MARKOV REGIME-SWITCHING HESTON MODEL WITH CIR MODEL FRAMEWORK AND PRICING VIX AND S&P500 AMERICAN PUT OPTIONS

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Communicated by Ionel Popescu

In this work, it is considered a combination of the CIR model framework with a Markov regime-switching model to price VIX and S&P500 American put options. We first present a closed-form formula for conditional higher moments of the stock return which are sensibly more straightforward than those obtained through a characteristic function approach. Next, we estimate the parameters of model by applying a complete maximum likelihood procedure. Then, we provide a Least Square Monte-Carlo (LSM) algorithm to determine S&P500 American option price in the regime-switching Heston model. Finally, by the binomial tree method as a benchmark, we provide some numerical experiments to illustrate the accuracy of proposed algorithm.

AMS 2020 Subject Classification: 91Gxx, 91G60.

Key words: Heston model, Markov regime-switching model, CIR model, american option, EM algorithm.

1. INTRODUCTION

One of the most noticeable and widely used stochastic volatility models is the Heston model [15]. The volatility of this model follows a mean reverting Cox-Ingersoll-Ross (CIR) process presented by Cox et al. [3]. The asset price dynamic under the Heston model is defined as follows

$$\begin{cases} \mathrm{d}S_t &= rS_t\ \mathrm{d}t + S_t\sqrt{V_t}\ \mathrm{d}W_t^1 \\ \mathrm{d}V_t &= \kappa\big(\theta - V_t\big)\ \mathrm{d}t + \sigma\sqrt{V_t}\ \mathrm{d}W_t^2, \end{cases} \tag{1.1}$$

where S_t and V_t represent the price and volatility of the underlying asset and also, W_t^1 and W_t^2 are the Brownian motions with correlation $\rho \in (-1, 1)$. Besides, κ, θ and σ are the mean reversion speed of the volatility, the long-run mean, and the volatility of volatility, respectively. r is the interest rate. We note that parameters should be chosen to satisfy the Feller condition, $2\kappa\theta > \sigma^2$. This will ensure that V_t stays strictly positive [1].

Among stochastic volatility models, the Heston model is an industry standard. Its parameters are known to exert clear and specific control over the implied volatility skew or smile, and it can mimic the implied volatilities of around-the-money options with a fair degree of accuracy. However, researches show that modification of regime should be indicated by the asset prices or the associated volatility process [8, 10]. In [14, 20, 23], according to [18], it is demonstrated that the model should have at least two regimes under the risk neutral measure. Moreover, a wide stream of research argues that index volatilities are subjected to regime switches under the physical measure. The economic consideration is one of the main motivations to exert regime switches using Markov chains instead of jump-diffusion, in order to deal with sudden changes in volatility.

The first incorporation of regime switching in the volatility process itself was achieved by [7, 9], which propose an extension of the Heston model, in which the mean-reverting level of volatility is modulated by an observable Markov chain and use it to derive the price of volatility derivatives, such as variance swaps and volatility swaps. In [12], it is generalized this approach by considering that a hidden Markov chain modulates the speed of mean reversion, the mean-reversion level, the volatility of volatility and the correlation with the stock index for the pricing and hedging of derivatives. In [13] and [19], it is showed that the regime-switching feature gives rise to a significant increase in the non-Gaussianity of conditional stock returns at short time horizons and also, the leptokurtotic of model is close to market data.

The first extension of the Heston model with regime switching to price VIX options was recently reported in [21], in which an observed Markov chain modeling the state of volatility modulates two components of the stock process: the intensity of jumps and an additional multiplicative factor for volatility. Although this model overcomes the shortcoming of VIX skew, regime shifts drive the stock returns, and thus, it is not clear how the dynamics of volatility itself can be monitored using this model. Moreover, as shown in [11], there is a very good consistency between forward variance swap rates estimates from S&P500 and VIX options, invalidating the existence of a jump premium priced in the market, since the unique underlying assumption behind the computation of VIX index is the continuity of stock price process.

The implied higher-order moment of stock return is one of the interesting issues in the study of asset price. In [5], it is provided a closed-form formula to the skewness and kurtosis of a stock price model with stochastic volatility, but these results can not be applied for the Heston model. In [25], the analytical solution for the skewness of stock return under Heston model obtained. Here, we give a formula to calculate the conditional higher moments of the stock return under the regime-switching Heston model.

In this work, it is considered a combination of the CIR framework with a Markov regime-switching model to price VIX and S&P500 American put options. Moreover, we estimate the parameters of the model by applying a complete maximum likelihood procedure.

The rest of the paper is organized as follows: in Section 2, we study the regime-switching Heston model and also, we estimate the parameters of the model by applying an organized iterative procedure. In Section 3, it is considered two algorithms, named LSM and binomial tree algorithms, to price S&P 500 American put option under the regime-switching Heston model. Some numerical studies are presented in Section 4.

2. REGIME-SWITCHING HESTON MODEL

We work on a probability space $(\Omega, \mathbb{F} := (\mathcal{F}_t)_{[0,T]}, \mathbb{P})$ where $\mathbb{P}$ is a risk-neutral or equivalent martingale measure. We assume that Z_t is a continuous-time Markov chain that represents the regime state of volatility where is independent of two Brownian motions $W^1 = \{W^1_t,\ t \geq 0\}$ and $W^2 = \{W^2_t,\ t \geq 0\}$ that the correlation between two Brownian motions is $\rho(Z_t)$. The regime variable can take three values, $Z_t \in E := \{1, 2, 3\}$ where $\{Z_t = 1\}$ means that volatility is in its low state, $\{Z_t = 2\}$ means it is in medium state, and $\{Z_t = 3\}$ means it is in a high state. Besides, we define $\mathcal{F}_t := \mathcal{F}^W_t \vee \mathcal{F}^Z_t$ where $\mathcal{F}^W_t$ is the filtration generated by two Brownian motions W^1 and W^2 and also, $\mathcal{F}^Z_t$ is the filtration generated by the process Z. We denote by Π the generator matrix of Z, which is given for all $i, j \in E$ by $\Pi_{i,j} \geq 0$ if $i \neq j$ and $\Pi_{ii} = -\sum_{i \neq j} \Pi_{i,j}$. We assume that the transition probabilities from a state $i \in E$ at time t towards a state $j \in E$ at time $t+h$ are stationary. This leads to an infinitesimal generator Π independent of time. Let us denote now by $T(h)$ the matrix of transition probabilities defined by $T_{ij}(h) = \mathbb{P}(Z_h = j | Z_0 = i)$, for all $i, j \in E$. Then, the following relation holds

$$\frac{\mathrm{d}T_{ij}(h)}{\mathrm{d}h} = \Pi_{ij} \qquad \text{for all } i, j \in E.$$

Note that this relation implies $T_{ij}(h) = (e^{\Pi h})_{ij}$. We denote by P the matrix of transition probabilities of the embedded discrete time Markov chain. P is given by

$$P_{ij} = \begin{cases} \frac{\Pi_{ij}}{\sum_{i \neq j} \Pi_{ij}} & \text{if } i \neq j \\ 0 & \text{otherwise.} \end{cases}$$

Assume that $S = \{S_t\}_{t \geq 0}$ and $V = \{V_t\}_{t \geq 0}$ be two stochastic processes in our probability space that model, respectively, the spot value of the S&P500 index and the instantaneous variance of S. The dynamic of our model follows paths generated by the following stochastic differential equations

$$\begin{cases} \mathrm{d}S_t &= rS_t\ \mathrm{d}t + S_t\sqrt{V_t}\ \mathrm{d}W_t^1, \quad S_0 = s \\ \mathrm{d}V_t &= \kappa(Z_t)\big(\theta(Z_t) - V_t\big)\ \mathrm{d}t + \sigma(Z_t)\sqrt{V_t}\ \mathrm{d}W_t^2, \quad V_0 = v_0, \end{cases} \tag{2.1}$$

where all the parameters of the volatility process V_t and the correlation factor between the S&P 500 index and its instantaneous variance V_t depend on the homogeneous continuous time Markov chain Z.

We now apply the maximum likelihood procedure in order to know how the parameters of the model change over time.

Definition 2.1. The VIX index is calculated as the strike of the one-month variance swap contract on the S&P500 index. Provided that S&P500 index has no jumps, the following relationship holds

$$VIX_t^2 = \mathbb{E}^{\mathbb{P}}\left[\frac{1}{\tau_{vix}}\int_t^{t+\tau_{vix}} V_s\ \mathrm{d}s \mid \mathcal{F}_t\right],$$

where τ_{vix} is the duration corresponding to one month [13].

THEOREM 2.2. *Let M be a discrete random variable in $\mathbb{N}^* := \mathbb{N}\backslash\{0\}$ such that $M-1$ is the random number of jumps of the Markov chain Z between 0 and τ_{vix}. Let $\big(\tau_1, \cdots, \tau_{M-1}\big)$ be the sequence of $M-1$ random jumps times, and $\Delta_{t_k} := \tau_{k+1} - \tau_k$ for each $k \in \{0, \cdots, M-1\}$ with $\tau_M = \tau_{vix}$ and $\tau_0 = 0$. For each $k \in \{0, \cdots, M-1\}$, there are two families of functions $\{f_k\}_{0\le k\le M-1}$ and $\{g_k\}_{0\le k\le M-1}$ defined from E^k to $\mathbb{R}$ such that*

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}\Big[\int_0^{\tau_{vix}} V_s\ \mathrm{d}s \,\Big|\, V_0 = v_0,\ \mathcal{F}^Z_{\tau_{vix}}\Big] &= \Big(\sum_{k=0}^{M-1} a(Z_{\tau_k}) f_k(Z_0, \cdots, Z_{\tau_{k-1}})\Big) v_0 \\ &\quad + \sum_{k=0}^{M-1} \big(b(Z_{\tau_k}) + a(Z_{\tau_k}) g_k(Z_0, \cdots, Z_{\tau_{k-1}})\big), \end{aligned} \tag{2.2}$$

where

$$f_k(Z_{\tau_0}, \cdots, Z_{\tau_{k-1}}) = \begin{cases} \displaystyle\prod_{j=0}^{k-1}(1-\kappa(Z_{\tau_j}))a(Z_{\tau_j}) & \text{if } k \in \{1, \cdots, M-1\} \\ 1 & \text{if } k = 0 \end{cases}$$

and

$$\begin{aligned} &g_k\big(Z_{\tau_0}, \cdots, Z_{\tau_{k-1}}\big) \\ &= \begin{cases} g_{k-1}\big(Z_{\tau_0}, \cdots, Z_{\tau_{k-2}}\big)\big(1 - \kappa(Z_{\tau_{k-2}})a(Z_{\tau_{k-2}})\big) & \\ +\kappa(Z_{\tau_{k-2}})\big(\theta(Z_{\tau_{k-2}})\Delta_{t_{k-2}} - b(Z_{\tau_{k-2}})\big), & \text{if } k \in \{2, \cdots, M-1\} \\ \kappa(Z_{\tau_0})\big(\theta(Z_{\tau_0})\Delta_{t_0} - b(Z_{\tau_0})\big) & \text{if } k = 1 \\ 0 & \text{if } k = 0 \end{cases} \end{aligned}$$

provided that

$$a(Z_{\tau_k}) = \frac{1-\exp(-\kappa(Z_{\tau_k})\Delta_{t_k})}{\kappa(Z_{\tau_k})},$$
$$b(Z_{\tau_k}) = \theta(Z_{\tau_k})(\Delta_{t_k} - a(Z_{\tau_k})).$$

THEOREM 2.3 (Implied skewness and kurtosis). *Let the expiration date $T>0$ and for all $0 \le t \le T$, define the compound stock return as $R_t^T := \ln(\frac{S_T}{S_t})$. Then, under the regime-switching Heston model (2.1), we have*

$$\mathbb{E}_t(V_s) = e^{-\kappa(Z_s)s}\Big(e^{\kappa(Z_t)t}V_t + \int_t^s \theta(Z_u)\kappa(Z_u)e^{\kappa(Z_u)u}\mathrm{d}u\Big),$$

where $\mathbb{E}_t(.)$ denotes the conditional expectation w.r.t $\mathcal{F}_t^W$ and also

$$\text{Skewness}\left(R_t^T\right) = \frac{\mathscr{A}(t) - \frac{3}{2}\mathscr{B}(t) + \frac{3}{4}\mathscr{C}(t) - \frac{1}{8}\mathscr{D}(t)}{\mathscr{G}^{3/2}(t)}, \tag{2.3}$$

$$\text{Kurtosis}\left(R_t^T\right) = \frac{\mathscr{H}(t) + \frac{3}{2}\mathscr{M}(t) - 2\mathscr{N}(t) - \frac{1}{2}\mathscr{O}(t) + \frac{1}{16}\mathscr{P}(t)}{\mathscr{G}^{2}(t)}, \tag{2.4}$$

given

$$\mathscr{A}(t) = 3\int_t^T \big(e^{-\kappa(Z_s)s}\int_t^s \sigma(Z_u)\rho(Z_u)e^{\kappa(Z_u)u}\mathbb{E}_t(V_u)\mathrm{d}u\big)\mathrm{d}s,$$

$$\begin{aligned}
\mathscr{B}(t) = &\int_t^T \Big[e^{-\kappa(Z_s)s}\int_t^s \big(\int_u^T e^{-\kappa(Z_s)s}\mathrm{d}s\big)\sigma^2(Z_u)e^{\kappa(Z_u)u}\mathbb{E}_t(V_u)\mathrm{d}u\Big]\mathrm{d}s\\
&+ 2\int_t^T \Big[\big(\int_s^T e^{-\kappa(Z_u)u}\mathrm{d}u\big)\\
&\big(e^{-\kappa(Z_s)s}\int_t^s \sigma(Z_u)\rho(Z_u)e^{\kappa(Z_u)u}\mathbb{E}_t(V_u)\mathrm{d}u\big)\rho(Z_s)\sigma(Z_s)e^{\kappa(Z_s)s}\Big]\mathrm{d}s,\\
\mathscr{C}(t) = &\int_t^T \Big(\big(\int_s^T e^{-\kappa(Z_u)u}\mathrm{d}u\big)^2\\
&\big(e^{-\kappa(Z_s)s}\int_t^s \sigma(Z_u)\rho(Z_u)e^{\kappa(Z_u)u}\mathbb{E}_t(V_u)\mathrm{d}u\big)\sigma^2(Z_s)e^{2\kappa(Z_s)s}\Big)\mathrm{d}s\\
&+ 2\int_t^T \Big[\Big(\big(\int_s^T e^{-\kappa(Z_u)u}\mathrm{d}u\big)\rho(Z_s)\sigma(Z_s)e^{\kappa(Z_s)s}\Big)\\
&\times \Big(\int_t^s \big(\int_u^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\big)\sigma^2(Z_u)e^{\kappa(Z_u)u}\mathbb{E}_t(V_u)\mathrm{d}u\Big)\Big]\mathrm{d}s,\\
\mathscr{D}(t) = 3&\int_t^T \Big[\Big(\big(\int_s^T e^{-\kappa(Z_u)u}\mathrm{d}u\big)^2\sigma^2(Z_s)e^{2\kappa(Z_s)s}\Big)\\
&\times \Big(e^{-\kappa(Z_s)s}\int_t^s \big(\int_u^T e^{-\kappa(Z_s)s}\mathrm{d}s\big)\sigma^2(Z_u)e^{\kappa(Z_u)u}\mathbb{E}_t(V_u)\mathrm{d}u\Big)\Big]\mathrm{d}s,
\end{aligned}$$

$$\begin{aligned}
\mathscr{H}(t) = 6\int_t^T & e^{-\kappa(Z_s)s}\Big(\big(e^{\kappa(Z_t)t}V_t + \int_t^s \theta(Z_u)\kappa(Z_u)e^{\kappa(Z_u)u}\mathrm{d}u\big) \\
& \big(\int_t^s \mathbb{E}_t(V_u)\mathrm{d}u\big)\Big)\mathrm{d}s, \\
\mathscr{M}(t) = \int_t^T & \Big[e^{-\kappa(Z_s)s}\Big(e^{\kappa(Z_t)t}V_t + \int_t^s \theta(Z_u)\kappa(Z_u)e^{-\kappa(Z_u)u}\mathrm{d}u\Big) \\
& \times\Big(\int_t^s \big(\int_u^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\big)^2\sigma^2(Z_u)e^{2\kappa(Z_u)u}\mathbb{E}_t(V_u)\mathrm{d}u\Big)\Big]\mathrm{d}s \\
& + \int_t^T \Big[\Big(\big(\int_s^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\big)^2\sigma^2(Z_s)e^{2\kappa(Z_s)s}\Big) \\
& \times e^{-\kappa(Z_s)s}\Big(\big(e^{\kappa(Z_t)t}V_t + \int_t^s \theta(Z_u)\kappa(Z_u)e^{\kappa(Z_u)u}\mathrm{d}u\big)\big(\int_t^s \mathbb{E}_t(V_u)\mathrm{d}u\big)\Big)\Big]\mathrm{d}s \\
& + 4\int_t^T \Big[\Big(\big(\int_s^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\big)\sigma(Z_s)\rho(Z_s)e^{\kappa(Z_s)s}\Big) \\
& \times e^{-\kappa(Z_s)s}\left(e^{\kappa(Z_t)t}V_t + \int_t^s \theta(Z_u)\kappa(Z_u)e^{\kappa(Z_u)u}\mathrm{d}u\right) \\
& \times\Big(\int_t^s \big(\int_u^s e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\big)\sigma(Z_u)\rho(Z_u)e^{\kappa(Z_u)u}\mathbb{E}_t(V_u)\mathrm{d}u\Big)\Big]\mathrm{d}s,
\end{aligned}$$

$$\begin{aligned}
\mathscr{N}(t) = 3\int_t^T & \Big[e^{-\kappa(Z_s)s}\Big(e^{\kappa(Z_t)t}V_t + \int_t^s \theta(Z_u)\kappa(Z_u)e^{\kappa(Z_u)u}\mathrm{d}u\Big) \\
& \times\Big(\int_t^s \big(\int_u^s e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\big)\sigma(Z_u)\rho(Z_u)e^{\kappa(Z_u)u}\mathbb{E}_t(V_u)\mathrm{d}u\Big)\Big]\mathrm{d}s \\
& + 3\int_t^T \Big[\Big(\big(\int_s^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\big)\sigma(Z_s)\rho(Z_s)e^{\kappa(Z_s)s}\Big) \\
& \times e^{-\kappa(Z_s)s}\Big(\big(e^{\kappa(Z_t)t}V_t + \int_t^s \theta(Z_u)\kappa(Z_u)e^{\kappa(Z_u)u}\mathrm{d}u\big)\big(\int_t^s \mathbb{E}_t(V_u)\mathrm{d}u\big)\Big)\Big]\mathrm{d}s, \\
\mathscr{O}(t) = 3\int_t^T & \Big[\Big(\big(\int_s^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\big)^2\sigma^2(Z_s)e^{2\kappa(Z_s)s}\Big) \\
& \times e^{-\kappa(Z_s)s}\Big(e^{\kappa(Z_t)t}V_t + \int_t^s \theta(Z_u)\kappa(Z_u)e^{\kappa(Z_u)u}\mathrm{d}u\Big) \\
& \times\Big(\int_t^s \big(\int_u^s e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\big)\sigma(Z_u)\rho(Z_u)e^{\kappa(Z_u)u}\mathbb{E}_t(V_u)\mathrm{d}u\Big)\Big]\mathrm{d}s \\
& + 3\int_t^T \Big[\Big(\big(\int_s^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\big)\sigma(Z_s)\rho(Z_s)e^{\kappa(Z_s)s}\Big) \\
& \times e^{-\kappa(Z_s)s}\Big(e^{\kappa(Z_t)t}V_t + \int_t^s \theta(Z_u)\kappa(Z_u)e^{-\kappa(Z_u)u}\mathrm{d}u\Big)
\end{aligned}$$

$$\times \Big(\int_t^s \big(\int_u^T e^{-\kappa(Z_\nu)\nu} \mathrm{d}\nu \big)^2 \sigma^2(Z_u) e^{2\kappa(Z_u)u} \mathbb{E}_t(V_u) \mathrm{d}u \Big) \Big] \mathrm{d}s,$$

$$\mathscr{P}(t) = 6 \int_t^T \Big[\Big(\big(\int_s^T e^{-\kappa(Z_\nu)\nu} \mathrm{d}\nu \big)^2 \sigma^2(Z_s) e^{2\kappa(Z_s)s} \mathrm{d}s \Big)$$

$$\times e^{-\kappa(Z_s)s} \Big(e^{\kappa(Z_t)t} V_t + \int_t^s \theta(Z_u)\kappa(Z_u) e^{-\kappa(Z_u)u} \mathrm{d}u \Big)$$

$$\times \Big(\int_t^s \big(\int_u^T e^{-\kappa(Z_\nu)\nu} \mathrm{d}\nu \big)^2 \sigma^2(Z_u) e^{2\kappa(Z_u)u} \mathbb{E}_t(V_u) \mathrm{d}u \Big) \Big] \mathrm{d}s,$$

$$\mathscr{G}(t) = \int_t^T \mathbb{E}_t(V_s) ds - \int_t^T \big(\int_s^T e^{-\kappa(Z_u)u} \mathrm{d}u \big) \sigma(Z_s)\rho(Z_s) e^{\kappa(Z_s)s} \mathbb{E}_t(V_s) \mathrm{d}s$$

$$+ \frac{1}{4} \int_t^T \big(\int_s^T e^{-\kappa(Z_u)u} \mathrm{d}u \big)^2 \sigma^2(Z_s) e^{2\kappa(Z_s)s} \mathbb{E}_t(V_s) \mathrm{d}s.$$

Proof. See Appendix A. □

Table 1 shows the results of comparing the implied and unconditional higher moments related to the compound returns of the S&P500 index under original Heston and regime-switching Heston models with real data.

	Heston	Regime-Switching Heston	Real Market
Imp. Skew. (t=45)	0.0307	0.1628	0.3700
Imp. Kurt. (t=45)	1.1618	1.3885	2.0024
Unconditional Skew. from 2000 to 2015	0.5192	0.1532	0.0168
Unconditional Kurt. from 2000 to 2015	1.7729	3.0544	2.9973

Table 1 – Implied and unconditional higher moments related to the compound returns.

2.1. **Expectation maximization algorithm**

Let us denote $\Lambda_t = VIX_t^2$, for all $0 \leq t \leq H$. Let Π be the generator matrix of Markov chain Z, i.e. if T is the transition probabilities matrix of Z, then $\Pi = \frac{1}{\delta}\log(T)$ where the time step δ is constant. We set $\Theta := \{(\kappa_i)_{i\in E}, (\theta_i)_{i\in E}, (\sigma_i)_{i\in E}, \Pi\}$. We now estimate Θ by Expectation Maximization (EM) algorithm [6].

The EM algorithm is an efficient iterative procedure to compute the Maximum Likelihood (ML) estimate in the presence of missing or hidden data. In ML estimation, we wish to estimate the model parameters for which the observed data are the most likely. Each iteration of the EM algorithm consists of two processes: The E-step, and the M-step. In the expectation, or E-step, the missing data are estimated given the observed data and current estimate of the model parameters. This is achieved using the conditional expectation, explaining the choice of terminology. In the M-step, the likelihood function is maximized under the assumption that the missing data are known. The estimate of the missing data from the E-step are used instead of the actual missing data.

Suppose that Θ_n and L are, respectively, the set of parameters in the nth step of the EM algorithm and the Likelihood function. Thus, by Jensen inequality we have [2]

$$\begin{aligned} L(\Theta) - L(\Theta_n) &= \log \mathbb{P}(\Lambda|\Theta) - \log \mathbb{P}(\Lambda|\Theta_n) \\ &= \log \sum_z \mathbb{P}(\Lambda|z,\Theta)\mathbb{P}(z|\Theta) - \mathbb{P}(\Lambda|\Theta_n) \\ &= \log \sum_z \mathbb{P}(z|\Lambda,\Theta_n) \left(\frac{\mathbb{P}(\Lambda|z,\Theta)\mathbb{P}(z|\Theta)}{\mathbb{P}(z|\Lambda,\Theta_n)} \right) - \mathbb{P}(\Lambda|\Theta_n) \\ &\geq \sum_z \mathbb{P}(z|\Lambda,\Theta_n) \log \left(\frac{\mathbb{P}(\Lambda|z,\Theta)\mathbb{P}(z|\Theta)}{\mathbb{P}(z|\Lambda,\Theta_n)\mathbb{P}(\Lambda|\Theta_n)} \right). \end{aligned}$$

The aim of the EM algorithm is to choose a Θ, so that L is maximized. Therefore,

$$\begin{aligned} \Theta_{n+1} &= \arg\max_{\Theta} \left[L(\Theta|\Theta_n) \right] \\ &= \arg\max_{\Theta} \left[L(\Theta_n) + \sum_z \mathbb{P}(z|\Lambda,\Theta_n) \log \left(\frac{\mathbb{P}(\Lambda|z,\Theta)\mathbb{P}(z|\Theta)}{\mathbb{P}(z|\Lambda,\Theta_n)\mathbb{P}(\Lambda|\Theta_n)} \right) \right]. \end{aligned}$$

Considering terms that are constant relative to Θ, we have

$$\begin{aligned} \Theta_{n+1} &= \arg\max_{\Theta} [\sum_z \mathbb{P}(z|\Lambda,\Theta_n) \log \left(\mathbb{P}(\Lambda|z,\Theta)\mathbb{P}(z|\Theta) \right)] \\ &= \arg\max_{\Theta} [\sum_z \mathbb{P}(z|\Lambda,\Theta_n) \log \mathbb{P}(\Lambda,z|\Theta)] \\ &= \arg\max_{\Theta} [\mathbb{E}_{Z|\Lambda,\Theta_n} \log \mathbb{P}(\Lambda,z|\Theta)], \end{aligned}$$

where $\mathbb{P}(\Lambda,z|\Theta) = \mathbb{P}(\Lambda_0 = y_0, \cdots, \Lambda_{t_H} = y_{t_H}, Z_0 = z_0, \cdots, Z_{t_H} = z_{t_H}|\Theta)$. So, the EM algorithm is asserted in each iteration as follows

Step E. Compute $\mathbb{E}_{Z|\Lambda,\Theta_n} \log \mathbb{P}(\Lambda,z|\Theta)$.

Step M. Maximizing the expression $\mathbb{E}_{Z|\Lambda,\Theta_n} \log \mathbb{P}(\Lambda,z|\Theta)$ relative to Θ.

Step E can be done according to the following theorem [13, 2].

THEOREM 2.4. *Let Θ_n and L are, respectively, the set of parameters after nth step and the likelihood function. Thus, we get*

$$
\begin{aligned}
L(\Theta,\Theta_n) &:= \mathbb{E}_{Z|\Lambda,\Theta_n} \log \mathbb{P}(\Lambda = y, Z|\Theta) \\
&= \sum_{j\in E} \log(\mathbb{P}(Z_0 = j))\mathbb{P}(Z_0 = j|\Lambda = y,\Theta_n) \\
&\quad + \sum_{(i,j)\in E^2} \sum_{k=1}^{H} \mathbb{P}(Z_{t_{k-1}} = i, Z_{t_k} = j|\Lambda = y,\Theta_n)\Big(\log(T_{i,j}(\delta)) \\
&\qquad + \log(f_\Lambda(y_k|Z_{t_{k-1}} = i, Z_{t_k} = j, y_{k-1},\Theta))\Big), \tag{2.5}
\end{aligned}
$$

where the density function of Λ_{t_k} with condition $(Z_{t_k} = j, Z_{t_{k-1}} = i, \Lambda_{t_{k-1}} = y_{k-1})$ is calculated as follows

$$
\begin{aligned}
f_\Lambda\big(y_k \bigm| Z_{t_k} = j, Z_{t_{k-1}} = i, y_{k-1},\Theta\big) &= \frac{1}{|\alpha_j|\sqrt{2\pi\delta\frac{y_{k-1}-\beta_i}{\alpha_i}}\sigma_j} \\
&\quad \times \exp\left\{ - \frac{\left(\frac{y_k-\beta j}{\alpha_j} - \theta_j\kappa_j\delta - (1-\kappa_j\delta)\frac{y_{k-1}-\beta_i}{\alpha_i}\right)^2}{2\delta\sigma_j^2\frac{y_{k-1}-\beta_i}{\alpha_i}} \right\}, \tag{2.6}
\end{aligned}
$$

and also,

$$
\alpha_i = \frac{1}{\tau_{VIX}}\mathbb{E}^{\mathbb{P}}\left[\sum_{k=0}^{M-1} a(Z_{\tau_k})f_k(Z_0,\cdots,Z_{\tau_{k-1}})|Z_0 = i\right],
$$

$$
\beta_i = \frac{1}{\tau_{VIX}}\mathbb{E}^{\mathbb{P}}\left[\sum_{k=0}^{M-1} b(Z_{\tau_k}) + a(Z_{\tau_k})g_k(Z_0,\cdots,Z_{\tau_{k-1}})|Z_0 = i\right].
$$

Besides, it can be confirmed that

$$
\begin{aligned}
&\mathbb{P}\big(Z_{t_{k-1}} = i, Z_{t_k} = j \bigm| \Lambda = y,\Theta_n\big) \\
&\qquad = \frac{w_i(t_{k-1})v_j(t_k)T_{n_{i,j}}(\delta)f_\Lambda(y_k|Z_{t_{k-1}} = i, Z_{t_k} = j, y_{k-1},\Theta_n)}{\sum_{k=1}^{H} w_i(t_k)}, \tag{2.7}
\end{aligned}
$$

where we have

$$
w_i(t_k) = \begin{cases} \sum_{j\in E} w_j(t_{k-1})f_\Lambda(y_k|y_{k-1}, Z_{t_{k-1}} = j, Z_{t_k} = i,\Theta_n)T_{n_{i,j}}(\delta), & \textit{if } k\in\{1,...,H\} \\ \mathbb{P}(Z_0 = j) & \textit{if } k = 0 \end{cases}
$$

and

$$v_i(t_k) = \begin{cases} \sum_{j\in E} v_j(t_{k+1}) f_\Lambda(y_{k+1}|y_k, Z_{t_k} = i, Z_{t_{k+1}} = j, \Theta_n) T_{n_{i,j}}(\delta), & \text{if } k \in \{0, ..., H-1\} \\ 1 & \text{if } k = H \end{cases}$$

In Table 2, with time step $\delta = \frac{1}{252}$, the number of iteration $N = 100$ of EM algorithm and initial vector $\Theta_0 := \{(\kappa_i)_0, (\theta_i)_0, (\sigma_i)_0, \Pi_0\}$ the parameters of the regime-switching Heston model are estimated. We use *fmincon* function of Matlab to calculate the maximum value in the nth step of the algorithm.

regimes	κ^{ML}	θ^{ML}	σ^{ML}	ρ^{ML}
state 1	8.3886	0.0021	0.789	-0.6484
state 2	12.125	0.1788	1.2571	0.1685
state 3	13.545	0.252	1.245	0.2861

Table 2 – Estimation of the parameters of the regime-switching Heston model

We know that some important features of the behaviour of the S&P 500 and its volatility as well as their joint dynamics can be reproduced by the Heston model. These main features are the excess skewness and kurtosis of the distribution of stock returns, the mean reversion of the volatility process and the so-called leverage effect for the stock-vol joint dynamics through the negative correlation between the two processes (see Figures 1 and 2).

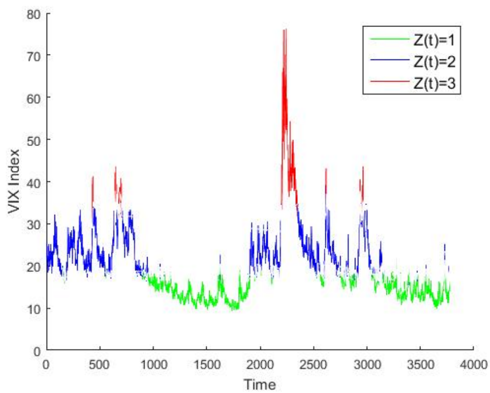

Figure 1 – The VIX index under various regimes from 2000 to 2015

Figure 1 illustrates how the VIX index varies under different regimes. In this figure, using parameters given in Table 2, we model the VIX index under three regimes from 2000 to 2015, where the first state is related to the regimes with low variance (green part), the second one is related to the forward regimes (blue part), and the third one is related to the high level of the variance (red part). Be noted that we are in the state $i \in \{1, 2, 3\}$, at time t_k, if the smoothed probability (2.7) corresponds to regime which has the maximum value.

In Figure 2, we model the S&P500 index where the VIX index is used as a proxy for its instantaneous variance.

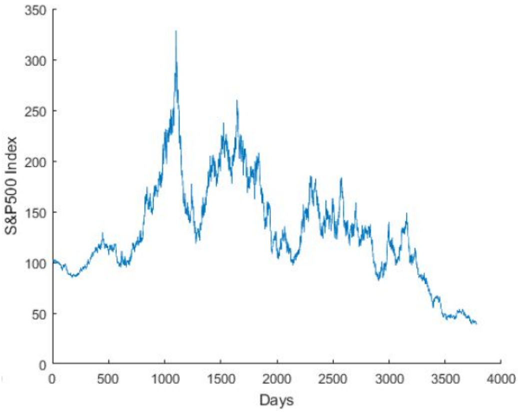

Figure 2 – The simulated paths of S&P500 index

3. PRICING S&P500 AMERICAN OPTIONS

Pricing an American option is generally equivalent to solve an optimal stopping problem by defining the optimal exercise rule. The value of an American option is thus, calculated by computing the expected discounted payoff under this rule.

Definition 3.1. Let Θ be a set of stopping times and S_t be a stock price. The price of the American put option is defined as follows

$$P(0, S(0)) = \sup\left\{\mathbb{E}\left[e^{-r\tau}\left(K - S(\tau)\right)_+\right]\right\}, \quad \tau \in \Theta, \tag{3.1}$$

where $S(0)$ is initial stock price. If $\tau = \infty$, then the value of the American put option is zero [22].

However, since American options do not have an analytical formula, simulation methods would be used as a helpful tool to determine the value of the option. In this section, our goal is to obtain the value of the S&P500 American put option when the volatility of this index follows the regime-switching Heston model. In the following, we verify two algorithms, named LSM and binomial tree, to obtain the value of the American put option.

3.1. **LSM algorithm**

LSM algorithm introduced by Longstaff and Schwartz [17] is a simple and widely used Monte Carlo method for calculating the value of the American style options by replacing the future expectation by a least squares interpolation.

In this work, LSM method used to S&P500 American put option can be interpreted as follows:

Algorithm 1 LSM algorithm

$S_j \leftarrow S\&P500\ AssetPath$
if $S_j < K$ **then**
  $C(j) \leftarrow (K - S_j)e^{-r\Delta t}$
else
  $C(j) \leftarrow 0$
end if
for $j = N - 1 : -1 : 1$ **do**
  $index = find(K - S_j > 0)$
  $X = [ones(index)\ S(index)\ S(index)^2]$
  $B \leftarrow (X^T X)^{0.5} X C(index)$
  $D \leftarrow X\ B$
  for $i = 1\ :\ \text{length}(index)$ **do**
    if $D \leq K - S_i$ **then**
      $C(index(i) = K - S_i$
    end if
    $C \leftarrow Ce^{-r\Delta t}$
  end for
  $AmericanPut \leftarrow C$
end for

3.2. **Binomial tree method**

The binomial tree method [4] is another commonly used method which can be applied to price the options (see Figure 3). The binomial pricing model traces the evolution of the option's key underlying variables in discrete-time. This is done by means of a binomial lattice (tree), for a number of time steps between the valuation and expiration dates. Each node in the lattice represents a possible price of the underlying at a given point in time. Valuation is performed iteratively, starting at each of the final nodes (those that may be reached at the time of expiration), and then working backwards through the tree towards the first node (valuation date). The value computed at each stage is the value of the option at that point in time.

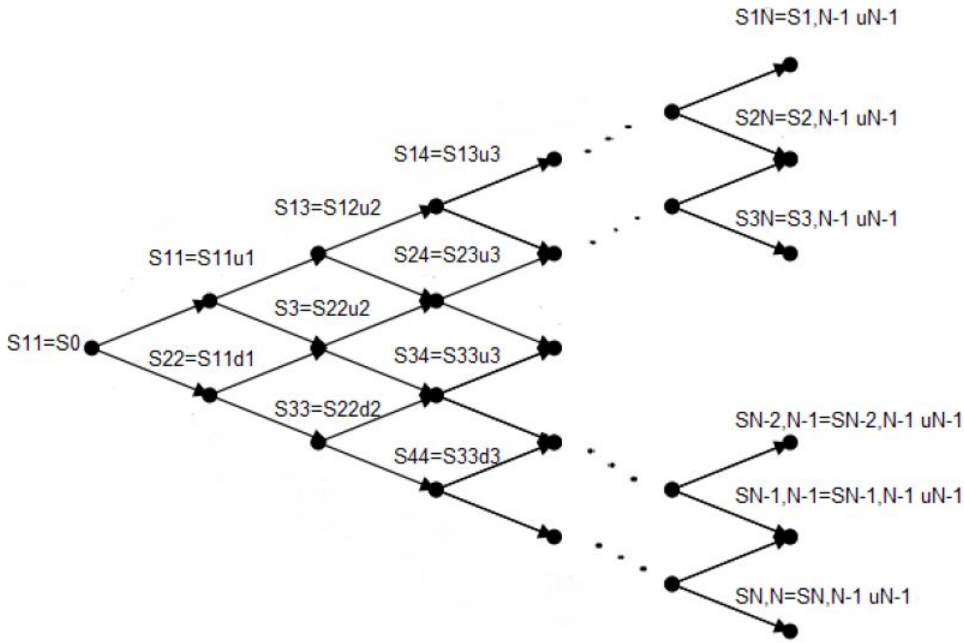

Figure 3 – Various paths of the binomial tree

In what follows, it is described S&P500 American put option valuation using this method:

Let S is the price of the S&P500 index. If T is the expiration date of option and δt is the time interval in each period. Then in this interval the value of the index either increases from S to Su or reduces to Sd. Since the volatility of the S&P500 index follows the stochastic volatility regime-switching Heston model,

$$u_i = \exp\{VIX_i\sqrt{\delta t}\}, \quad d_i = \exp\{-VIX_i\sqrt{\delta t}\},$$

where the value of the VIX index follows a CIR process with regimes-switching. It is necessary to check at each node whether the earlier exercise will exceed its maintenance over a longer period of time. Thus, the value of the put American option in the node (i,j) is as follows

$$AmericanPut1_{ij} = K - S_{i,j}$$

$$AmericanPut2_{i,j} = e^{-r\delta t}\big[P_i AmericanPut_{i,j+1} + (1-P_i)AmericanPut_{i+1,j+1}\big]$$

$$AmericanPut_{i,j} = \max\big(AmericanPut1_{i,j}, AmericanPut2_{i,j}\big),$$

where $P_i = \frac{e^{rt}-d_i}{u_i-d_i}$. Note that the value of the American option in the node (i, N), starting node, is as follows

$$AmericanPut_{i,N} = \max\left(0, K - S_{i,N}\right)$$

where N is the number of periods.

Algorithm 2 Binomial tree algorithm

$N \leftarrow$ *number of periods of the Binomial tree*
$T \leftarrow$ *time*
$K \leftarrow$ *strike price*
$S_0 \leftarrow$ *price*
$\delta t \leftarrow T_{\bar{N}}$
$S_{11} \leftarrow S_0$
for $i = 1, \cdots, N$ **do**
  $u_i = \exp\{VIX_i\sqrt{\delta t}\}$
  $d_i = (u_i)^{-1}$
  $P_i = \frac{e^{rT}-d_i}{u_i-d_i}$
end for
for $i = 2 : 1 : N$ **do**
  for $j = 1, \cdots, i-1$ **do**
    $S_{i,j} = S_{j,i-1} + S_{j,i-1} \times u_{i-1}$
  end for
  $S_{i,i} = S_{i-1,i-1} - Si - 1, i - 1 \times d_{i-1}$
  $AmericanPut_{i,N} = \max(K - S_{i,N}, 0)$
end for
for $i = N-1 : -1 : 1$ **do**
  $AmericanPut_{j,N} = \max\left(K - S_{j,i}, e^{-r\delta t}(AmericanPut_{j,i+1} \times P_i) + (AmericanPut_{j+1,i+1} \times (1 - P_i))\right)$
end for
$AmericanPut = AmericanPut_{1,1}$

4. NUMERICAL RESULTS

In this section, we calculate the value of the S&P500 American put option where its volatility follows the regime-switching Heston model. Moreover, we study the binomial tree algorithm with various periods.

In Table 3, we present the value of the American put option calculated by LSM and binomial tree algorithms with different values of the strike price and interest rate. Besides, Figure 4 illustrates more our achieved results.

In Table 4, it is considered various values of the strike price and periods. See Figure 5 for more illustration.

Results show that the interest rate has a negative effect on the options price, i.e. the smaller interest rate r implies higher option price at any time. This is reasonable, since if interest rates are high, you will hold the asset for a longer time to deliver it under the put option. Plainly selling the asset and using the proceeds to invest at a higher rate would be a better option. This makes the put option less attractive and thus, less costly when interest rates are high. Moreover, we can see that growing up the strike price causes a drop in the value of the put option. Another intriguing outcome is that as the number of periods of binomial tree algorithm increases, the obtained results coincide much more with the results achieved from the LSM algorithm. Convergence of binomial tree and LSM algorithms is investigated as [24] and [16].

Table 3 – Comparison of the S&P500 American put option price obtained by the LSM and binomial tree algorithms with $N = 1000$

r	K	LSM	Binomial	| LSM - Binomial |
0.02	40	5.0656	4.8410	0.2246
	60	7.6561	7.3056	0.3505
	80	10.2578	9.7682	0.4896
	100	12.8374	12.2373	0.6001
0.04	40	5.0541	4.7019	0.3522
	60	7.6424	7.1022	0.5402
	80	10.2431	9.5033	0.7396
	100	12.8185	11.9443	0.8742
0.06	40	5.0505	4.5108	0.5397
	60	7.6296	6.7962	0.8334
	80	10.2259	9.1551	0.7081
	100	12.7982	11.5472	1.2510
0.08	40	5.0470	4.2234	0.8236
	60	7.6180	6.4541	1.1639
	80	10.2090	8.7473	1.4617
	100	12.7775	11.1392	1.6383

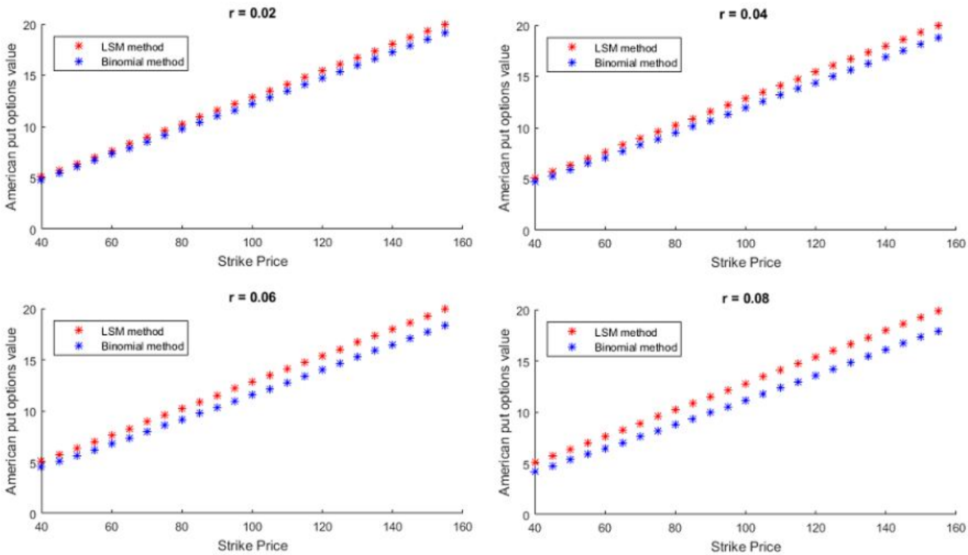

Figure 4 – Value of the S&P500 American put option obtained by the LSM and binomial tree algorithms with $N = 1000$.

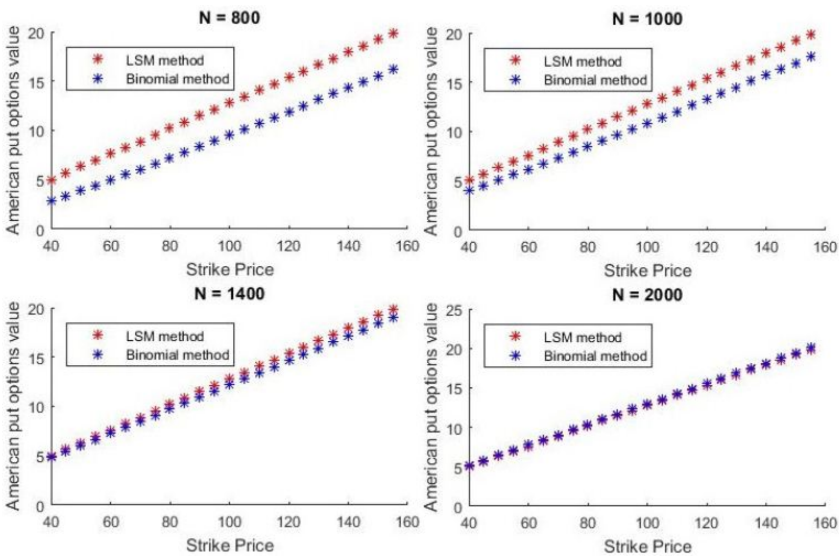

Figure 5 – Value of the S&P500 American put option obtained by the LSM and binomial tree algorithms with $r = 0.08$

Table 4 – Comparison of the S&P500 American put option price obtained by the LSM and binomial tree algorithms with $r = 0.08$

N	K	LSM	Binomial	| LSM - Binomial |
800	40	5.0239	2.8368	2.1871
	60	7.6035	4.9597	2.6438
	80	10.1970	7.1739	3.0231
	100	12.7670	9.4995	3.2675
1000	40	5.0297	3.9994	1.0303
	60	7.6072	6.1669	1.4403
	80	10.1934	8.4450	1.7484
	100	12.7584	10.8313	1.9271
1400	40	5.0122	4.8263	0.1859
	60	7.6059	7.2733	0.3326
	80	10.1892	9.7286	0.4606
	100	12.7651	12.1697	0.5954
2000	40	4.9890	5.1625	0.1735
	60	7.5615	7.7511	0.1897
	80	10.1898	10.3447	0.1549
	100	12.7491	12.9403	0.1912

APPENDIX A

Applying Itô's Lemma to Eq. (2.1) gives

$$R_t^T = \int_t^T (r - \frac{1}{2}V_u)\mathrm{d}u + \int_t^T \sqrt{V_u}\mathrm{d}W_u^1.$$

and also,

$$\mathbb{E}_t(R_t^T) = \int_t^T (r - \frac{1}{2}\mathbb{E}_t(V_u))\mathrm{d}u.$$

To simplify, we set

$$X_T \equiv \int_t^T \sqrt{V_u}\mathrm{d}W_u^1 \quad \text{and} \quad Y_T \equiv \int_t^T (V_u - \mathbb{E}_t(V_u)\mathrm{d}u,$$

so

$$R_t^T - \mathbb{E}_t(R_t^T) = X_T - \frac{1}{2}Y_T.$$

By Itô's Lemma, we have

$$V_s = e^{-\kappa(Z_s)s}\Big(e^{\kappa(Z_t)t}V_t + \int_t^s \theta(Z_u)\kappa(Z_u)e^{\kappa(Z_u)u}\mathrm{d}u + \int_t^s \sigma(Z_u)e^{\kappa(Z_u)u}\sqrt{V_u}\mathrm{d}W_u^2\Big),$$

for which

$$\mathbb{E}_t(V_s) = e^{-\kappa(Z_s)s}\Big(e^{\kappa(Z_t)t}V_t + \int_t^s \theta(Z_u)\kappa(Z_u)e^{\kappa(Z_u)u}\mathrm{d}u\Big).$$

Hence,

$$V_s - \mathbb{E}_t(V_s) = e^{-\kappa(Z_s)s}\int_t^s \sigma(Z_u)e^{\kappa(Z_u)u}\sqrt{V_u}\mathrm{d}W_u^2,$$

and

$$\begin{aligned}Y_T &= \int_t^T \big(e^{-\kappa(Z_s)s}\int_t^s \sigma(Z_u)e^{\kappa(Z_u)u}\sqrt{V_u}\mathrm{d}W_u^2\big)\mathrm{d}s\\ &= \int_t^T \big(\int_u^T e^{-\kappa(Z_s)s}\mathrm{d}s\big)\sigma(Z_u)e^{\kappa(Z_u)u}\sqrt{V_u}\mathrm{d}W_u^2.\end{aligned}$$

Now, we define the process Y_s^* as

$$Y_s^* := \int_t^s \left(\int_u^T e^{-\kappa(Z_s)s}\mathrm{d}s\right)\sigma(Z_u)e^{\kappa(Z_u)u}\sqrt{V_u}\mathrm{d}W_u^2.$$

Note that $Y_T^* = Y_T$ but the weight function in Y_s^* is $\int_u^T e^{-\kappa(Z_s)s}\mathrm{d}s$ which is independent of s while the weight function in Y_s is $\int_u^s e^{-\kappa(Z_s)s}\mathrm{d}s$ which depends on s. This difference determines that Y_s^* is an Itô process (martingale) and Y_s is not.

Indeed,

$$\begin{aligned}&\mathbb{E}_t(R_t^T - \mathbb{E}_t(R_t^T))^2\\ &= \mathbb{E}_t(X_T - \frac{1}{2}Y_T)^2\\ &= \mathbb{E}_t(X_T^2) - \mathbb{E}_t(X_TY_T) + \frac{1}{4}\mathbb{E}_t(Y_T^2)\\ &= \int_t^T \mathbb{E}_t(V_s)\mathrm{d}s - \int_t^T \big(\int_s^T e^{-\kappa(Z_u)u}\mathrm{d}u\big)\sigma(Z_s)\rho(Z_s)e^{\kappa(Z_s)s}\mathbb{E}_t(V_s)\mathrm{d}s\\ &\qquad + \frac{1}{4}\int_t^T \big(\int_s^T e^{-\kappa(Z_u)u}\mathrm{d}u\big)^2\sigma^2(Z_s)e^{2\kappa(Z_s)s}\mathbb{E}_t(V_s)\mathrm{d}s,\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}_t(R_t^T - \mathbb{E}_t(R_t^T))^3 &= \mathbb{E}_t(X_T - \frac{1}{2}Y_T)^3\\ &= \mathbb{E}_t(X_T^3) - \frac{3}{2}\mathbb{E}_t(X_T^2Y_T) + \frac{3}{4}\mathbb{E}_t(X_TY_T^2) - \frac{1}{8}\mathbb{E}_t(Y_T^3).\end{aligned} \tag{4.1}$$

It is worth noting that

$$\mathbb{E}_t(R_t^T - \mathbb{E}_t(R_t^T))^2 = \mathscr{G}(t). \tag{4.2}$$

By Itô's Lemma, we conclude that

$$\begin{aligned}\mathrm{d}X_s^3 &= 3X_s^2\mathrm{d}X_s + 3X_s\mathrm{d}X_s\mathrm{d}X_s = 3X_s^2\sqrt{V_s}\mathrm{d}W_s^1 + 3X_sV_s\mathrm{d}s,\\ \mathrm{d}(X_s^2Y_s^*) &= 2X_sY_s^*\mathrm{d}X_s + Y_s^*\mathrm{d}X_s\mathrm{d}X_s + X_s^2\mathrm{d}Y_s^* + 2X_sdX_s\mathrm{d}Y_s^*\\ &= 2X_sY_s^*\sqrt{V_s}\mathrm{d}W_s^1 + Y_s^*V_s\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&+ X_s^2\Big(\int_s^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)\sigma(Z_s)e^{\kappa(Z_s)s}\sqrt{V_s}\mathrm{d}W_s^2\\
&+ 2X_s\Big(\int_s^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)\rho(Z_s)\sigma(Z_s)e^{\kappa(Z_s)s}V_s\mathrm{d}s,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{d}(X_sY^{*2}_s) &= (Y_s^*)^2\mathrm{d}X_s + 2Y_s^*X_s\mathrm{d}Y_s^* + X_s\mathrm{d}Y_s^*\mathrm{d}Y_s^* + 2Y_s^*\mathrm{d}X_s\mathrm{d}Y_s^*\\
&= (Y_s^*)^2\sqrt{V_s}\mathrm{d}W_s^1 + 2Y_s^*X_s\Big(\int_s^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)\sigma(Z_s)e^{\kappa(Z_s)s}\sqrt{V_s}\mathrm{d}W_s^2\\
&\quad + X_s\Big(\int_s^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)^2\sigma^2(Z_s)e^{2\kappa(Z_s)s}V_s\mathrm{d}s\\
&\quad + 2Y_s^*\Big(\int_s^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)\rho(Z_s)\sigma(Z_s)e^{\kappa(Z_s)s}V_s\mathrm{d}s,\\
\mathrm{d}(Y_s^*)^3 &= 3(Y_s^*)^2\mathrm{d}Y_s^* + 3Y_s^*\mathrm{d}Y_s^*\mathrm{d}Y_s^*\\
&= 3(Y_s^*)^2\Big(\int_s^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)\sigma(Z_s)e^{\kappa(Z_s)s}\sqrt{V_s}\mathrm{d}W_s^2\\
&\quad + 3Y_s^*\Big(\int_s^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)^2\sigma^2(Z_s)e^{2\kappa(Z_s)s}V_s\mathrm{d}s.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\mathbb{E}_t(X_sV_s) &= e^{-\kappa(Z_s)s}\int_t^s \sigma(Z_u)\rho(Z_u)e^{\kappa(Z_u)u}\mathbb{E}_t(V_u)\mathrm{d}u,\\
\mathbb{E}_t(Y_s^*V_s) &= e^{-\kappa(Z_s)s}\int_t^s \Big(\int_u^T e^{-\kappa(Z_s)s}\mathrm{d}s\Big)\sigma^2(Z_u)e^{\kappa(Z_u)u}\mathbb{E}_t(V_u)\mathrm{d}u.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\mathbb{E}_t(X_T^3) &= 3\int_t^T \mathbb{E}_t(X_sV_s)\mathrm{d}s\\
&= 3\int_t^T \Big(e^{-\kappa(Z_s)s}\int_t^s \sigma(Z_u)\rho(Z_u)e^{\kappa(Z_u)u}\mathbb{E}_t(V_u)\mathrm{d}u\Big)\mathrm{d}s,
\end{aligned}
\tag{4.3}
$$

(4.4)

$$
\begin{aligned}
\mathbb{E}_t(X_T^2Y_T) &= \mathbb{E}_t(X_T^2Y_T^*)\\
&= \int_t^T \mathbb{E}_t(Y_s^*V_s)\mathrm{d}s\\
&\quad + 2\int_t^T \Big(\int_s^T e^{-\kappa(Z_u)u}\mathrm{d}u\Big)\rho(Z_s)\sigma(Z_s)e^{\kappa(Z_s)s}\mathbb{E}_t(X_sV_s)\mathrm{d}s\\
&= \int_t^T \Big[e^{-\kappa(Z_s)s}\int_t^s \Big(\int_u^T e^{-\kappa(Z_s)s}\mathrm{d}s\Big)\sigma^2(Z_u)e^{\kappa(Z_u)u}\mathbb{E}_t(V_u)\mathrm{d}u\Big]\mathrm{d}s
\end{aligned}
$$

$$+ 2\int_t^T \Big[\big(\int_s^T e^{-\kappa(Z_u)u}\mathrm{d}u\big)\Big(e^{-\kappa(Z_s)s}$$
$$\int_t^s \sigma(Z_u)\rho(Z_u)e^{\kappa(Z_u)u}\mathbb{E}_t(V_u)\mathrm{d}u\Big)\rho(Z_s)\sigma(Z_s)e^{\kappa(Z_s)s}\Big]\mathrm{d}s,$$

(4.5)

$$\begin{aligned}
\mathbb{E}_t(X_T Y_T^2) &= \mathbb{E}_t(X_T {Y^*}_T^2)\\
&= \int_t^T \Big(\big(\int_s^T e^{-\kappa(Z_u)u}\mathrm{d}u\big)^2\sigma^2(Z_s)e^{2\kappa(Z_s)s}\mathbb{E}_t(X_sV_s)\Big)\mathrm{d}s\\
&\quad + 2\int_t^T \Big(\big(\int_s^T e^{-\kappa(Z_u)u}\mathrm{d}u\big)\rho(Z_s)\sigma(Z_s)e^{\kappa(Z_s)s}\mathbb{E}_t(Y_s^*V_s)\Big)\mathrm{d}s\\
&= \int_t^T \Big(\big(\int_s^T e^{-\kappa(Z_u)u}\mathrm{d}u\big)^2\\
&\quad \Big(e^{-\kappa(Z_s)s}\int_t^s \sigma(Z_u)\rho(Z_u)e^{\kappa(Z_u)u}\mathbb{E}_t(V_u)\mathrm{d}u\Big)\,\sigma^2(Z_s)e^{2\kappa(Z_s)s}\Big)\mathrm{d}s\\
&\quad + 2\int_t^T \Big[\Big(\big(\int_s^T e^{-\kappa(Z_u)u}\mathrm{d}u\big)\rho(Z_s)\sigma(Z_s)e^{\kappa(Z_s)s}\Big)\\
&\quad \times\Big(\int_t^s \big(\int_u^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\big)\sigma^2(Z_u)e^{\kappa(Z_u)u}\mathbb{E}_t(V_u)\mathrm{d}u\Big)\Big]\mathrm{d}s,
\end{aligned}$$

(4.6)

$$\begin{aligned}
\mathbb{E}_t(Y_T^3) = \mathbb{E}_t({Y^*}_T^2) &= 3\int_t^T \Big(\big(\int_s^T e^{-\kappa(Z_u)u}\mathrm{d}u\big)^2\sigma^2(Z_s)e^{2\kappa(Z_s)s}\mathbb{E}_t({Y^*}_sV_s)\Big)\mathrm{d}s\\
&= 3\int_t^T \Big[\Big(\big(\int_s^T e^{-\kappa(Z_u)u}\mathrm{d}u\big)^2\sigma^2(Z_s)e^{2\kappa(Z_s)s}\Big)\\
&\quad \times\Big(e^{-\kappa(Z_s)s}\int_t^s \big(\int_u^T e^{-\kappa(Z_s)s}\mathrm{d}s\big)\sigma^2(Z_u)e^{\kappa(Z_u)u}\mathbb{E}_t(V_u)\mathrm{d}u\Big)\Big]\mathrm{d}s.
\end{aligned}$$

Finally, we get from Eqs. (4.3) - (4.6) that

$$\mathbb{E}_t(R_t^T - \mathbb{E}_t(R_t^T))^3 = \mathscr{A}(t) - \frac{3}{2}\mathscr{B}(t) + \frac{3}{4}\mathscr{C}(t) - \frac{1}{8}\mathscr{D}(t), \tag{4.7}$$

and consequently,

$$\text{Skewness}(R_t^T) = \frac{\mathbb{E}_t(R_t^T - \mathbb{E}_t(R_t^T))^3}{(\mathbb{E}_t(R_t^T - \mathbb{E}_t(R_t^T))^2)^{3/2}} \tag{4.8}$$

In what follows, we affirm the proof of the *Kurtosis*(R_t^T): One more time, applying Itô's Lemma gives

$$\mathbb{E}_t(R_t^T - \mathbb{E}_t(R_t^T))^4 = \mathbb{E}_t(X_T - \frac{1}{2}Y_T)^4$$

$$
\begin{aligned}
&= \mathbb{E}_t(X_T^4) + \frac{3}{2}\mathbb{E}_t(X_T^2Y_T^2) - 2\mathbb{E}_t(X_T^3Y_T) - \frac{1}{2}\mathbb{E}_t(X_TY_T^3) \\
&+ \frac{1}{16}\mathbb{E}_t(Y_T^4), \qquad (4.9)
\end{aligned}
$$

and also,

$$
\begin{aligned}
\mathrm{d}X_s^4 &= 4X_s^3\mathrm{d}X_s + 6X_s^2\mathrm{d}X_s\mathrm{d}X_s = 4X_s^3\sqrt{V_s}\mathrm{d}W_s^1 + 6X_s^2V_s\mathrm{d}s, \\
\mathrm{d}(X_s^2{Y^*}_s^2) &= 2X_s{Y^*}_s^2\mathrm{d}X_s + {Y^*}_s^2\mathrm{d}X_s\mathrm{d}X_s + 2Y_s^*X_s^2\mathrm{d}Y_s^* \\
&\quad + X_s^2\mathrm{d}Y_s^*\mathrm{d}Y_s^* + 4X_sY_s^*\mathrm{d}X_s\mathrm{d}Y_s^* \\
&= 2X_s{Y^*}_s^2\sqrt{V_s}\mathrm{d}W_s^1 + {Y^*}_s^2V_s\mathrm{d}s \\
&\quad + 2Y_s^*X_s^2\Big(\int_s^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)\sigma(Z_s)e^{\kappa(Z_s)s}\sqrt{V_s}\mathrm{d}W_s^2 \\
&\quad + X_s^2\Big(\int_s^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)^2\sigma^2(Z_s)e^{2\kappa(Z_s)s}V_s\mathrm{d}s \\
&\quad + 4X_sY_s^*\Big(\int_s^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)\sigma(Z_s)\rho(Z_s)e^{\kappa(Z_s)s}V_s\mathrm{d}s, \\
\mathrm{d}(X_s^3Y_s^*) &= 3X_s^2Y_s^*\mathrm{d}X_s + 3X_sY_s^*\mathrm{d}X_s\mathrm{d}X_s + X_s^3\mathrm{d}Y_s^* + 3X_s^2\mathrm{d}X_s\mathrm{d}Y_s^* \\
&= 3X_s^2Y_s^*\sqrt{V_s}\mathrm{d}W_s^1 + 3X_sY_s^*V_s\mathrm{d}s \\
&\quad + X_s^3\Big(\int_s^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)\sigma(Z_s)e^{\kappa(Z_s)s}\sqrt{V_s}\mathrm{d}W_s^2 \\
&\quad + 3X_s^2\Big(\int_s^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)\sigma(Z_s)\rho(Z_s)e^{\kappa(Z_s)s}V_s\mathrm{d}s, \\
\mathrm{d}(X_s{Y^*}_s^3) &= 3{Y^*}_s^2X_s\mathrm{d}Y_s^* + 3Y_s^*X_s\mathrm{d}Y_s^*\mathrm{d}Y_s^* + {Y^*}_s^3\mathrm{d}X_s + 3{Y^*}_s^2\mathrm{d}Y_s^*\mathrm{d}X_s \\
&= 3{Y^*}_s^2X_s\Big(\int_s^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)\sigma(Z_s)e^{\kappa(Z_s)s}\sqrt{V_s}\mathrm{d}W_s^2 \\
&\quad + 3Y_s^*X_s\Big(\int_s^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)^2\sigma^2(Z_s)e^{2\kappa(Z_s)s}V_s\mathrm{d}s + {Y^*}_s^3\sqrt{V_s}\mathrm{d}W_s^1 \\
&\quad + 3{Y^*}_s^2\Big(\int_s^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)\sigma(Z_s)\rho(Z_s)e^{\kappa(Z_s)s}V_s\mathrm{d}s, \\
\mathrm{d}{Y^*}_s^4 &= 4{Y^*}_s^3\mathrm{d}{Y^*}_s + 6{Y^*}_s^2\mathrm{d}{Y^*}_s\mathrm{d}{Y^*}_s \\
&= 4{Y^*}_s^3\Big(\int_s^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)\sigma(Z_s)e^{\kappa(Z_s)s}\sqrt{V_s}\mathrm{d}W_s^2 \\
&\quad + 6{Y^*}_s^2\Big(\int_s^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)^2\sigma^2(Z_s)e^{2\kappa(Z_s)s}V_s\mathrm{d}s.
\end{aligned}
$$

On the other hand, we have

$$\begin{aligned}
\mathbb{E}_t(X_s^2 V_s) &= e^{-\kappa(Z_s)s}\Big[\big(e^{\kappa(Z_t)t}V_t + \int_t^s \theta(Z_u)\kappa(Z_u)e^{\kappa(Z_u)u}\mathrm{d}u\big)\big(\int_t^s \mathbb{E}_t(V_u)\mathrm{d}u\big) \\
&\quad + \mathbb{E}_t\Big(\big(\int_t^s \sqrt{V_u}\mathrm{d}W_u^1\big)^2\big(\int_t^s \sigma(Z_u)e^{\kappa(Z_u)u}\sqrt{V_u}\mathrm{d}W_u^2\big)\Big)\Big] \\
&= e^{-\kappa(Z_s)s}\Big(\big(e^{\kappa(Z_t)t}V_t + \int_t^s \theta(Z_u)\kappa(Z_u)e^{\kappa(Z_u)u}\mathrm{d}u\big)\big(\int_t^s \mathbb{E}_t(V_u)\mathrm{d}u\big)\Big), \\
\mathbb{E}_t(Y^{*2}_s V_s) &= e^{-\kappa(Z_s)s}\Big(e^{\kappa(Z_t)t}V_t + \int_t^s \theta(Z_u)\kappa(Z_u)e^{-\kappa(Z_u)u}\mathrm{d}u\Big) \\
&\qquad \times \Big(\int_t^s \big(\int_u^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\big)^2\sigma^2(Z_u)e^{2\kappa(Z_u)u}\mathbb{E}_t(V_u)\mathrm{d}u\Big) \\
&\quad + e^{-\kappa(Z_s)s}\mathbb{E}\Big(\big(\int_t^s \big(\int_u^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\big)\sigma(Z_u)e^{\kappa(Z_u)u}\sqrt{V_u}\mathrm{d}W_u^2\big)^2 \\
&\qquad \times \big(\int_t^s \sigma(Z_u)e^{\kappa(Z_u)u}\sqrt{V_u}\mathrm{d}W_u^2\big)\Big) \\
&= e^{-\kappa(Z_s)s}\Big(e^{\kappa(Z_t)t}V_t + \int_t^s \theta(Z_u)\kappa(Z_u)e^{-\kappa(Z_u)u}\mathrm{d}u\Big) \\
&\qquad \times \Big(\int_t^s \big(\int_u^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\big)^2\sigma^2(Z_u)e^{2\kappa(Z_u)u}\mathbb{E}_t(V_u)\mathrm{d}u\Big), \\
\mathbb{E}_t(X_s Y_s^* V_s) &= \mathbb{E}_t\Big(\int_t^s \sqrt{V_u}\mathrm{d}W_u^1 \times \Big[\int_t^s \big(\int_u^s e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\big)\sigma(Z_u)e^{\kappa(Z_u)u}\sqrt{V_u}\mathrm{d}W_u^2\Big] \\
&\quad \times \Big[e^{-\kappa(Z_s)s}\Big(e^{\kappa(Z_t)t}V_t + \int_t^s \theta(Z_u)\kappa(Z_u)e^{\kappa(Z_u)u}\mathrm{d}u \\
&\quad + \int_t^s \sigma(Z_u)e^{\kappa(Z_u)u}\sqrt{V_u}\mathrm{d}W_u^2\Big)\Big]\Big) \\
&= e^{-\kappa(Z_s)s}\Big(e^{\kappa(Z_t)t}V_t + \int_t^s \theta(Z_u)\kappa(Z_u)e^{\kappa(Z_u)u}\mathrm{d}u\Big) \\
&\quad \times \Big(\int_t^s \big(\int_u^s e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\big)\sigma(Z_u)\rho(Z_u)e^{\kappa(Z_u)u}\mathbb{E}_t(V_u)\mathrm{d}u\Big).
\end{aligned}$$

Therefore, we can write

$$\begin{aligned}
\mathbb{E}_t(X_T^4) &= 6\int_t^T \mathbb{E}_t(X_s^2 V_s)\mathrm{d}s \\
&= 6\int_t^T e^{-\kappa(Z_s)s}\Big(\big(e^{\kappa(Z_t)t}V_t + \int_t^s \theta(Z_u)\kappa(Z_u)e^{\kappa(Z_u)u}\mathrm{d}u\big) \\
&\qquad \big(\int_t^s \mathbb{E}_t(V_u)\mathrm{d}u\big)\Big)\mathrm{d}s, \qquad (4.10)
\end{aligned}$$

$$
\begin{aligned}
\mathbb{E}_t(X_T^2 Y_T^{*2}) &= \int_t^T \mathbb{E}_t(Y^{*2}_s V_s)\mathrm{d}s \\
&\quad + \int_t^T \Big(\int_s^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)^2 \sigma^2(Z_s) e^{2\kappa(Z_s)s}\mathbb{E}_t(X_s^2 V_s)\mathrm{d}s \\
&\quad + 4\int_t^T \Big(\int_s^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)\sigma(Z_s)\rho(Z_s)e^{\kappa(Z_s)s}\mathbb{E}_t(V_s X_s Y_s^*)\mathrm{d}s \\
&= \int_t^T \Big[e^{-\kappa(Z_s)s}\Big(e^{\kappa(Z_t)t}V_t + \int_t^s \theta(Z_u)\kappa(Z_u)e^{-\kappa(Z_u)u}\mathrm{d}u\Big) \\
&\qquad \times \Big(\int_t^s \Big(\int_u^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)^2\sigma^2(Z_u)e^{2\kappa(Z_u)u}\mathbb{E}_t(V_u)\mathrm{d}u\Big)\Big]\mathrm{d}s \\
&\quad + \int_t^T \Big[\Big(\Big(\int_s^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)^2\sigma^2(Z_s)e^{2\kappa(Z_s)s}\Big)\times \\
&\qquad \times e^{-\kappa(Z_s)s}\Big(\Big(e^{\kappa(Z_t)t}V_t + \int_t^s \theta(Z_u)\kappa(Z_u)e^{\kappa(Z_u)u}\mathrm{d}u\Big) \\
&\qquad \Big(\int_t^s \mathbb{E}_t(V_u)\mathrm{d}u\Big)\Big)\Big]\mathrm{d}s \\
&\quad + 4\int_t^T \Big[\Big(\Big(\int_s^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)\sigma(Z_s)\rho(Z_s)e^{\kappa(Z_s)s}\Big) \\
&\qquad \times e^{-\kappa(Z_s)s}\Big(e^{\kappa(Z_t)t}V_t + \int_t^s \theta(Z_u)\kappa(Z_u)e^{\kappa(Z_u)u}\mathrm{d}u\Big) \\
&\qquad \times \Big(\int_t^s \Big(\int_u^s e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)\sigma(Z_u)\rho(Z_u)e^{\kappa(Z_u)u}\mathbb{E}_t(V_u)\mathrm{d}u\Big)\Big]\mathrm{d}s,
\end{aligned} \tag{4.11}
$$

$$
\begin{aligned}
\mathbb{E}_t(X_T^3 Y_T^*) &= 3\int_t^T \mathbb{E}_t(X_s Y_s^* V_s)\mathrm{d}s \\
&\quad + 3\int_t^T \Big(\int_s^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)\sigma(Z_s)\rho(Z_s)e^{\kappa(Z_s)s}\mathbb{E}_t(X_s^2 V_s)\mathrm{d}s \\
&= 3\int_t^T \Big[e^{-\kappa(Z_s)s}\Big(e^{\kappa(Z_t)t}V_t + \int_t^s \theta(Z_u)\kappa(Z_u)e^{\kappa(Z_u)u}\mathrm{d}u\Big) \\
&\qquad \times \Big(\int_t^s \Big(\int_u^s e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)\sigma(Z_u)\rho(Z_u)e^{\kappa(Z_u)u}\mathbb{E}_t(V_u)\mathrm{d}u\Big)\Big]\mathrm{d}s \\
&\quad + 3\int_t^T \Big[\Big(\Big(\int_s^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)\sigma(Z_s)\rho(Z_s)e^{\kappa(Z_s)s}\Big)e^{-\kappa(Z_s)s}\Big(\Big(e^{\kappa(Z_t)t}V_t \\
&\qquad + \int_t^s \theta(Z_u)\kappa(Z_u)e^{\kappa(Z_u)u}\mathrm{d}u\Big)\Big(\int_t^s \mathbb{E}_t(V_u)\mathrm{d}u\Big)\Big)\Big]\mathrm{d}s,
\end{aligned} \tag{4.12}
$$

$$
\begin{aligned}
\mathbb{E}_t(X_T {Y^*}_T^3) &= 3\int_t^T \Big(\int_s^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)^2\sigma^2(Z_s)e^{2\kappa(Z_s)s}\mathbb{E}_t(Y_s^* X_s V_s)\mathrm{d}s\\
&\quad + 3\int_t^T \Big(\int_s^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)\sigma(Z_s)\rho(Z_s)e^{\kappa(Z_s)s}\mathbb{E}_t({Y^*}_s^2 V_s)\mathrm{d}s\\
&= 3\int_t^T \Big[\Big(\Big(\int_s^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)^2\sigma^2(Z_s)e^{2\kappa(Z_s)s}\Big)\\
&\qquad\qquad \times e^{-\kappa(Z_s)s}\Big(e^{\kappa(Z_t)t}V_t + \int_t^s \theta(Z_u)\kappa(Z_u)e^{\kappa(Z_u)u}\mathrm{d}u\Big)\\
&\qquad\qquad \times \Big(\int_t^s \Big(\int_u^s e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)\sigma(Z_u)\rho(Z_u)e^{\kappa(Z_u)u}\mathbb{E}_t(V_u)\mathrm{d}u\Big)\Big]\mathrm{d}s\\
&\quad + 3\int_t^T \Big[\Big(\Big(\int_s^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)\sigma(Z_s)\rho(Z_s)e^{\kappa(Z_s)s}\Big)\\
&\qquad\qquad \times e^{-\kappa(Z_s)s}\Big(e^{\kappa(Z_t)t}V_t + \int_t^s \theta(Z_u)\kappa(Z_u)e^{-\kappa(Z_u)u}\mathrm{d}u\Big)\\
&\qquad\qquad \times \Big(\int_t^s \Big(\int_u^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)^2\sigma^2(Z_u)e^{2\kappa(Z_u)u}\mathbb{E}_t(V_u)\mathrm{d}u\Big)\Big]\mathrm{d}s,
\end{aligned}
\tag{4.13}
$$

$$
\begin{aligned}
\mathbb{E}_t({Y^*}_T^4) &= 6\int_t^T \Big(\int_s^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)^2\sigma^2(Z_s)e^{2\kappa(Z_s)s}\mathbb{E}_t({Y^*}_s^2 V_s)\mathrm{d}s\\
&= 6\int_t^T \Big[\Big(\Big(\int_s^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)^2\sigma^2(Z_s)e^{2\kappa(Z_s)s}\mathrm{d}s\Big)\\
&\qquad\qquad \times e^{-\kappa(Z_s)s}\Big(e^{\kappa(Z_t)t}V_t + \int_t^s \theta(Z_u)\kappa(Z_u)e^{-\kappa(Z_u)u}\mathrm{d}u\Big)\\
&\qquad\qquad \times \Big(\int_t^s \Big(\int_u^T e^{-\kappa(Z_\nu)\nu}\mathrm{d}\nu\Big)^2\sigma^2(Z_u)e^{2\kappa(Z_u)u}\mathbb{E}_t(V_u)\mathrm{d}u\Big)\Big]\mathrm{d}s.
\end{aligned}
\tag{4.14}
$$

Finally, we get from Eqs. (4.10) - (4.14) that

$$\mathbb{E}_t(R_t^T - \mathbb{E}_t(R_t^T))^4 = \mathscr{H}(t) + \frac{3}{2}\mathscr{M}(t) - 2\mathscr{N}(t) - \frac{1}{2}\mathscr{O}(t) + \frac{1}{16}\mathscr{P}(t), \tag{4.15}$$

and consequently,

$$\text{Kurtosis}(R_t^T) = \frac{\mathbb{E}_t(R_t^T - \mathbb{E}_t(R_t^T))^4}{(\mathbb{E}_t(R_t^T - \mathbb{E}_t(R_t^T))^2)^2}. \tag{4.16}$$

The proof is done.

5. CONCLUSION

In this paper, we studied the regime-switching Heston model. We obtained a closed-form formula for conditional higher moments of the stock return. Our achieved results confirmed that the regime-switching Heston model is a better choice for analyzing the conditional higher moments of the real market date than the original Heston model. We exert a maximum likelihood procedure to estimate the parameters of model. Next, providing a LSM algorithm, we calculate the value of the S&P500 American put option under this model where its volatility managed by the VIX index. Additionally, we apply binomial tree method as a certain criteria to verify the exactness of our presented algorithm. Based on our achieved results, by increasing the periods of the binomial tree method, the results of this algorithm converge to the results of the LSM algorithm.

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Received January 26, 2019

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