtained analogous results. In [21], we have examined some relationships between convergence and uniform statistical convergence of a given sequence and its subsequences. The related notions of statistical limit superior and inferior and statistical cluster points have been studied in recent papers including [11], [12] by Fridy and Orhan and [17] by Miller and Miller-Van Wieren. In [22] we have obtained some results concerning the uniform statistical limit superior and inferior, the set of uniform statistical cluster points of a given sequence and its subsequences.

Now, we pause to collect some known notions. Let $K \subseteq \mathbb{N}$ where $\mathbb{N}$ is the set of natural numbers. If $m, n \in \mathbb{N}$, by $K(m,n)$ we denote the cardinality of the set of numbers i in K such that $m \leq i \leq n$. The numbers

$$\underline{d}(K) = \liminf_{n\to\infty} \frac{K(1,n)}{n}, \quad \overline{d}(K) = \limsup_{n\to\infty} \frac{K(1,n)}{n}$$

are called the lower and the upper asymptotic density of the set K, respectively. If $\underline{d}(K) = \overline{d}(K)$, then it is said that $d(K) = \underline{d}(K) = \overline{d}(K)$ is the asymptotic density of K. The uniform density of $K \subseteq \mathbb{N}$ has been introduced in [3], [4] as follows:

$$\underline{u}(K) = \lim_{n\to\infty} \frac{\min\limits_{i\geq 0} K(i+1, i+n)}{n}, \quad \overline{u}(K) = \lim_{n\to\infty} \frac{\max\limits_{i\geq 0} K(i+1, i+n)}{n}$$

are, respectively, called the lower and the upper uniform density of the set K (the existence of these bounds is also mentioned in [2]). If $\underline{u}(K) = \overline{u}(K)$, then $u(K) = \underline{u}(K) = \overline{u}(K)$ is called the uniform density of K. It is clear that for each $K \subseteq \mathbb{N}$, we have $\underline{u}(K) \leq \underline{d}(K) \leq \overline{d}(K) \leq \overline{u}(K)$. The concept of statistical convergence has been introduced in [9] as follows: Let $x = \{x_n\}$ be a sequence of complex numbers. The sequence x is said to be statistically convergent to a complex number L provided that for every $\varepsilon > 0$ we have $d(K_\varepsilon) = 0$, where $K_\varepsilon = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}$. If $x = \{x_n\}$ converges statistically to L, then we write $st - \lim x = L$.

Now let us recall the concept of uniform statistical convergence which is the primary topic of this paper. A generalized approach to convergence has been obtained by means of the notion of an ideal I of subsets of $\mathbb{N}$, i.e., I is an additive and hereditary class of sets. A sequence x is said to be I-convergent to L if for every ε the set $K_\varepsilon = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}$ belongs to I, and we write $I - lim x = L$. If $I = I_d = \{A \subseteq \mathbb{N} : d(A) = 0\}$, then I_d-convergence coincides with statistical convergence. In the case $I = I_u = \{A \subseteq \mathbb{N} : u(A) = 0\}$ we obtain uniform statistical convergence to L or I_u- convergence to L. Then we write $st_u - \lim x = L$.

Definition 1. γ is called a uniform statistical cluster point of $x = \{x_k\}$ if for every $\varepsilon > 0$, the set $\{k : |x_k - \gamma| < \varepsilon\}$ does not have uniform density 0.

The definition of a statistical cluster point is obtained by simply omitting the word uniform and replacing uniform density by asymptotic density in the above definition (see [17], [18]).

Let Γ_u denote the set of all uniform statistical cluster points of x and Γ_s denote the set of all statistical cluster points. Clearly $\Gamma_s \subseteq \Gamma_u$. It has been shown in [22] that $\Gamma_s \subsetneqq \Gamma_u$, for example if

$$x = 0, 1, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, ..., 0, 0, 0, 1, 1, 1, ...$$

where segments of 0's of length 2^k , $k = 0, 1, 2, 3, ...$ and 1's of length $k+1$, $k = 0, 1, 2, 3, ...$ alternate. Then one can see that x is statistically convergent to 0 so $\Gamma_s = \{0\}$ but $\Gamma_u = \{0, 1\}$.

Definition 2. Given $x = \{x_k\}$, l is called a uniform statistical limit point of $x = \{x_k\}$ if there exists a sequence $\{n_k\}$, with $\bar{u}(\{n_k : k \in \mathbb{N}\}) > 0$ and $\lim_{k\to\infty} x_{n_k} = l$.

Likewise, the definition of a statistical limit point is obtained by replacing upper uniform density by upper asymptotic density. Statistical limit points have been extensively studied by Kostyrko et al. [15]. Let $\lambda_u(x)$ denote the set of all uniform statistical limit points of x and $\lambda(x)$ denote the set of all statistical limit points of x. Easily $\lambda(x) \subseteq \lambda_u(x)$. Kostyrko et al. [15] have proved that for any sequence x, $\lambda(x)$ is an F_σ set and that given any F_σ set X there exists a sequence x for which $\lambda(x) = X$. Now for $0 < d \leq 1$ define

$$\lambda_u(x, d) = \{l : \exists n_k,\ \bar{u}(\{n_k : k \in \mathbb{N}\}) > d,\ \lim_{k\to\infty} x_{n_k} = l\}.$$

It is easy to check $\lambda_u(x, d)$ is closed for $d > 0$ and $\lambda_u(x) = \bigcup_j \lambda_u(x, \frac{1}{j})$, so we have:

PROPOSITION 1. *Suppose x is a bounded sequence. Then $\lambda_u(x)$ is a F_σ set.*

The following lemma is simple but very useful; therefore, we omit the proof.

LEMMA 1. *$l \in \lambda_u(x)$ if and only if there exists a $d > 0$ so that $\bar{u}(\{k : |x_k - l| < \varepsilon\}) \geq d$ holds for every positive ε.*

2. MAIN RESULTS

In this section, we present some results on the relationship between the set of uniform statistical limit points of a sequence and its subsequences. We also study the construction of a (sub)sequence with a prescribed set of uniform statistical limit points. Now we are ready to prove our first main result.

THEOREM 1. *Suppose $x = \{x_n\}$ is a bounded sequence, L the set of its limit points and $M \subset L$ a nonempty F_σ set. Then there exists a subsequence $y = \{y_n\}$ of x such that $\lambda_u(y) = M$.*

Proof. Since M is a nonempty F_σ set, we can write $M=\bigcup_{j=1}^{\infty}M_j$ where M_j are nonempty closed sets (if M is closed the proof is simpler). For each j, we can find a sequence a_{ij}, $i=1,2,3,...$ such that $\overline{\{a_{ij}:i\in\mathbb{N}\}}=M_j$. Now for each fixed pair i,j, fix a subsequence of x, $x_{n_k,i,j}$, $k=1,2,...$ such that $\lim_{k\to\infty}x_{n_k,i,j}=a_{ij}$ and $\{x_{n_k,i,j}\}\subseteq(a_{ij}-\frac{1}{j2^i},a_{ij}+\frac{1}{j2^i})\}$. Now we will construct $\{y_n\}$ from the sequences $\{x_{n_k,i,j}\}$, $i,j=1,2,...$ the following way:

$$\begin{array}{cccccl} y_1 & y_3 & y_5 & \cdots & y_{2m+1} & \text{... will be chosen from} \\ y_2 & y_6 & y_{10} & \cdots & y_{2(2m+1)} & \text{... will be chosen from} \\ y_4 & y_{12} & y_{20} & \cdots & y_{4(2m+1)} & \text{... will be chosen from} \\ \cdots & & & \cdots & & \\ y_{2^{j-1}} & y_{2^{j-1}3} & y_{2^{j-1}5} & \cdots & y_{2^{j-1}(2m+1)} & \text{... will be chosen from} \end{array}$$

$\{x_{n_k,i,1}\}_{k=1}^{\infty}$, $\{x_{n_k,i,2}\}_{k=1}^{\infty}$, $\{x_{n_k,i,3}\}_{k=1}^{\infty}$,.., $\{x_{n_k,i,j}\}_{k=1}^{\infty}$;$i=1,2,3,...$ respectively, in the following way: when choosing y_n its index with respect to the sequence $\{x_n\}$ will be larger than the indices of $y_1,y_2,...,y_{n-1}$ and for each j, the sequence $y_{2^{j-1}},y_{2^{j-1}3},y_{2^{j-1}5},...$ is made of "blocks" of terms of the form $1,1,2,1,2,3,1,2,3,4,...$ so that each block "1" consists of distinct members of $\{x_{n_k,1,j}\}$, each block "2" consists of members of $\{x_{n_k,2,j}\}$ (not used before), ..., each block "i" consists of distinct members of $\{x_{n_k,i,j}\}$ (not used before) and also that each block has length equal to the sum of the lengths of all previous blocks combined. As mentioned at the same time, when constructing $\{y_n\}$ we also make sure that its indices in terms of x are increasing. Now we will check that $\lambda_u(\{y_n\})=M$. First we check $\lambda_u(\{y_n\})\supseteq M$. Suppose $a\in M$. Then $a\in M_j$ for some j. Let $\varepsilon>0$ be arbitrary. Then we can fix some a_{ij} such that $|a_{ij}-a|<\frac{\varepsilon}{2}$. Then all but finitely many $x_{n_k,i,j}$ must be inside $(a-\varepsilon,a+\varepsilon)$. So all but finitely many members of "i" blocks inside of $\{y_{2^{j-1}(2m-1)}\}_{m=1}^{\infty}$ are inside of $(a-\varepsilon,a+\varepsilon)$. But from our construction the members of "i" blocks have upper uniform density 1 inside of $\{y_{2^{j-1}(2m-1)}\}$, (since the lengths of "i" blocks go to ∞) while $\{y_{2^{j-1}(2m-1)}\}$ has uniform density $\frac{1}{2^j}$ in $\{y_n\}$. Therefore, $\bar{u}(\{n:y_n\in(a-\varepsilon,a+\varepsilon)\})\geq\frac{1}{2^j}$. Since ε is arbitrary, by our lemma we get $a\in\lambda_u(\{y_n\})$. Hence, $\lambda_u(\{y_n\})\supseteq M$. Now we check $\lambda_u(\{y_n\})\subseteq M$. Suppose $a\notin M$. Since $M=\bigcup_{j=1}^{\infty}M_j$, for each j there exists $\varepsilon_j>0$ such that $[a-\varepsilon_j,a+\varepsilon_j]\bigcap[M_1\cup M_2\cup...\cup M_j]=\emptyset$. Now for each j, $(a-\varepsilon_j,a+\varepsilon_j)$ can contain finitely many elements from

$$\begin{array}{cccccc} y_1 & y_3 & y_5 & \cdots & y_{2m+1} & \cdots \\ y_2 & y_6 & y_{10} & \cdots & y_{2(2m+1)} & \cdots \\ y_4 & y_{12} & y_{20} & \cdots & y_{4(2m+1)} & \cdots \\ \cdots & & & \cdots & & \\ y_{2^{j-1}} & y_{2^{j-1}3} & y_{2^{j-1}5} & \cdots & y_{2^{j-1}(2m+1)} & \cdots \end{array}$$

since if not, $(a-\varepsilon_j,a+\varepsilon_j)$ would contain infinitely many from some $\{y_{2^k(2m-1)}\}$

for some $k, 0 \leq k \leq j-1$ and then $[a-\varepsilon_j, a+\varepsilon_j]$ would intersect M_k which is a contradiction. Since $(a-\varepsilon_j, a+\varepsilon_j)$ contains only finitely many elements from

$$\begin{array}{cccccc} y_1 & y_3 & y_5 & \cdots & y_{2m+1} & \cdots \\ y_2 & y_6 & y_{10} & \cdots & y_{2(2m+1)} & \cdots \\ y_4 & y_{12} & y_{20} & \cdots & y_{4(2m+1)} & \cdots \\ \cdots & & & \cdots & & \\ y_{2^{j-1}} & y_{2^{j-1}3} & y_{2^{j-1}5} & \cdots & y_{2^{j-1}(2m+1)} & \cdots \end{array}$$

we can conclude that $\bar{u}(\{n : y_n \in (a-\varepsilon_j, a+\varepsilon_j)\}) < \frac{1}{2^j}$, for each j. But then from Lemma 1, we get that a is not a uniform statistical limit point of $\{y_n\}$, i.e., $a \notin \lambda_u(\{y_n\})$. Hence, $\lambda_u(\{y_n\}) \subseteq M$. □

Remark 1. An obvious corollary of the above theorem is that given any F_σ set λ, there exists a sequence x, $\lambda_u(x) = \lambda$.

But the analogous lemma and theorem hold also if uniform density and uniform statistical limit points are replaced by just asymptotic density and statistical limit points. The lemma is easy to prove and also:

THEOREM 2. *Suppose $x = \{x_n\}$ is a bounded sequence, L the set of its limit points and $\lambda \subset L$ is a nonempty F_σ set. Then there exists a subsequence $y = \{y_n\}$ of x such that $\lambda(y) = \lambda$.*

Proof. The construction of the sequence is identical to the one in the proof of the previous theorem. For the $y = \{y_n\}$ constructed in the proof, clearly $\lambda(y) \subseteq \lambda_u(y) = \lambda$. Also, from the sequence construction the "i " blocks have upper asymptotic density $> \frac{1}{2}$ inside of $\{y_{2^{j-1}(2m-1)}\}$ (since each block has length equal to the sum of lengths of the previous blocks) and $\{y_{2^{j-1}(2m-1)}\}$ has uniform and asymptotic density $\frac{1}{2^j}$ in $\{y_n\}$ so we get that the upper asymptotic density of $\{n : y_n \in (a-\varepsilon_j, a+\varepsilon_j)\} \geq \frac{1}{2^{j+1}}$ (for arbitrary ε and a as in the previous proof) and hence, we obtain that $\lambda \subseteq \lambda(y)$. So in the case of this sequence $\lambda_u(y) = \lambda(y) = \lambda$. □

However in general, for sequences x, $\lambda(x) \subseteq \lambda_u(x)$.

Subsequences of a sequence x can be naturally identified with numbers $t \in (0,1]$ written by a binary expansion with infinitely many 1's. Thus, we can denote by $\{x(t)\}$ the subsequence of x corresponding to t. Next, similar to earlier results about the set of subsequences of a given sequence with the same set of statistical cluster or limit points ([18]) or the same set of uniform statistical cluster points ([22]), we have proved the following result:

Theorem 3. *If $x = \{x_n\}$ is a bounded sequence, the set of $t \in (0,1]$ for which $\lambda_u(x) = \lambda_u(x(t))$ has Lebesgue measure 1 or 0 (both may occur).*

Proof. Let T denote the set of all $t \in (0,1]$ for which $\lambda_u(x) = \lambda_u(x(t))$. Clearly T is a tail set. Next, we will show T is measurable. First, we show that $\lambda_u(x) \subseteq \lambda_u(x(t))$ for almost all $t \in (0,1]$. As previously defined $\lambda_u(x) = \bigcup_j \lambda_u(x, \frac{1}{j})$. Suppose $j \in \mathbb{N}$ is fixed. Since $\lambda_u(x, \frac{1}{j})$ is closed and separable, there exists a set $\{l_{ij} : i \in \mathbb{N}\}$ such that its closure is $\lambda_u(x, \frac{1}{j})$. Fix any l_{ij}. Since $l_{ij} \in \lambda_u(x, \frac{1}{j})$, there exist positive integers $N_1 < N_2 < N_3 < ... < N_{2k-1} < N_{2k} < ...$ such that $(N_{2k-1} - N_{2k}) \to \infty$ and a subsequence $\{n_r\} \subseteq \bigcup_k \{n : N_{2k-1} \le n \le N_{2k}\}$ such that $\lim_{r\to\infty} x_{n_r} = l_{ij}$ and

$$\lim_{k\to\infty} \frac{|\{r : N_{2k-1} \le n_r \le N_{2k}\}|}{N_{2k} - N_{2k-1}} \ge \frac{1}{j}.$$

Now due to the Law of Large Numbers, for almost all $t \in (0,1]$, $d(\{r : t(n_r) = 1\}) = \frac{1}{2}$ where $t(n_r)$ is the n_r digit of t after the decimal point (i.e. $t = 0.t(1)t(2)...t(n)...$) and d is the asymptotic density. Then it is easy to check that

$$\limsup_{k\to\infty} \frac{|\{r : t(n_r) = 1,\ N_{2k-1} \le n_r \le N_{2k}\}|}{N_{2k} - N_{2k-1}} \ge \frac{1}{4j}$$

for almost all $t \in (0,1]$ which means that for all $t \in (0,1]$, $x(t)$ has a subsequence of upper uniform density at least $\frac{1}{4j}$ that converges to l_{ij}. Thus $l_{ij} \in \lambda_u(x(t), \frac{1}{4j})$ for almost all $t \in (0,1]$. Hence, we can conclude that for almost all $t \in (0,1]$, $l_{ij} \in \lambda_u(x(t), \frac{1}{4j})$ for all $i = 1,2,3,....$ Now since $\lambda_u(x(t), \frac{1}{4j})$ is closed for each t we have

$$\lambda_u(x, \frac{1}{j}) = \overline{\{l_{ij} : i \in \mathbb{N}\}} \subseteq \lambda_u(x(t), \frac{1}{4j}) \subseteq \lambda_u(x(t))$$

for almost all t. The above holds for each j so

$$\lambda_u(x) = \bigcup_j \lambda_u(x, \frac{1}{j}) \subseteq \lambda_u(x(t))$$

for almost all $t \in (0,1]$. In the second part, we will see that the set of $t \in (0,1]$ where $\lambda_u(x(t)) \subseteq \lambda_u(x)$ is measurable (may have measure 0 or 1).

Now, let L denote the set of limit points of x, $L \subseteq [\liminf x, \limsup x]$. The following is easy to verify: For $a \in [\liminf x, \limsup x]$, the set $\lambda_a = \{t \in (0,1] : a \in \lambda_u(x(t))\}$ is a tail set and is measurable so it has Lebesgue measure 0 or 1 (if $a \notin L$, of course $\lambda_a = \emptyset$).

There are two possibilities:

1. There exists a. $a \in L \backslash \lambda_u(x)$ such that λ_a has Lebesgue measure 1 or
2. For every $a \in L \backslash \lambda_u(x)$, λ_a has Lebesgue measure 0.

In case 1, it is clear that $\{t : \lambda_u(x(t)) \subseteq \lambda_u(x)\}$ must have measure 0, since it is disjoint from λ_a.

Suppose the second case holds. We know that

$$\lambda_u(x) = \bigcup_{n=1}^{\infty} F_n$$

where F_n are closed subsets of L. Let $G_n = [\liminf x, \limsup x] \backslash F_n$. Then for each n, we can write $G_n = \bigcup_{i=1}^{\infty} I_{ni}$ where I_{ni} are open (or half open, in case $\liminf x, \limsup x$ are endpoints) and have length less than or equal to $\frac{1}{n}$ (the intervals do not have to be disjoint). For each n, i, let

$$d_{ni} = \max_{\substack{d, \\ 0 \leq d \leq 1}} [\{t : \bar{u}(\{x(t)\} \cap I_{ni}) \geq d\} \text{ has measure 1}]$$

(the set $\{t : \bar{u}(\{x(t)\} \cap I_{ni}) \geq d\}$ is easily a tail set and measurable, so has measure 0 or 1). Now if $a \in [\liminf x, \limsup x] \backslash \bigcup_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} G_n$ we have $a \in \bigcap_{n=1}^{\infty} I_{ni_n}$ for some i_n, $n = 1, 2, 3, ...$ Then it is easy to see that $d_{ni_n} \to 0$, since otherwise, as the lengths of I_{ni_n} go to 0, a would be a uniform statistical limit point for almost all t. This is a contradiction, since λ_a has Lebesgue measure 0. Now, for n, i, let $d^*_{ni} = min(2d_{ni}, 1)$. Then whenever $d_{ni} < 1$, $\{t : \bar{u}(\{x(t)\} \cap I_{ni}) \geq d^*_{ni}\}$ has measure 0. Hence, for either $d_{ni} < 1$ or $d_{ni} = 1$ the set $X_{ni} = \{t : \bar{u}(\{x(t)\} \cap I_{ni}) \leq d^*_{ni}\}$ has measure 1. Then $X = \cap_i \cap_n X_{ni}$ has measure 1. Suppose $a \in L \backslash \lambda_u(x)$. We know that $a \in \cap_n I_{ni_n}$ and $d_{ni_n} \to 0$, and consequently $d^*_{ni_n} \to 0$. For $t \in X$ we have $\bar{u}(\{x(t)\} \cap I_{ni}) \leq d^*_{ni} \to 0$ and the lengths of I_{ni_n} go to 0, so $t \notin \lambda_a$, i.e., $a \notin \lambda_u(x(t))$. Hence, $a \in L \backslash \lambda_u(x)$ implies $a \notin \lambda_u(x(t))$ for every $t \in X$, i.e., for every $t \in X$, $\lambda_u(x(t)) \subseteq \lambda_u(x)$. Since X has Lebesgue measure 1, then $\{t : \lambda_u(x(t)) \subseteq \lambda_u(x)\}$ has measure 1 in this case. This completes the proof. □

By the above theorem, the following corollary is apparent.

COROLLARY 1. *Suppose x is a bounded sequence, L the set of its limit points. The Lebesgue measure of the set $\{t \in (0, 1] : \lambda_u(x(t)) = \lambda_u(x)\}$ is 0 if and only if there exists $a \in L \backslash \lambda_u(x)$ such that $\{t \in (0, 1] : a \in \lambda_u(x(t))\}$ has Lebesgue measure 1.*

Remark 2. Examples of both cases can be easily given. For measure 1, any convergent sequence is an example, for measure 0 we can refer to the earlier example of [16]. This example is also in accord with the corollary.

REFERENCES

[1] R.P. Agnew, *Summability of subsequences*. Bull. Amer. Math. Soc. **50** (1944), 596-598.

[2] V. Balàž and T. Šalát, *Uniform density u and corresponding I_u- convergence*. Math. Commun. **11** (2006), *1*, 1-7.

[3] T.C. Brown and A.R. Freedman, *Arithmetic progressions in lacunary sets*. Rocky Mountain J. Math. **17** (1987), 587-596.

[4] T.C. Brown and A.R. Freedman, *The uniform density of sets of integers and Fermat's last theorem*. C. R. Math. Acad. Sci. Soc. R. Can. **12** (1990), *1*, 1-6.

[5] R.C. Buck, *A note on subsequences*. Bull. Amer. Math. Soc. **49** (1943), 924-931.

[6] R.C. Buck, *An addentum to "a note on subsequences"*. Proc. Amer. Math. Soc. **7** (1956), 1074-1075.

[7] R.C. Buck and H. Pollard, *Convergence and summability properties of subsequences*. Bull. Amer. Math. Soc. **49** (1943), 924-931.

[8] D.F. Dawson, *Summability of subsequences and strechings of sequences*. Pacific J. Math. **44** (1973), 455-460.

[9] H. Fast, *Sur la convergence statistique*. Colloq. Math. **2** (1951), 241-244.

[10] J.A. Fridy, *Summability of rearrangements of sequences*. Math. Z. **143** (1975), 187-192.

[11] J.A. Fridy and C. Orhan, *Statistical limit superior and limit inferior*. Proc. Amer. Math. Soc. **125** (1997), *12*, 3625-3631.

[12] J.A. Fridy and C. Orhan, *Statistical core theorems*. J. Math. Anal. Appl. **208** (1997), *2*, 520-527.

[13] R.G. Antonini and G. Grekos, *Weighted uniform densities*. J. Théor. Nombres Bordeaux **19** (2007), *1*, 191-204.

[14] P. Kostyrko, T. Šalát and W. Wilczyński, *I-convergence*. Real Anal. Exchange **26** (2000/2001), *2*, 669-686.

[15] P. Kostyrko, M. Mačaj, T. Šalát and O. Strauch, *On statistical limit points*. Proc. Amer. Math. Soc. **129** (2000), *9*, 2647-2654.

[16] H.I. Miller and C. Orhan, *On almost convergence and statistically convergent subsequences*. Acta. Math. Hungar. **93** (2001), 135-151.

[17] H.I. Miller and L. Miller-Wan Wieren, *Some statistical cluster point theorems*. Hacet. J. Math. Stat. **44** (2015), *6*, 1405-1409.

[18] H.I. Miller and L. Miller-Wan Wieren, *Statistical cluster point and statistical limit point sets of subsequences of a given sequence*. Hacet. J. Math. Stat. **49** (2020), *2*, 494-497.

[19] S. Pehlivan, *Strongly almost convergent sequences defined by a modulus and uniformly statistical convergence*. Soochow J. Math. **20** (1994), *2*, 205-211.

[20] E. Tas and T. Yurdakadim, *Characterization of uniform statistical convergence for double sequences*. Miskolc Math. Notes **13** (2012), *2*, 543-553.

[21] T. Yurdakadim and L. Miller-Wan Wieren, *Subsequential results on uniform statistical convergence*. Sarajevo J. Math. **12(25)** (2016), *2*, 1-9.

[22] T. Yurdakadim and L. Miller-Wan Wieren, *Some results on uniform statistical cluster points*. Turkish J. Math. **41** (2017), *5*, 1133-1139.

[23] J. Zeager, *Buck-type theorems for statistical convergence.* Radovi Math. **9** (1999), *1*, 59-69.

Received January 25, 2019

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