CONTINUED FRACTIONS FOR THE n^{th} ROOT OF RATIONAL FUNCTIONS IN CHARACTERISTIC p

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The aim of this paper is to exhibit the continued fractions expansions of n^{th} root of rational functions. We are particularly interested to the expansion of $(1-\frac{1}{T})^{\frac{1}{n}}$ in the field of power series of characteristic a prime integer p, for a suitable n prime with p.

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1. INTRODUCTION

A central question in Diophantine approximation is concerned with how algebraic numbers can be approximated by rationals. This problem is ultimately connected with the behavior of their continued fraction expansion. In [4], Khintchine conjectured that if x is a real algebraic number of degree > 2 then x has a continued fraction expansion whose sequence of partial quotients is unbounded. The answer to this conjecture is far from being tractable. This is due to the fact that no explicit continued fraction expansion of algebraic number of degree $n > 2$ is known. However, more things are known in the case of algebraic power series over a finite field of characteristic p. The continued fraction expansion of many algebraic, and nonquadratic, elements are completely described. The first explicit description of continued fraction for algebraic power series over a finite field goes back to the work of Baum and Sweet [2], in the middle of the 1970's. They could give the first example of an algebraic power series, of degree 3 over $\mathbb{F}_2$, with all partial quotients of bounded degree. Moreover, they have exhibited other families of formal series with unbounded partial quotients. Later, this work was carried on by Mills and Robbins in their paper [10]. They presented several examples of algebraic continued fractions, also, they introduced a particular subset of algebraic power series called hyperquadratic. Let $q = p^t$ with $t \geq 0$, we say that α belonging to $\mathbb{F}((T^{-1}))$ is hyperquadratic if α is irrational and satisfies an algebraic equation of the particular form $A\alpha^{q+1} + B\alpha^q + C\alpha + D = 0$, where A, B, C and D belong to $\mathbb{F}[T]$. Note that quadratic elements are hyperquadratic. Many explicit continued fractions are known for nonquadratic but hyperquadratic elements; see for example [1], [6] and [12]. Even though the pattern of hyperquadratic expansions can sometimes be very sophisticated, it is yet doubtful whether this description, even partial, is possible for all hyperquadratic power series.

Now we consider the case of the equation algebraic irreducible

$$x^n = R, \tag{1.1}$$

where n is a positive integer, not divisible by p, and $R \in \mathbb{F}(T)$. Such an equation has a root in $\mathbb{F}((T^{-1}))$ if (and only if) $\deg R$ is a multiple of n and the first coefficient of R belongs to $\mathbb{F}^n$. Osgood [11], Voloch [14], de Mathan [3] and Lasjaunias [5] have studied the rational approximation of the solution of the equation (1.1). For instance, we know that it is well approximable by rationals for suitable R. However, the explicit continued fraction expansion of the solutions of (1.1) is not yet described.

In this work, we are interested in giving explicitly the continued fraction expansion for the solution of equation (1.1) for $R = 1 - \dfrac{1}{T}$, and for the cases when $n = q+1$ and $n = q-1$ respectively, where q is a power of p. Note that in each case, it is easy to see that the solution is hyperquadratic. Moreover, on the way, we will discuss the solutions of a Diophantine equation. We recall that $(1 - \dfrac{1}{T})^{\frac{1}{n}}$ has a continued fraction expansion in $\mathbb{Q}((T^{-1}))$ with all partial quotient of degree 1, which is completely described (see [5] p. 225). Of course, this is different from the case of field of power series in positive characteristic. We note that for n an integer prime to p, the equation $x^n = 1 - \dfrac{1}{T}$ has a unique root α in $\mathbb{F}((T^{-1}))$ with $|\alpha - 1| < 1$. Further, in [2, Theorem 11] Baum and Sweet have given the continued fraction expansion, in characteristic 2, of $(1 - \dfrac{1}{T})^{\frac{1}{2^n-1}}$ for all $n \geq 1$. However, their method cannot be extended to odd characteristic, see [2, Remark p. 610]. We will exhibit this expansion using another method which is valid for all positive characteristic.

Therefore, by this work we add other examples to the explicitly known hyperquadratic continued fractions. Furthermore, for these examples, Liouville's theorem is sharp, and thus a Thue-Siegel-Roth theorem cannot hold for such examples. Indeed, in Corollary 4.2, we will improve the following result of Baum and Sweet [2].

THEOREM 1.1. *Let $d, n \in \mathbb{N}\backslash\{0\}$. Then there exist an algebraic formal* *power series $\theta \in \mathbb{F}_2((T^{-1}))$ of degree 2^n+1 such that the equation*

$$|\theta - \frac{P}{Q}| = \frac{2^{-d}}{|Q|^{2^n+1}}$$

has infinitely many solutions $(P, Q) \in \mathbb{F}_2[T] \times \mathbb{F}_2[T]$.

2. PRELIMINARIES

Let p be a given prime number and $\mathbb{F}$ be a finite field of characteristic p. We denote by $\mathbb{F}[T]$ the ring of polynomials with coefficients in $\mathbb{F}$ and by $\mathbb{F}(T)$ the field of fractions of $\mathbb{F}[T]$. Let $\mathbb{F}((T^{-1}))$ be the field of formal power series:

$$\mathbb{F}((T^{-1})) = \left\{\alpha = \textstyle\sum_{i\geq n_0} c_i T^{-i} \quad : \quad n_0 \in \mathbb{Z} \quad \text{and} \quad c_i \in \mathbb{F}\right\}.$$

Let $\alpha = \sum c_i T^{-i}$ be any formal power series, we define its polynomial part, denoted $[\alpha]$, by $[\alpha] := \sum_{i\leq 0} c_i T^{-i}$. If $\alpha \neq 0$, then the degree of α is $\deg(\alpha) = \sup\{-i; c_i \neq 0\}$ and $\deg(0) = -\infty$. Thus, we define the non-Archimedean absolute value over $\mathbb{F}((T^{-1}))$ by $|\alpha| = e^{\deg(\alpha)}$ and $|0| = 0$.

As in the classical context of real numbers, we have a continued fraction algorithm in $\mathbb{F}((T^{-1}))$. If $\alpha \in \mathbb{F}((T^{-1}))$ we can write

$$\alpha = a_0 + 1/(a_1 + 1/(a_2 + \cdots = [a_0, a_1, a_2, \ldots]$$

where $a_i \in \mathbb{F}[T]$. The a_i are called the partial quotients and we have $\deg a_i > 0$ for $i > 0$. This continued fraction is finite if and only if $\alpha \in \mathbb{F}(T)$.

We define two sequences of polynomials (P_n) and (Q_n) by $P_0 = a_0$, $Q_0 = 1$, $P_1 = a_0 a_1 + 1$, $Q_2 = a_1$ and, for any $n \geq 2$,

$$P_n = a_n P_{n-1} + P_{n-2}, \quad Q_n = a_n Q_{n-1} + Q_{n-2}.$$

We easily check that $P_{n+1}Q_n - Q_{n+1}P_n = (-1)^n$, whence P_n and Q_n are coprime polynomials. The rational function P_n/Q_n is called a convergent to α and we have $P_n/Q_n = [a_0, a_1, \ldots, a_n]$. It is easily to see that $\deg Q_{n+1} = \deg a_{n+1} + \deg Q_n$, thus $\deg Q_n = \sum_{j=1}^{n} \deg a_j$. Furthermore, we have the following important equality

$$\left|\alpha - \frac{P_n}{Q_n}\right| = |a_{n+1}|^{-1}|Q_n|^{-2}.$$

Moreover, we have for $n \geq 0$ the identity:

$$\alpha = [a_0, a_1, \ldots, a_n, \alpha_{n+1}] = \frac{P_n \alpha_{n+1} + P_{n-1}}{Q_n \alpha_{n+1} + Q_{n-1}}, \tag{2.2}$$

where $\alpha_{n+1} = [a_{n+1}, a_{n+2}, \ldots]$ is called the complete quotient of α.

Mahler [8] proved the equivalence of Liouville's theorem [7] in the formal series case:

Theorem 2.1. *Let $\alpha \in \mathbb{F}((T^{-1}))$ algebraic of degree $d \geq 2$, then there exists $c > 0$ such that for all P , Q $\in \mathbb{F}[T]$:*

$$|\alpha - \frac{P}{Q}| \geq \frac{c}{|Q|^d}.$$

He introduced the first example of an $\alpha \in \mathbb{F}_p((T^{-1}))$ satisfying the irreducible polynomial $T\alpha^p - T\alpha + 1 = 0$, for which, the exponent d in the previous theorem cannot be reduced. So Thue-Siegel-Roth theorem cannot hold for such example. The reader who is interested in a survey of the different contributions to this topic and full references can consult for example [5], [12] and [13, Chap. 9].

We need to introduce the following two lemmas. The proof of the first is easy so we omit it, and the proof of the second can be found in [2, p. 600].

Lemma 2.2. *Let $A \in \mathbb{F}[T]$ and $\alpha \in \mathbb{F}((T^{-1}))$ such that $\alpha = [a_0, a_1, \cdots, a_n, a_{n+1} \cdots]$. Suppose that A divides a_i for all $i \geq 0$. Then*

$$A^{-1}\alpha = [b_0, \cdots, b_n, \cdots]$$

where for all $k \geq 0$

$$b_{2k} = A^{-1}a_{2k} \quad \textit{and} \quad b_{2k+1} = Aa_{2k+1}. \tag{2.3}$$

Note that the formula (2.3) does not give the usual continued fraction expansion of $A^{-1}\alpha$. Indeed, $A^{-1}a_{2k}$ may be in $\mathbb{F}^*$ for some index $k \geq 1$. However, the usual continued fraction expansion of $A^{-1}\alpha$ can be deduced from it. Let $\delta = A^{-1}a_{2k} \in \mathbb{F}^*$, then

$$\begin{aligned}\beta &= [b_0, \ldots, b_{2k-1}, \delta, b_{2k+1}, \ldots] \\ &= [b_0, \ldots, b_{2k-1} + \frac{1}{\delta}, -\delta^2 b_{2k+1} - \delta, -\frac{b_{2k+2}}{\delta^2}, -\delta^2 b_{2k+3}, \ldots],\end{aligned}$$

because

$$b_{2k-1} + \frac{1}{\delta + (1/\beta_{2k+1})} = b_{2k-1} + \frac{1}{\delta} - \frac{1}{\delta^2\beta_{2k+1} + \delta}.$$

Lemma 2.3. *Let $P(x) = \sum_{0 \leq i \leq n} A_i x^i$ with $A_i \in \mathbb{F}[T] \backslash \{0\}$ and $n \geq 1$. Let $\alpha = [a_0, \cdots, a_n, \cdots] \in \mathbb{F}((T^{-1}))$ be an algebraic formal power series such that $P(\alpha) = 0$. Let U be a nonconstant polynomial of $\mathbb{F}[T]$. Then $\beta = [a_0(U), \cdots, a_n(U), \cdots] = \alpha(U)$ satisfies $\sum_{0 \leq i \leq n} B_i \beta^i = 0$ where $B_i = A_i[U]$.*

We will also use a lemma relying on an idea about real continued fractions due to Mendès France. This lemma seems to appear for the first time in his work on finite continued fraction in the context of real numbers, see [9, p. 209]. Let $P_n/Q_n \in \mathbb{F}(T)$ such that $P_n/Q_n := [a_1, a_2, \ldots, a_n]$. For all $x \in \mathbb{F}(T)$, we will write

$$\big[[a_1, a_2, \ldots, a_n], x\big] := \frac{P_n}{Q_n} + \frac{1}{x}$$

Lemma 2.4. *Let $a_1, \ldots, a_n, x \in \mathbb{F}(T)$. We have the following equality:*

$$\big[[a_1, a_2, \ldots, a_n], x\big] = [a_1, a_2, \ldots, a_n, y], \text{ where } \ y = (-1)^{n-1}Q_n^{-2}x - Q_{n-1}Q_n^{-1}.$$

Particulary, we have

$$\big[[a_1, a_2, a_3], x\big] = [a_1, a_2, a_3, y], \text{ where } \ y = (a_2a_3 + 1)^{-2}x - a_2(a_2a_3 + 1)^{-1}.$$

The proof of this lemma can be found in Lasjaunias's article [6, p. 336].

3. CONTINUED FRACTION EXPANSION OF THE $(q + 1)^{th}$ ROOT OF $1 - \frac{1}{T}$

Theorem 3.1. *Let $\mathbb{F}$ be a field of characteristic p. Let $t \geq 1$ be an integer and $q = p^t > 2$. Let $\alpha \in \mathbb{F}((T^{-1}))$ be the irrational solution of the equation:*

$$x^{q+1} = 1 - \frac{1}{T^{q+1}}. \tag{3.4}$$

Then, the continued fraction expansion of α is $[a_0, \cdots, a_n, \cdots]$, where $a_0 = 1$, $a_1 = -T^{q+1}$, and, for $n \geq 0$,

$$a_{n+2} = -T^{q^n+(-1)^{n+1}}\Big(1 + T^{q+1} + T^{2(q+1)} + \cdots$$
$$+T^{(q-2)(q+1)}\Big)^{q^n}(T^{q+1} - 1)^{\frac{q^n+(-1)^{n+1}}{q+1}}.$$

Proof. Let $\alpha = [a_0, \cdots, a_n, \cdots]$ be the irrational solution of the equation (3.4). Then $a_0 = [\alpha] = 1$. Put $\beta = T\alpha$. Let $[b_0, \cdots, b_n, \cdots]$ the continued fraction expansion of β. Then β satisfies the equation:

$$\beta^{q+1} = T^{q+1} - 1 \tag{3.5}$$

and $[\beta] = b_0 = T$. We have $\beta = T + \beta_1^{-1}$. By replacing β by $T + \beta_1^{-1}$ in (3.5), we get that β_1 satisfies the equation:

$$\beta_1^{q+1} + T^q\beta_1^q + T\beta_1 + 1 = 0. \tag{3.6}$$

So $[\beta_1] = b_1 = -T^q$. Moreover, we can write (3.6) as:

$$\beta_1^q = \frac{-T\beta_1 - 1}{\beta_1 + T^q}. \tag{3.7}$$

We know that $\beta_1 = -T^q + \beta_2^{-1}$. So the equation (3.7) becomes:

$$\beta_1^q = \frac{-T(-T^q + \beta_2^{-1}) - 1}{\beta_2^{-1}} = (T^{q+1} - 1)\beta_2 - T. \tag{3.8}$$

The identity (3.8) gives that:

$$b_1^q + \beta_2^{-q} = (T^{q+1} - 1)\beta_2 - T.$$

So

$$\begin{aligned}\beta_2 &= \frac{b_1^q + T}{T^{q+1} - 1} + (T^{q+1} - 1)^{-1}\beta_2^{-q} \\ &= \frac{-T^{q^2} + T}{T^{q+1} - 1} + (T^{q+1} - 1)^{-1}\beta_2^{-q}.\end{aligned}$$

Since

$$-T^{q^2} + T = -T(T^{q+1} - 1)\big(1 + T^{q+1} + T^{2(q+1)} + \ldots + T^{(q-2)(q+1)}\big),$$

then $b_2 = -T(1 + T^{q+1} + T^{2(q+1)} + \ldots + T^{(q-2)(q+1)})$ and

$$\beta_3 = (T^{q+1} - 1)\beta_2^q. \tag{3.9}$$

Again, knowing that $\beta_2 = b_2 + \beta_3^{-1}$, it follows from the identity (3.9) that $\beta_3 = (T^{q+1} - 1)b_2^q + (T^{q+1} - 1)\beta_3^{-q}$. Then

$$b_3 = (T^{q+1} - 1)b_2^q = -T^q(T^{q+1} - 1)\big(1 + T^{q+1} + T^{2(q+1)} + \ldots + T^{(q-2)(q+1)}\big)^q,$$

and $\beta_4 = (T^{q+1} - 1)^{-1}\beta_3^q$. The last identity gives that $\beta_4 = b_4 + \beta_5^{-1} = (T^{q+1} - 1)^{-1}b_3^q + (T^{q+1} - 1)^{-1}\beta_4^{-q}$. Then

$$b_4 = -T^{q^2}(T^{q+1} - 1)^{q-1}\big(1 + T^{q+1} + T^{2(q+1)} + \ldots + T^{(q-2)(q+1)}\big)^{q^2},$$

and $\beta_5 = (T^{q+1} - 1)\beta_4^q$. So by a simple recursion we prove that for all $k \geq 1$

$$\beta_{2k+1} = (T^{q+1} - 1)\beta_{2k}^q \quad \text{and} \quad \beta_{2k+2} = (T^{q+1} - 1)^{-1}\beta_{2k+1}^q.$$

Hence for all $k \geq 1$

$$b_{2k+1} = (T^{q+1} - 1)b_{2k}^q \quad \text{and} \quad b_{2k+2} = (T^{q+1} - 1)^{-1}b_{2k+1}^q.$$

Thus for all $k \geq 1$

$$b_{2k+1} = -T^{q^{2k-1}}\big(1 + T^{q+1} + T^{2(q+1)} + \ldots + T^{(q-2)(q+1)}\big)^{q^{2k-1}}\big(T^{q+1} - 1\big)^{\frac{q^{2k-1}+1}{q+1}}$$

$$b_{2k+2} = -T^{q^{2k}}\big(1 + T^{q+1} + T^{2(q+1)} + \ldots + T^{(q-2)(q+1)}\big)^{q^{2k}}\big(T^{q+1} - 1\big)^{\frac{q^{2k}-1}{q+1}}.$$

Now, since $\alpha = T^{-1}\beta$ and T divides b_{2k} for all $k \geq 0$, following Lemma 2.2, we get the continued fraction expansion of α. This is $a_0 = 1$, $a_1 = -T^{q+1}$, $a_2 = -(1 + T^{q+1} + T^{2(q+1)} + \ldots + T^{(q-2)(q+1)})$ and for all $k \geq 1$

$$a_{2k+1} = -T^{q^{2k-1}+1}\big(1+T^{q+1}+T^{2(q+1)} + \ldots + T^{(q-2)(q+1)}\big)^{q^{2k-1}}\big(T^{q+1} - 1\big)^{\frac{q^{2k-1}+1}{q+1}}$$

$$a_{2k+2} = -T^{q^{2k}-1}\left(1+T^{q+1}+T^{2(q+1)}+\ldots+T^{(q-2)(q+1)}\right)^{q^{2k}}\left(T^{q+1}-1\right)^{\frac{q^{2k}-1}{q+1}}.$$

So we obtain the desired result. $\square$

COROLLARY 3.1. *Let $\mathbb{F}$ be a field of characteristic p. Let $t \geq 1$ be an integer and $q = p^t > 2$. Let $\alpha \in \mathbb{F}((T^{-1}))$ be the irrational solution of the equation:*

$$x^{q+1} = 1 - \frac{1}{T}. \tag{3.10}$$

Then, the continued fraction expansion of α is:

$$[a_0, \cdots, a_n, \cdots],$$

where

$$a_0 = 1, a_1 = -T, a_2 = 1 + T + T^2 + \ldots + T^{q-2},$$

and for $n \geq 1$

$$a_{n+2} = -T^{\frac{q^n+(-1)^{n+1}}{q+1}}\left(1+T+T^2+\ldots+T^{q-2}\right)^{q^n}(T-1)^{\frac{q^n+(-1)^{n+1}}{q+1}}.$$

Proof. We know that for all $n \geq 0$: $q^n + (-1)^{n+1} \equiv 0 \mod (q+1)$. So we can write the partial quotients of α solution of the equation (3.4) as:

$$a_{n+2} = -(T^{q+1})^{\frac{q^n+(-1)^{n+1}}{q+1}}\left(1+(T^{q+1})+(T^{q+1})^2+\cdots\right.$$
$$\left.+(T^{q+1})^{q-2}\right)^{q^n}(T^{q+1}-1)^{\frac{q^n+(-1)^{n+1}}{q+1}}.$$

So the result is a consequence of Lemma 2.3. $\square$

The case of cubic root of $(1 - \frac{1}{T})$ will be deduced from the first theorem of the Section 5.

4. ON A DIOPHANTINE EQUATION

In the following theorem, we will study the solutions of a Diophantine equation ultimately related to the equation (3.10).

THEOREM 4.1. *Let q be a power of p. Let B be a nonconstant polynomial of $\mathbb{F}[T]$. The Diophantine equation*

$$BP^{q+1} - (B-1)Q^{q+1} = 1 \tag{4.11}$$

have infinitely many solution $(P, Q) \in \mathbb{F}[T] \times \mathbb{F}[T]$, which are the even convergents of $\theta = \alpha(B)$ where α is the solution of the equation (3.10).

Proof. Let $\theta = [b_0, b_1, \ldots]$ be the continued fraction expansion of θ. As $\theta = \alpha(B)$ then $b_i = a_i(B)$. Let $(P_n/Q_n)_{n\geq 0}$ be the sequence of convergent of θ. Let $H(Y,Z) = BY^{q+1} - (B-1)Z^{q+1}$ and θ be the unique root of $L(Y) = H(Y,1)$ satisfying $|\theta| = 1$. Then by writing $H(Y,1)$ in the form

$$H(Y,1) = B\theta^q(Y-\theta) + BY(Y-\theta)^q,$$

we can conclude, for all integer $s \geq 0$, that

$$|H(\frac{P_{2s}}{Q_{2s}},1)| = |B||\theta - \frac{P_{2s}}{Q_{2s}}|. \tag{4.12}$$

On the other hand, a simple calculation gives that

$$|Q_{2s}| = \prod_{i=1}^{2s} |b_i| = |B||B|^{2\frac{q^{2s-1}-q}{(q-1)(q+1)}}|B|^{(q-2)\frac{q^{2s-1}-1}{q-1}},$$

and

$$|b_{2s+1}| = |B|^{2\frac{q^{2s-1}+1}{q+1}}|B|^{(q-2)q^{2s-1}}.$$

We can easily check that $|b_{2s+1}| = |B||Q_{2s}|^{q-1}$. This gives that

$$|\theta - \frac{P_{2s}}{Q_{2s}}| = \frac{1}{|b_{2s+1}||Q_{2s}|^2} = \frac{1}{|B||Q_{2s}|^{q+1}}. \tag{4.13}$$

So the equality (4.12) becomes

$$|H(\frac{P_{2s}}{Q_{2s}},1)| = |B|\frac{1}{|B||Q_{2s}|^{q+1}} = \frac{1}{|Q_{2s}|^{q+1}}.$$

Since

$$H(P,Q) = Q^{q+1}H(\frac{P}{Q},1)$$

we obtain that $|H(P_{2s}, Q_{2s})| = 1$ and so $H(P_{2s}, Q_{2s}) \in \mathbb{F}^*$. Since $P_0 = Q_0 = 1$ then $H(P_0, Q_0) = 1$. This gives that $H(P_{2s}, Q_{2s}) = 1$ for all $s \geq 0$. $\square$

The aim of the following corollary is to improve Baum and Sweet's theorem stated in Theorem (1.1).

COROLLARY 4.2. *Let $d \in \mathbb{N}\backslash\{0\}$ and q be a power of p. Then there exist an algebraic formal power series $\alpha \in \mathbb{F}((T^{-1}))$ of degree $q+1$ such that the equation*

$$|\alpha - \frac{P}{Q}| = \frac{e^{-d}}{|Q|^{q+1}}$$

has infinitely many solutions $(P,Q) \in \mathbb{F}[T] \times \mathbb{F}[T]$.

Proof. The proof is directly deduced from the equality (4.13). $\square$

5. CONTINUED FRACTION EXPANSION OF THE $(q-1)^{th}$ ROOT OF $1-\frac{1}{T}$

THEOREM 5.1. *Let $\mathbb{F}$ be a field of characteristic 2. Let $t \geq 2$ be an integer and $q = 2^t$. Let $\alpha \in \mathbb{F}((T^{-1}))$ be the irrational solution of the equation:*

$$x^{q-1} = 1 - \frac{1}{T}. \tag{5.14}$$

Then, the continued fraction expansion of α is:

$$[a_0, \cdots, a_n, \cdots],$$

where

$$a_0 = 1, a_1 = T+1, a_2 = T(T+1)(T^{q-4}+T^{q-6}+\ldots+T^2+1)+1, a_3 = T(T+1)$$

and for all $k \geq 2$

$$a_{2k} = T^{\frac{(q-2).q^{k-1}+1}{q-1}}(T+1)^{\frac{(q-2).q^{k-1}+1}{q-1}}\left(T^{q-4}+T^{q-6}+\ldots+T^2+1\right)^{q^{k-1}},$$

and

$$a_{2k+1} = T^{\frac{q^k-1}{q-1}}(T+1)^{\frac{q^k-1}{q-1}}.$$

Proof. We begin by computing the partial quotients of the continued fraction expansion of the solution γ of the equation:

$$x^{q-1} = 1 - \frac{1}{T^{q-1}}. \tag{5.15}$$

Put $\gamma = [a_0, \cdots, a_n, \cdots]$ and $\beta = [b_0, \cdots, b_n, \cdots]$ such that $\beta = T\gamma$. Then β satisfies the equation:

$$\beta^{q-1} = T^{q-1} - 1. \tag{5.16}$$

We have $[\beta] = b_0 = T$. Furthermore, the equation (5.16) is equivalent to

$$\beta^q = (T^{q-1}-1)\beta. \tag{5.17}$$

We know that $\beta = T + \beta_1^{-1}$. So by replacing β by $T + \beta_1^{-1}$ in the equation (5.17), we get that β_1 satisfies the following equation:

$$T\beta_1^q + (T^{q-1}+1)\beta_1^{q-1} + 1 = 0. \tag{5.18}$$

We can see that $\frac{T^{q-1}+1}{T}$ is a convergent to β_1. In fact

$$|\beta_1 - \frac{T^{q-1}+1}{T}| = \frac{1}{|\beta_1|^{q-1}}\frac{|T\beta_1^q - (T^{q-1}+1)\beta_1^{q-1}|}{|T|} = \frac{1}{|T||\beta_1|^{q-1}} < \frac{1}{|T|^2}.$$

So we obtain that $b_1 = [\beta_1] = T^{q-2}$ and $b_2 = T$. Moreover, the equation (5.18) can be written as:

$$\beta_1^q = \frac{\beta_1}{T\beta_1 + (T^{q-1}+1)}. \tag{5.19}$$

We also have, from identity (2.2) that

$$\beta_1 = \frac{(T^{q-1}+1)\beta_3 + T^{q-2}}{T\beta_3 + 1}. \tag{5.20}$$

So, combining (5.19) and (5.20) we get the following identity:

$$\beta_1^q = (T^{q-1}+1)\beta_3 + T^{q-2}. \tag{5.21}$$

Since $\beta_1 = b_1 + \beta_2^{-1}$, the identity (5.21) can be written as:

$$\frac{b_1^q + T^{q-2}}{T^{q-1}+1} + (T^{q-1}+1)^{-1}\beta_2^{-q} = \beta_3 = [b_3, \beta_4].$$

We have

$$\begin{aligned}
\frac{b_1^q + T^{q-2}}{T^{q-1}+1} &= \frac{T^{q(q-2)} + T^{q-2}}{T^{q-1}+1} = T^{q-2}\frac{T^{q^2-3q+2}+1}{T^{q-1}+1} \\
&= T^{q-2}(1 + T^{q-1} + T^{2(q-1)} + \ldots + T^{(q-1)(q-3)}).
\end{aligned}$$

So we get that

$$\begin{aligned}
b_3 &= T^{q-2}(1 + T^{q-1} + T^{2(q-1)} + \ldots + T^{(q-3)(q-1)}) \\
&= T^{q-2}(T^{q-1}+1)(T^{(q-4)(q-1)} + T^{(q-6)(q-1)} + \ldots + T^{2(q-1)} + 1),
\end{aligned}$$

and

$$\beta_4 = (T^{q-1}+1)\beta_2^q.$$

This last identity is equivalent to $b_4 + \beta_5^{-1} = (T^{q-1}+1)b_2^q + (T^{q-1}+1)\beta_3^{-q}$. Hence $b_4 = (T^{q-1}+1)T^q$ and $\beta_5 = (T^{q-1}+1)^{-1}\beta_3^q$.
We apply again the same reason and we can prove by recursion for all $k \geq 1$

$$\beta_{2k+2} = (T^{q-1}+1)\beta_{2k}^q,$$

and

$$\beta_{2k+3} = (T^{q-1}+1)^{-1}\beta_{2k+1}^q.$$

This gives for all $k \geq 1$ that

$$\begin{aligned}
b_{2k+2} &= (T^{q-1}+1)b_{2k}^q, \\
b_{2k+3} &= (T^{q-1}+1)^{-1}b_{2k+1}^q.
\end{aligned}$$

Thus for all $k \geq 1$

$$\begin{aligned}
b_{2k+1} = T^{(q-2).q^{k-1}}\left(T^{q-1}+1\right)^{\frac{(q-2).q^{k-1}+1}{q-1}}\big(T^{(q-4)(q-1)} \\
+ T^{(q-6)(q-1)} + \cdots + T^{2(q-1)} + 1\big)^{q^{k-1}},
\end{aligned}$$

$$b_{2k+2} = T^{q^k}\left(T^{q-1}+1\right)^{\frac{q^k-1}{q-1}}.$$

Now, we have $\gamma = T^{-1}\beta$. In this case we have that $b_2 = T$ then $T^{-1}b_2 = 1 \in \mathbb{F}^*$. So using Lemma 2.2 and the note that follows, we get the usual continued fraction expansion of γ satisfying (5.15) that is: $a_0 = 1$, $a_1 = T^{q-1}+1$, $a_2 = T^{q-1}(T^{q-1}+1)(T^{(q-4)(q-1)} + T^{(q-6)(q-1)} + \ldots + T^{2(q-1)} + 1) + 1$, $a_3 = T^{q-1}(T^{q-1}+1)$ and for all $k \geq 2$

$$\begin{aligned} a_{2k} = T^{(q-2).q^{k-1}+1}\left(T^{q-1}+1\right)^{\frac{(q-2).q^{k-1}+1}{q-1}} &\left(T^{(q-4)(q-1)}\right. \\ &\left. + T^{(q-6)(q-1)} + \ldots + T^{2(q-1)} + 1\right)^{q^{k-1}}, \end{aligned}$$

$$a_{2k+1} = T^{q^k-1}(T^{q-1}+1)^{\frac{q^k-1}{q-1}}.$$

Since for all $k \geq 0$: $(q-2).q^k + 1 \equiv 0 \mod (q-1)$ and $q^k - 1 \equiv 0 \mod (q-1)$, we can write:

$$\begin{aligned} a_{2k} = \left(T^{q-1}\right)^{\frac{(q-2).q^{k-1}+1}{q-1}}\left(T^{q-1}+1\right)^{\frac{(q-2).q^{k-1}+1}{q-1}} &\left(T^{(q-4)(q-1)}\right. \\ &\left. + T^{(q-6)(q-1)} + \ldots + T^{2(q-1)} + 1\right)^{q^{k-1}}, \end{aligned}$$

$$a_{2k+1} = \left(T^{q-1}\right)^{\frac{q^k-1}{q-1}}\left(T^{q-1}+1\right)^{\frac{q^k-1}{q-1}},$$

for all $k \geq 2$. Finally, as $\gamma = \alpha(T^{q-1})$ we deduce from Lemma 2.3 the desired result. $\square$

Example 5.1. We consider a field $\mathbb{F}$ of characteristic 2, and $\alpha \in \mathbb{F}((T^{-1}))$ be the irrational solution of the equation:

$$x^3 = 1 - \frac{1}{T}. \tag{5.22}$$

Then, by applying the result of the previous Theorem with $q = 4$, we get that the continued fraction expansion of α is $[a_0, \cdots, a_n, \cdots]$, with $a_0 = 1$, $a_1 = T+1$, $a_2 = T^2+T+1$ and for $n \geq 1$

$$a_{n+2} = \left(T^2+T\right)^{\frac{2^{n+1}+(-1)^n}{3}}.$$

When $\mathbb{F}$ is a field of characteristic $p > 2$ and q is a power of p, the determination of the partial quotients of the root of the equation $x^{q-1} = 1 - \dfrac{1}{T}$ is more complicated. This can none-the-less be achieved using the method used to prove Theorem 5.1. We will be satisfied with giving the beginning of the continued fraction of the $(q-1)^{th}$ root of $1 - 1/T$ in characteristic $p \neq 2$.

Theorem 5.2. *Let $\mathbb{F}$ be a field of characteristic $p > 2$. Let $t \geq 1$ be an integer and $q = p^t > 3$. Let $\alpha \in \mathbb{F}((T^{-1}))$ be the irrational solution of the* *equation:*

$$x^{q-1} = 1 - \frac{1}{T}. \tag{5.23}$$

Then, the continued fraction expansion of α begins with:

$$\alpha = [1, T-1, -T^{q-2} - T^{q-3} + \ldots - T - 1, -2^{-1}T + 2^{-1}, 2^{-5}T - 2^{-5}, \ldots].$$

Proof. We begin by computing the partial quotients of the continued fraction expansion of the solution α of the equation:

$$x^{q-1} = 1 - \frac{1}{T^{q-1}}. \tag{5.24}$$

Put $\alpha = [a_0, \cdots, a_n, \cdots]$ and $\beta = [b_0, \cdots, b_n, \cdots]$ such that $\beta = T\alpha$. Then β satisfies the equation:

$$\beta^{q-1} = T^{q-1} - 1. \tag{5.25}$$

We have $[\beta] = b_0 = T$. Furthermore, the equation (5.25) is equivalent to

$$\beta^q = (T^{q-1} - 1)\beta. \tag{5.26}$$

We know that $\beta = T + \beta_1^{-1}$. So by replacing β by $T + \beta_1^{-1}$ in the equation (5.26), we get that β_1 satisfies the following equation:

$$-T\beta_1^q + (T^{q-1} - 1)\beta_1^{q-1} - 1 = 0. \tag{5.27}$$

We can see that $\dfrac{T^{q-1} - 1}{T}$ is a convergent to β_1. In fact

$$|\beta_1 - \frac{T^{q-1} - 1}{T}| = \frac{1}{|\beta_1|^{q-1}} \frac{|T\beta_1^q - (T^{q-1} - 1)\beta_1^{q-1}|}{|T|} = \frac{1}{|T||\beta_1|^{q-1}} < \frac{1}{|T|^2}.$$

So we obtain that $b_1 = [\beta_1] = T^{q-2}$ and $b_2 = -T$. Moreover, the equation (5.27) can be written as:

$$\beta_1^q = \frac{\beta_1}{-T\beta_1 + (T^{q-1} - 1)}. \tag{5.28}$$

We also have, from identity (2.2) that

$$\beta_1 = \frac{(-T^{q-1} + 1)\beta_3 + T^{q-2}}{-T\beta_3 + 1}. \tag{5.29}$$

So, combining (5.28) and (5.29) we get the following identity:

$$\beta_1^q = (T^{q-1} - 1)\beta_3 - T^{q-2}. \tag{5.30}$$

Since $\beta_1 = b_1 + \beta_2^{-1}$, the identity (5.30) can be written as:

$$\frac{b_1^q + T^{q-2}}{T^{q-1} - 1} + (T^{q-1} - 1)^{-1}\beta_2^{-q} = \beta_3.$$

We have $T^{q^2-2q}=\big(T^{q^2-3q+1}+T^{q^2-4q+2}+\ldots+T^{q-2}\big)(T^{q-1}-1)+T^{q-2}$. So

$$\begin{aligned} b_1^q+T^{q-2} &= T^{q^2-2q}+T^{q-2} \\ &= \big(T^{q^2-3q+1}+T^{q^2-4q+2}+\ldots+T^{q-2}\big)\big(T^{q-1}-1\big)+2T^{q-2}. \end{aligned}$$

Then

$$\begin{aligned} \frac{b_1^q+T^{q-2}}{T^{q-1}-1} &= (T^{q^2-3q+1}+T^{q^2-4q+2}+\ldots+T^{q-2})+\frac{2T^{q-2}}{T^{q-1}-1} \\ &= [T^{q^2-3q+1}+T^{q^2-4q+2}+\ldots+T^{q-2},2^{-1}T,-2T^{q-2}]. \end{aligned}$$

Hence $\big[[T^{q^2-3q+1}+T^{q^2-4q+2}+\ldots+T^{q-2},2^{-1}T,-2T^{q-2}],(T^{q-1}-1)\beta_2^q\big]=\beta_3$. By Lemma 2.4 we get

$$[T^{q^2-3q+1}+T^{q^2-4q+2}+\ldots-T^{q-2},2^{-1}T,-2T^{q-2},\beta']=\beta_3,$$

with

$$\beta^{'}=\frac{\beta_2^q}{(T^{q-1}-1)}+\frac{2^{-1}T}{(T^{q-1}-1)}.$$

Since $|\beta^{'}|>1$ then $\beta^{'}=\beta_6$. So we obtain that

$$\begin{aligned} &b_3=T^{q^2-3q+1}+T^{q^2-4q+2}+\ldots+T^{q-2},\quad b_4=2^{-1}T,\quad b_5=-2T^{q-2} \\ &\beta_6=\frac{\beta_2^q}{(T^{q-1}-1)}+\frac{2^{-1}T}{(T^{q-1}-1)}. \end{aligned}$$

We apply again the same reasoning to get the following relation:

$$\beta_6=\frac{b_2^q+2^{-1}T}{(T^{q-1}-1)}+\frac{1}{(T^{q-1}-1)\beta_3^q}=\frac{-T^q+2^{-1}T}{(T^{q-1}-1)}+\frac{1}{(T^{q-1}-1)\beta_3^q}.$$

So we obtain

$$\big[[-T,-2T^{q-2},2^{-1}T],(T^{q-1}-1)\beta_3^q\big]=\beta_6.$$

Then from Lemma 2.4 we get

$$[-T,-2T^{q-2},2^{-1}T,\beta'']=\beta_6,$$

with $\beta''=\dfrac{\beta_3^q}{(T^{q-1}-1)}+\dfrac{-2T^{q-2}}{(T^{q-1}-1)}$. So we obtain that $b_6=-T$, $b_7=-2T^{q-2}$, $b_8=2^{-1}T$ and $\beta''=\beta_9$. Thus

$$\beta_9=\frac{\beta_3^q}{(T^{q-1}-1)}+\frac{-2T^{q-2}}{(T^{q-1}-1)}.$$

Note that

$$\begin{aligned} b_3 &= T^{q-2}(T^{q^2-4q+3}+T^{q^2-5q+4}+\ldots+1) \\ &= T^{q-2}(T^{(q-1)(q-3)}+T^{(q-1)(q-4)}+\ldots+T^{q-1}+1). \end{aligned}$$

We conclude that the continued fraction expansion of β begins with

$$\beta = [T, T^{q-2}, -T, T^{q-2}(T^{(q-1)(q-3)} + T^{(q-1)(q-4)} + \ldots + T^{q-1} - 1),\\ 2^{-1}T, -2T^{q-2}, -T, -2T^{q-2}, 2^{-1}T, \cdots].$$

Hence, according to Lemma 2.2 and the note that follows, the continued fraction expansion of $\alpha = \beta/T$ is:

$$\alpha = [1, T^{q-1} - 1, -T^{q-1}(T^{(q-1)(q-3)} + T^{(q-1)(q-4)} + \ldots + T^{q-1} + 1) - 1,\\ -2^{-1}T^{q-1} + 2^{-2}, 2^{-5}T^{q-1} - 2^{-5}, \cdots].$$

So from Lemma 2.3 we deduce the desired result. □

REFERENCES

[1] K. Ayadi, *On the approximation exponent of some hyperquadratic power series*. Bull. Belg. Math. Soc. Simon Stevin **22** (2015), 511–520.

[2] L. Baum and M. Sweet, *Continued fractions of algebraic power series in characteristic 2*. Ann. of Math. **103** (1976), 593–610.

[3] B. de Mathan, *Approximation exponents for algebraic functions*. Acta Arith. **60** (1992), 359–370.

[4] A. Khintchine, *Continued Fractions (in Russian)*. Gosudarst. Izdat. Tecn.-Teor. Lit, Moscow-Liningrad, 2nd edition, 1949.

[5] A. Lasjaunias, *A survey of Diophantine approximation in fields of power series*. Monatsh. Math. **130** (2000), 211–229.

[6] A. Lasjaunias, *Continued fractions for hyperquadratic power series over finite field*. Finite Fields Appl. **14** (2008), 329–350.

[7] J. Liouville, *Nouvelle démonstration d'un théorème sur les irrationelles algébriques*. C. R. Math. Acad. Sci. Paris **18** (1844), 910–911.

[8] K. Mahler, *On a theorem of Liouville in fields of positive characteristic*. Canadian J. Math. **1** (1949), 397–400.

[9] M. Mendès France, *Sur les fractions continues limitées*. Acta Arith. **23** (1973), 207–215.

[10] W. Mills and D. Robbins, *Continued fractions for certain algebraic power series*. J. Number Theory **23** (1986), 388–404.

[11] C. Osgood, *Effective bounds on the "diophantine approximation" of algebraic functions over fields of arbitrary characteristic and applications to differential equations*. Indag. Math. **37** (1975), 105–119.

[12] W. Schmidt, *On continued fractions and diphantine approximation in power series fields*. Acta Arith. **95** (2000), 139–166.

[13] D. Thakur, *Function Field Arithmetic*. World Scientific, 2004.

[14] J.F. Voloch, *Diophantine approximation in positive characteristic*. Period. Math. Hungar. **19** (1988), 217–225.

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