ON GENERALIZED BERNOULLI-BARNES POLYNOMIALS

YAMILET QUINTANA, JOSÉ L. RAMÍREZ, and VÍCTOR F. SIRVENT

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The main purpose of this paper is to introduce some generalizations of the Bernoulli-Barnes polynomials. These generalizations come from suitable modifications of the Mittag-Leffler type function linked to the generating function corresponding to the Bernoulli-Barnes polynomials. We provide several algebraic and combinatorial properties for these new classes of polynomials involving the Nørlund polynomials, Frobenius-Euler functions and Stirling numbers of second kind. Also, we deduce some connection formulae between a subclass of generalized Apostol-type Bernoulli-Barnes polynomials and the Jacobi polynomials, generalized Bernoulli polynomials, Genocchi polynomials and Apostol-Euler polynomials, respectively.

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1. INTRODUCTION

For a fixed $N \in \mathbb{N}$ and $\mathbf{a} = (a_1, \dots, a_N) \in \mathbb{C}^N \setminus \{0\}$, the *Bernoulli-Barnes polynomials* $\mathcal{B}_r(x; \mathbf{a})$ in the variable x and multi-dimensional parameter $\mathbf{a}$ can be defined by means of the generating function

$$E(z; \mathbf{a}) e^{xz} = \sum_{r=0}^{\infty} \mathcal{B}_r(x; \mathbf{a}) \frac{z^r}{r!}, \quad |z| < \min_{1 \le j \le N} \left\{ \frac{2\pi}{|a_j|} \right\}, \tag{1}$$

where

$$E(z; \mathbf{a}) := \prod_{j=1}^{N} \frac{z}{e^{a_j z} - 1}, \tag{2}$$

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is the Mittag-Leffler type function associated to the multi-dimensional parameter $\mathbf{a}$. As usual, the numbers $\mathcal{B}_r(\mathbf{a}) := \mathcal{B}_r(0;\mathbf{a})$ are called Bernoulli-Barnes numbers.

This class of polynomials was introduced by Barnes in a series of papers in which he developed the initial theory for the so-called multiple zeta and gamma functions [4, 5, 6, 7, 8].

The motivation behind Barnes' articles was to study in a comprehensive way the following functions:

- A multi-dimensional version of the Riemann ζ-function given by

$$\begin{aligned} \zeta_N(s,t;\mathbf{a}) &:= \sum_{\mathbf{m}\in\mathbb{Z}^N_{\geq 0}} \frac{1}{(t+\langle \mathbf{m},\mathbf{a}\rangle)^s} \\ &= \sum_{m_1,m_2,\dots,m_N=0}^{\infty} \frac{1}{(t+m_1a_1+\cdots+m_Na_N)^s}, \end{aligned} \tag{3}$$

where $\mathbf{a}\in\mathbb{R}^N_{>0}$, the sets $\mathbb{Z}^N_{\geq 0}$ and $\mathbb{R}^N_{>0}$ are given by

$$\begin{aligned} \mathbb{Z}^N_{\geq 0} &:= \{\mathbf{m}=(m_1,\dots,m_N)\in\mathbb{Z}^N : m_j\geq 0,\ \text{for } j=1,\dots,N\}, \\ \mathbb{R}^N_{>0} &:= \{\mathbf{a}=(a_1,\dots,a_N)\in\mathbb{R}^N : a_j> 0,\ \text{for } j=1,\dots,N\}, \end{aligned}$$

$\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbb{R}^N$, and the complex numbers t, s are such that $\mathrm{Re}(t)>0$ and $\mathrm{Re}(s)>N$, respectively.

- A multi-dimensional version of the gamma function, denoted by $\Gamma^B(t)$ and given in terms of the s-derivative of $\zeta_N(s,t;\mathbf{a})$ at $s=0$, $\partial_s\zeta_N(s,t;\mathbf{a})|_{s=0}$ (cf. [24] and the references therein).

The connection between the Barnes ζ-function and the Bernoulli-Barnes polynomials $\mathcal{B}_r(x;\mathbf{a})$ is given by the relation.

$$\zeta_N(-k,x;\mathbf{a}) = \frac{(-1)^N k!}{(N+k)!}\mathcal{B}_{N+k}(x;\mathbf{a}), \quad k\in\mathbb{N}. \tag{4}$$

Also, it is well-known that the Bernoulli-Barnes polynomials generalize other families of classical polynomials, for instance, when $N=1$, $\mathbf{a}=1$ and the variable x is real in (1), we obtain $\mathcal{B}_r(x;\mathbf{a}) = B_r(x)$, where $B_r(x)$ is the r-th Bernoulli polynomial, $r\geq 0$. When $\mathbf{a}=\mathbf{1}=(1,1,\dots,1)$ and the variable x is real in (1), we obtain $\mathcal{B}_r(x;\mathbf{a}) = B_r^{(N)}(x)$, where $B_r^{(N)}(x)$ is the r-th Nørlund polynomial [18], i.e., $B_r^{(N)}(x)$ is defined by means of the generating function

$$\left(\frac{z}{e^z-1}\right)^N e^{xz} = \sum_{r=0}^{\infty} B_r^{(N)}(x)\frac{z^r}{r!}, \quad |z|<2\pi. \tag{5}$$

Recently, the interest for studying and analyzing these polynomials has been renewed as it is shown in the interesting articles [2, 10, 12, 14, 24, 28].

In this article, we focus our attention on some light perturbations of the Mittag-Leffler type function (2) by adding new parameters in order to define some generalizations of the Bernoulli-Barnes polynomials. So, we prove that such new polynomial classes preserve some similar algebraic and combinatorial properties as the Bernoulli-Barnes polynomials and as an immediate consequence we recover many known algebraic and combinatorial properties of such polynomials. Also, several algebraic and combinatorial properties for these new classes of polynomials involving the Nørlund polynomials, Frobenius-Euler functions and Stirling numbers of second kind are provided. Furthermore, we deduce an inversion formula for the Bernoulli-Barnes polynomials which allows to connect them with certain extensions of generalized Apostol-type polynomials, Jacobi polynomials, generalized Bernoulli polynomials and Genocchi polynomials. As far as we know, the generalizations expounded here are not available in the current literature.

The article is organized as follows. Section 2 has an auxiliary character, however, it provides some (old and new) properties of the Bernoulli-Barnes polynomials, for instance, to the best of our knowledge, the inversion formula (9) and the relation (13) (see Theorem 1 below) are new. In Section 3, we define the generalized Bernoulli-Barnes polynomials and the generalized Apostol-type Bernoulli Barnes polynomials and prove several algebraic and combinatorial properties for these new classes of polynomials involving the Nørlund polynomials, Frobenius-Euler functions and Stirling numbers of second kind. This section also contains some connection formulae between a subclass of generalized Apostol-type Bernoulli-Barnes polynomials and the Jacobi polynomials, generalized Bernoulli polynomials, Genocchi polynomials and Apostol-Euler polynomials, respectively.

2. IDENTITIES INVOLVING THE BERNOULLI-BARNES POLYNOMIALS AND NUMBERS

Throughout the article, we will denote by $\left\{ {n \atop m} \right\}$ the Stirling numbers of the second kind, which are defined by

$$\frac{(e^x-1)^m}{m!} = \sum_{n=m}^{\infty} \left\{ {n \atop m} \right\} \frac{x^n}{n!}. \tag{6}$$

Throughout the article the variable x is real.

The following result summarizes some (old and new) properties of the Bernoulli-Barnes polynomials and their corresponding numbers.

THEOREM 1. *For a fixed* $N \in \mathbb{N}$ *and* $\mathbf{a} = (a_1, \ldots, a_N) \in \mathbb{C}^N \setminus \{0\}$, *let* $\{\mathcal{B}_r(x; \mathbf{a})\}_{r \geq 0}$ *be the sequence of Bernoulli-Barnes polynomials in the variable* x *and multi-dimensional parameter* $\mathbf{a}$. *Then the following identities hold.*

(a) *Summation formula. For every* $r \geq 0$,

$$\mathcal{B}_r(x; \mathbf{a}) = \sum_{k=0}^{r} \binom{r}{k} \mathcal{B}_k(\mathbf{a})\, x^{r-k}. \tag{7}$$

(b) *Differential relations (Appell polynomial sequences). For* $r, j \geq 0$ *with* $0 \leq j \leq r$, *we have*

$$[\mathcal{B}_r(x; \mathbf{a})]^{(j)} = \frac{r!}{(r-j)!} \mathcal{B}_{r-j}(x; \mathbf{a}). \tag{8}$$

(c) *Inversion formula. The Bernoulli-Barnes polynomials satisfy the following inversion formula*

$$x^n = \sum_{\ell_1 + \cdots + \ell_{N+1} = n} \binom{n}{\ell_1, \ell_2, \cdots, \ell_{N+1}} \frac{a_1^{\ell_1+1}}{\ell_1+1} \frac{a_2^{\ell_2+1}}{\ell_2+1} \cdots \frac{a_N^{\ell_N+1}}{\ell_N+1} \mathcal{B}_{\ell_{N+1}}(x; \mathbf{a}), \;\; n \geq 0. \tag{9}$$

(d) *Integral formulas.*

$$\int_{x_0}^{x_1} \mathcal{B}_r(x; \mathbf{a})\, dx = \frac{1}{r+1} \left[\mathcal{B}_{r+1}(x_1; \mathbf{a}) - \mathcal{B}_{r+1}(x_0; \mathbf{a})\right] \tag{10}$$

$$= \sum_{k=0}^{r} \frac{1}{r-k+1} \binom{r}{k} \mathcal{B}_k(\mathbf{a}) (x_1^{r-k+1} - x_0^{r-k+1}). \tag{11}$$

$$\mathcal{B}_r(x; \mathbf{a}) = r \int_0^x \mathcal{B}_{r-1}(t; \mathbf{a})\, dt + \mathcal{B}_r(\mathbf{a}). \tag{12}$$

(e) *Relation with the Stirling numbers. The Bernoulli-Barnes numbers* $\mathcal{B}_m(\mathbf{a})$ *can be expressed in terms of Stirling numbers of the second kind as follows*

$$\mathcal{B}_m(\mathbf{a}) = \sum_{i_1 + \cdots + i_n = m,\, i_\ell \geq 0} \binom{m}{i_1, \ldots, i_n} \prod_{\ell=1}^{n} \sum_{\ell=0}^{i_\ell} \frac{(-1)^\ell \ell! \left\{ {i_\ell \atop \ell} \right\}}{\ell+1} a_\ell^{i_\ell - 1}. \tag{13}$$

Proof. The summation formula (7) is an immediate consequence of the identities

$$e^{xz} \prod_{j=1}^{n} \frac{z}{e^{a_j z} - 1} = e^{xz} \left[\sum_{r=0}^{\infty} \mathcal{B}_r(\mathbf{a}) \frac{z^r}{r!}\right] = \left[\sum_{k=0}^{\infty} x^k \frac{z^k}{k!}\right] \left[\sum_{r=0}^{\infty} \mathcal{B}_r(\mathbf{a}) \frac{z^r}{r!}\right],$$

and the appropriate use of the Cauchy product series.

Using (7) the proof of differential relations (8) is straightforward, so it is omitted.

From Equation (1) we have

$$z^N e^{xz} = \prod_{j=1}^{N}(e^{a_j z}-1)\sum_{n=0}^{\infty}\mathcal{B}_n(x;\mathbf{a})\frac{z^n}{n!} = z^N\prod_{j=1}^{N}\left[\sum_{n=0}^{\infty}\frac{a_j^{n+1}}{n+1}\frac{z^n}{n!}\right]\left[\sum_{n=0}^{\infty}\mathcal{B}_n(x;\mathbf{a})\frac{z^n}{n!}\right]$$
$$= z^N \sum_{\ell_1+\cdots+\ell_N+\ell_{N+1}=n}\binom{n}{\ell_1,\ell_2,\cdots,\ell_N,\ell_{N+1}}\frac{a_1^{\ell_1+1}}{\ell_1+1}\frac{a_2^{\ell_2+1}}{\ell_2+1}\cdots\frac{a_N^{\ell_N+1}}{\ell_N+1}$$
$$\times\,\mathcal{B}_{\ell_{N+1}}(x;\mathbf{a})\frac{z^n}{n!}.$$

Comparing the coefficients on both sides, we get the inversion formula.

The integral formulas (10) and (12) are an immediate consequence of (8). And taking into account (7) the formulae (11) is obtained.

Finally, from (6) we have

$$\sum_{m=0}^{\infty}\mathcal{B}_m(\mathbf{a})\frac{z^n}{n!} = \prod_{j=1}^{n}\frac{z}{e^{a_j z}-1} = \prod_{j=1}^{n}\frac{1}{a_j}\left(\frac{a_j z}{e^{a_j z}-1}\right) = \prod_{j=1}^{n}\frac{1}{a_j}\left(\frac{\ln(1+(e^{a_j z}-1))}{e^{a_j z}-1}\right)$$
$$= \prod_{j=1}^{n}\frac{1}{a_j}\sum_{s=0}^{\infty}\frac{(-1)^s(e^{a_j z}-1)^s}{s+1} = \prod_{j=1}^{n}\frac{1}{a_j}\sum_{s=0}^{\infty}\frac{(-1)^s}{s+1}s!\sum_{\ell=s}^{\infty}\left\{{\ell \atop s}\right\}\frac{(a_j z)^\ell}{\ell!}$$
$$= \prod_{j=1}^{n}\left(\sum_{\ell=0}^{\infty}\left(\sum_{s=0}^{\ell}\frac{(-1)^s s!\left\{{\ell \atop s}\right\}}{s+1}a_j^{\ell-1}\right)\frac{z^\ell}{\ell!}\right).$$

Comparing the coefficients on both sides, we get (13). □

Remark 2.1. Note that (7) and (9) imply that the sequence $\{\mathcal{B}_r(x;\mathbf{a})\}_{r\geq 0}$ is a linearly independent set such that $\deg(\mathcal{B}_r(x;\mathbf{a})) = r$, and hence it is a basis for the linear space of polynomials in the variable x. Consequently, in this setting, it is possible to consider approaches involving operational matrix methods (see [21]).

Remark 2.2. For $N = 1$, i.e., $\mathbf{a} = a_1$, the explicit inversion formula is given by

$$x^n = \frac{1}{n+1}\sum_{\ell=0}^{n}\binom{n+1}{\ell+1}a_1^{\ell+1}\mathcal{B}_{n-\ell}(x;\mathbf{a}).$$

So, for $a_1 = 1$ we recover the familiar inversion formula for the Bernoulli polynomials [1, 18].

While for $N = 2, 3$ the explicit inversion formulas are given by:

$$x^n = \frac{1}{(n+1)(n+2)} \sum_{j=0}^{n} \sum_{\ell=0}^{j} \binom{j+2}{\ell+1} \binom{n+2}{j+2} a_1^{\ell+1} a_2^{j+1-\ell} \mathcal{B}_{n-j}(x; \mathbf{a}),$$

$$x^n = \frac{1}{(n+1)(n+2)(n+3)} \sum_{r=0}^{n} \sum_{s=0}^{r} \sum_{\ell=0}^{s} \binom{s+2}{\ell+1} \binom{r+3}{s+2} \binom{n+3}{r+3} \times a_1^{\ell+1} a_2^{s+1-\ell} a_3^{r-s+1} \mathcal{B}_{n-r}(x; \mathbf{a}).$$

3. GENERALIZED BERNOULLI-BARNES TYPE POLYNOMIALS

In this section, we will follow a similar methodology to those given in [9, 11, 13, 15, 17, 19, 20, 26] and the references therein, in order to define generalizations of the Bernoulli-Barnes polynomials. More precisely, we will make light perturbations to the Mittag-Leffler type function (2) by adding of some new parameters. In addition, we will prove that such new polynomial classes preserve some similar algebraic and combinatorial properties as the Bernoulli-Barnes polynomials.

Definition 2. For a fixed $N \in \mathbb{N}$, $\mathbf{a} = (a_1, \ldots, a_N) \in \mathbb{C}^N \setminus \{0\}$ and $b, c, d \in \mathbb{R}_{>0}$, the generalized Bernoulli-Barnes type polynomials $\mathcal{B}_r(x; \mathbf{a}, b, c, d)$ in the variable x, multi-dimensional parameter $\mathbf{a}$ and parameters b, c, d, are defined by means of the following generating function

$$E(z; \mathbf{a}, b, c) d^{xz} = \sum_{r=0}^{\infty} \mathcal{B}_r(x; \mathbf{a}, b, c, d) \frac{z^r}{r!}, \quad |z| < \min_{1 \le j \le N} \left| \frac{2\pi}{\log\left(\frac{b^{a_j}}{c}\right)} \right|, \tag{14}$$

where

$$E(z; \mathbf{a}, b, c) := \prod_{j=1}^{N} \frac{z}{b^{a_j z} - c^z}$$

is a Mittag-Leffler type function associated to the parameters $\mathbf{a}$, b and c.

Example 3. For $N = 2$, i.e., $\mathbf{a} = (a_1, a_2)$, the first three generalized Bernoulli-Barnes type polynomials are

$$\begin{aligned}
\mathcal{B}_0(x; \mathbf{a}, b, c, d) &= \frac{1}{(a_1 \ln(b) - \ln(c))(a_2 \ln(b) - \ln(c))}, \\
\mathcal{B}_1(x; \mathbf{a}, b, c, d) &= \frac{2x \ln(d) - (a_1 + a_2) \ln(b) - 2 \ln(c)}{2(a_1 \ln(b) - \ln(c))(a_2 \ln(b) - \ln(c))}, \\
\mathcal{B}_2(x; \mathbf{a}, b, c, d) &= \frac{\left(a_1^2 + 3a_1 a_2 + a_2^2\right) \ln^2(b) + (a_1 + a_2) \ln(b)(7 \ln(c) - 6x \ln(d))}{6(a_1 \ln(b) - \ln(c))(a_2 \ln(b) - \ln(c))}
\end{aligned}$$

$$+\frac{5\ln^2(c) - 12x\ln(c)\ln(d) + 6x^2\ln^2(d)}{6(a_1\ln(b) - \ln(c))(a_2\ln(b) - \ln(c))}.$$

Definition 4. For a fixed $N \in \mathbb{N}$, $\mathbf{a} = (a_1, \ldots, a_N) \in \mathbb{C}^N \setminus \{0\}$, $\lambda \in \mathbb{C}$ and $b, c, d \in \mathbb{R}_{>0}$, the generalized Apostol-type Bernoulli-Barnes polynomials (in short, Apostol-Bernoulli-Barnes polynomials) $\mathcal{B}_r(x; \lambda; \mathbf{a}, b, c, d)$ in the variable x, multi-dimensional parameter $\mathbf{a}$ and parameters λ, b, c, d, are defined by means of the generating function

$$E(z; \lambda; \mathbf{a}, b, c)d^{xz} = \sum_{r=0}^{\infty} \mathcal{B}_r(x; \lambda; \mathbf{a}, b, c, d)\frac{z^r}{r!}, \tag{15}$$

where

$$E(z; \lambda; \mathbf{a}, b, c) := \prod_{j=1}^{N} \frac{z}{\lambda b^{a_j z} - c^z}, \tag{16}$$

is a Mittag-Leffler type function associated to the parameters $\lambda, \mathbf{a}, b$ and c.

Furthermore,

$$|z| < \min_{1\le j\le N} \left| \frac{2\pi}{\ln\left(\frac{b^{a_j}}{c}\right)} \right|, \text{ when } \lambda = 1,$$

and

$$|z| < \min_{1\le j\le N} \left| \frac{\ln(\lambda)}{\ln\left(\frac{b^{a_j}}{c}\right)} \right|, \text{ when } \lambda \in \mathbb{C} \setminus \{1\}.$$

The numbers given by

$$\begin{aligned} \mathcal{B}_r(\mathbf{a}, b, c) &:= \mathcal{B}_r(0; \mathbf{a}, b, c, d), \\ \mathcal{B}_r(\lambda; \mathbf{a}, b, c) &:= \mathcal{B}_r(0; \lambda; \mathbf{a}, b, c, d), \end{aligned}$$

denote the corresponding generalized Bernoulli-Barnes numbers of parameters $\mathbf{a}, b, c$, and the generalized Apostol-type Bernoulli-Barnes numbers of parameters $\mathbf{a}, \lambda, b, c$, respectively. Note that for $N = 1 = a$, $b = e = d$ and $c = 1$ we obtain the classical Apostol-Bernoulli polynomials.

Example 5. For $N = 2$ and $\lambda \in \mathbb{C}\setminus\{1\}$, the first four generalized Apostol-type Bernoulli-Barnes polynomials are

$$\begin{aligned} \mathcal{B}_0(x; \lambda; \mathbf{a}, b, c, d) &= \mathcal{B}_1(x; \lambda; \mathbf{a}, b, c, d) = 0, \\ \mathcal{B}_2(x; \lambda; \mathbf{a}, b, c, d) &= \frac{2}{(\lambda - 1)^2}, \\ \mathcal{B}_3(x; \lambda; \mathbf{a}, b, c, d) &= \frac{-6\lambda(a_1 + a_2)\ln(b) + 12\ln(c) + 6(\lambda - 1)x\ln(d)}{(\lambda - 1)^3}. \end{aligned}$$

3.1. **Some basic combinatorial identities**

THEOREM 6. *The Apostol-Bernoulli-Barnes polynomials satisfy the following relation*

$$\mathcal{B}_r(x;\lambda;\mathbf{a},b,c,d)=\sum_{i=0}^{r}\binom{r}{i}\mathcal{B}_i(\lambda;\mathbf{a},b,c)(\ln d)^{r-i}x^{r-i}.$$

In particular, the generalized Bernoulli-Barnes type polynomials satisfy the relation

$$\mathcal{B}_r(x;\mathbf{a},b,c,d)=\sum_{i=0}^{r}\binom{r}{i}\mathcal{B}_i(\mathbf{a},b,c)(\ln d)^{r-i}x^{r-i}.$$

Proof. From Equation (15) we have

$$\begin{aligned}\sum_{r=0}^{\infty}\mathcal{B}_r(x;\lambda;\mathbf{a},b,c,d)\frac{z^r}{r!}&=\prod_{j=1}^{N}\left(\frac{z}{\lambda b^{a_j z}-c^z}\right)e^{xz\ln d}\\&=\left(\sum_{r=0}^{\infty}\mathcal{B}_r(\lambda;\mathbf{a},b,c)\frac{z^r}{r!}\right)e^{xz\ln d}\\&=\sum_{r=0}^{\infty}\left(\sum_{i=0}^{r}\binom{r}{i}\mathcal{B}_i(\lambda,\mathbf{a},b,c)(\ln d)^{r-i}x^{r-i}\right)\frac{z^r}{r!}.\end{aligned}$$

Comparing the coefficients we obtain the desired result. □

In particular, if $b=d=e$ and $\lambda=1=c$ we recover Equation (7).

THEOREM 7. *The Apostol-Bernoulli-Barnes polynomials satisfy the following relation*

$$\mathcal{B}_r(x+1;\lambda;\mathbf{a},b,c,d)=\sum_{i=0}^{r}\binom{r}{i}\mathcal{B}_i(x;\lambda;\mathbf{a},b,c,d)(\ln d)^{r-i}.$$

In particular, the Bernoulli-Barnes polynomials satisfy the relation

$$\mathcal{B}_r(x+1;\mathbf{a})=\sum_{i=0}^{r}\binom{r}{i}\mathcal{B}_i(x;\mathbf{a}).$$

Proof. From Equation (15) we have

$$\begin{aligned}\sum_{r=0}^{\infty}\mathcal{B}_r(x+1;\lambda;\mathbf{a},b,c,d)\frac{z^r}{r!}&=\prod_{j=1}^{N}\left(\frac{z}{\lambda b^{a_j z}-c^z}\right)e^{(x+1)z\ln d}\\&=\left(\sum_{r=0}^{\infty}\mathcal{B}_r(x;\lambda;\mathbf{a},b,c,d)\frac{z^r}{r!}\right)e^{z\ln d}\end{aligned}$$

$$= \sum_{r=0}^{\infty} \left(\sum_{i=0}^{r} \binom{r}{i} \mathcal{B}_i(x; \lambda; \mathbf{a}, b, c, d)(\ln d)^{r-i} \right) \frac{z^r}{r!}.$$

Comparing the coefficients, we obtain the result. □

A similar argument to the previous theorem allows to obtain the following addition formula.

THEOREM 8. *The Apostol-Bernoulli-Barnes polynomials satisfy the relation*

$$\mathcal{B}_r(x+y; \lambda; \mathbf{a}, b, c, d) = \sum_{i=0}^{r} \binom{r}{i} \mathcal{B}_i(x; \lambda; \mathbf{a_1}, b, c, d) \mathcal{B}_{r-i}(y; \lambda; \mathbf{a_2}, b, c, d),$$

where $\mathbf{a} = (a_1, \ldots, a_{N+M}), \mathbf{a_1} = (a_1, \ldots, a_N)$ *and* $\mathbf{a_2} = (a_{N+1}, \ldots, a_{N+M})$.

From Theorem 6 we obtain the following identities.

THEOREM 9. *For* $r, j \geq 0$ *with* $0 \leq j \leq r$, *we have*

$$[\mathcal{B}_r(x; \lambda; \mathbf{a}, b, c, d)]^{(j)} = \frac{r!}{(r-j)!} (\ln d)^j \mathcal{B}_{r-j}(x; \lambda; \mathbf{a}, b, c, d),$$

$$[\mathcal{B}_r(x; \mathbf{a}, b, c, d)]^{(j)} = \frac{r!}{(r-j)!} (\ln d)^j \mathcal{B}_{r-j}(x; \mathbf{a}, b, c, d).$$

In particular, for $b = d = e$ and $\lambda = 1 = c$ we obtain (8). If $d = e$ in the above theorem we obtain that $\{\mathcal{B}_r(x; \lambda; \mathbf{a}, b, c, e)\}_{r \geq 0}$ is an Appel sequence [23]. Therefore, we have the following basic relations.

THEOREM 10. *If* $r \geq 0$ *then*

(i) $\mathcal{B}_r(x+y; \lambda; \mathbf{a}, b, c, e) = \sum_{i=0}^{r} \binom{r}{i} \mathcal{B}_i(x; \lambda; \mathbf{a}, b, c, e) y^{r-i}$.

(ii) $\mathcal{B}_r(mx; \lambda; \mathbf{a}, b, c, e) = \sum_{i=0}^{r} \binom{r}{i} \mathcal{B}_i(x; \lambda; \mathbf{a}, b, c, e)(m-1)^{r-i} x^{r-i}$, $m \geq 1$.

(iii) $\mathcal{B}_r(x+1; \lambda; \mathbf{a}, b, c, e) - \mathcal{B}_r(x; \lambda; \mathbf{a}, b, c, e) = \sum_{i=0}^{r-1} \binom{r}{i} \mathcal{B}_i(x; \lambda; \mathbf{a}, b, c, e)$.

If $\lambda = 1 = c$ and $b = e$ we obtain the following basic relation for the Bernoulli-Barnes polynomials.

COROLLARY 11. *If* $r \geq 0$ *then*

1. $\mathcal{B}_r(x+y; \mathbf{a}) = \sum_{i=0}^{r} \binom{r}{i} \mathcal{B}_i(x; \mathbf{a}) y^{r-i}$.

2. $\mathcal{B}_r(mx;\mathbf{a}) = \sum_{i=0}^{r} \binom{r}{i} \mathcal{B}_i(x;\mathbf{a})(m-1)^{r-i}x^{r-i}, \; m \geq 1.$

3. $\mathcal{B}_r(x+1;\mathbf{a}) - \mathcal{B}_r(x;\mathbf{a}) = \sum_{i=0}^{r-1} \binom{r}{i} \mathcal{B}_i(x;\mathbf{a}).$

THEOREM 12. *The Apostol-Bernoulli-Barnes polynomials satisfy the relation*

$$\mathcal{B}_r(x;\lambda;\mathbf{a},b,c,d) = \sum_{i=0}^{r} \binom{r}{i} \mathcal{B}_i(\lambda;\overline{\mathbf{a}})(x\ln d - N\ln c)^{r-i}.$$

where $\overline{\mathbf{a}} = (a_1 \ln b - \ln c, \dots, a_N \ln b - \ln c)$ *and*

$$\sum_{r=0}^{\infty} \mathcal{B}_r(\lambda;\mathbf{a})\frac{z^r}{r!} = \prod_{j=1}^{N} \frac{z}{\lambda e^{a_j z} - 1}. \tag{17}$$

The polynomials $\mathcal{B}_r(\lambda;\mathbf{a})$ *are called the Apostol-Bernoulli-Barnes numbers.*

In particular,

$$\mathcal{B}_r(x;\mathbf{a},b,c,d) = \sum_{i=0}^{r} \binom{r}{i} \mathcal{B}_i(\overline{\mathbf{a}})(x\ln d - N\ln c)^{r-i}.$$

Proof. From Equation (15) we have

$$\begin{aligned}
\sum_{r=0}^{\infty} \mathcal{B}_r(x;\lambda;\mathbf{a},b,c,d)\frac{z^r}{r!} &= \prod_{j=1}^{N} \left(\frac{z}{\lambda b^{a_j z} - c^z} \right) e^{xz\ln d} \\
&= \prod_{j=1}^{N} \left(\frac{z}{e^{z\ln c}\left(e^{\ln\lambda + a_i z\ln b - z\ln c} - 1\right)} \right) e^{xz\ln d} \\
&= \prod_{j=1}^{N} \left(\frac{z}{e^{\ln\lambda + a_i z\ln b - z\ln c} - 1} \right) e^{(x\ln d - N\ln c)z} \\
&= \left(\sum_{r=0}^{\infty} \mathcal{B}_r(\overline{\mathbf{a}};\lambda)\frac{z^r}{r!} \right) e^{(x\ln d - N\ln c)z} \\
&= \sum_{r=0}^{\infty} \left(\sum_{i=0}^{r} \binom{r}{i} \mathcal{B}_i(\lambda;\overline{\mathbf{a}})(x\ln d - N\ln c)^{n-i} \right) \frac{z^r}{r!}.
\end{aligned}$$

Comparing the coefficients, we obtain the desired result. □

If $\lambda = 1 = c$ and $b = e = d$ we obtain (7).

3.2. **Relations with the Stirling numbers and some special polynomials**

We will show some relations of the Apostol-Bernoulli-Barnes polynomials with the Stirling numbers and also with certain special polynomials.

THEOREM 13. *The Apostol-Bernoulli-Barnes polynomials satisfy the relation*

$$\mathcal{B}_r(x;\lambda;\mathbf{a},b,c,d)=\sum_{\ell=0}^{\infty}\sum_{s=\ell}^{r}\binom{r}{s}\left\{ {s \atop \ell} \right\}(\ln d)^s\mathcal{B}_{r-s}(-\ell;\lambda;\mathbf{a},b,c,d)(x)^{(\ell)}, \tag{18}$$

where

$$(x)^{(m)}=x(x+1)\cdots(x+m-1)\quad (m\geq 1)\quad \textit{with}\quad (x)^{(0)}=1.$$

In particular,

$$\mathcal{B}_r(x;\mathbf{a},b,c,d)=\sum_{\ell=0}^{\infty}\sum_{s=\ell}^{r}\binom{r}{s}\left\{ {s \atop \ell} \right\}(\ln d)^s\mathcal{B}_{r-s}(-\ell;\mathbf{a},b,c,d)(x)^{(\ell)}.$$

Proof. From (15) and (6), by the binomial series

$$\frac{1}{(1-x)^c}=\sum_{n=0}^{\infty}\binom{c+n-1}{n}x^n$$

and by the relation

$$\binom{x+\ell-1}{s}=\frac{(x)^{(\ell)}}{s!}$$

we get:

$$\begin{aligned}
\sum_{r=0}^{\infty}\mathcal{B}_r(x;\lambda;\mathbf{a},b,c,d)\frac{z^r}{r!}&=\prod_{j=1}^{N}\left(\frac{z}{\lambda b^{a_jz}-c^z}\right)e^{xz\ln d}\\
&=\prod_{j=1}^{N}\left(\frac{z}{\lambda b^{a_jz}-c^z}\right)(1-(1-e^{-z\ln d}))^{-x}\\
&=\prod_{j=1}^{N}\left(\frac{z}{\lambda b^{a_jz}-c^z}\right)\sum_{\ell=0}^{\infty}\binom{x+\ell-1}{\ell}(1-e^{-z\ln d})^{\ell}\\
&=\sum_{\ell=0}^{\infty}\frac{(x)^{(\ell)}}{\ell!}(e^{z\ln d}-1)^{\ell}\prod_{j=1}^{N}\left(\frac{z}{\lambda b^{a_jz}-c^z}\right)e^{-\ell z\ln d}\\
&=\sum_{\ell=0}^{\infty}(x)^{(\ell)}\left(\sum_{r=0}^{\infty}\left\{ {r \atop \ell} \right\}(\ln d)^r\frac{z^r}{r!}\right)
\end{aligned}$$

$$\times\left(\sum_{r=0}^{\infty}\mathcal{B}_r(-\ell;\lambda,\mathbf{a},b,c,d)\frac{z^r}{r!}\right)$$
$$=\sum_{\ell=0}^{\infty}(x)^{(\ell)}\sum_{r=0}^{\infty}\left(\sum_{s=0}^{r}\binom{r}{s}\left\{{s \atop \ell}\right\}(\ln d)^s\mathcal{B}_{r-s}(-\ell;\lambda,\mathbf{a},b,c,d)\right)\frac{z^r}{r!}$$
$$=\sum_{r=0}^{\infty}\left(\sum_{\ell=0}^{\infty}\sum_{s=0}^{r}\binom{r}{s}\left\{{s \atop \ell}\right\}(\ln d)^s\mathcal{B}_{r-s}(-\ell;\lambda,\mathbf{a},b,c,d)(x)^{(\ell)}\right)\frac{z^r}{r!}.$$

Comparing the coefficients on both sides, we have (18). □

If $\lambda = 1 = c$ and $b = e = d$ we obtain the next corollary.

COROLLARY 14. *The Bernoulli-Barnes polynomials satisfy the relation*

$$\mathcal{B}_r(x;\mathbf{a}) = \sum_{\ell=0}^{\infty}\sum_{s=\ell}^{r}\binom{r}{s}\left\{{s \atop \ell}\right\}\mathcal{B}_{r-s}(-\ell;\mathbf{a})(x)^{(\ell)}, \quad r \geq 0.$$

THEOREM 15. *For $r \geq 0$, the Apostol-Bernoulli-Barnes polynomials satisfy the relation*

$$\mathcal{B}_r(x;\lambda;\mathbf{a},b,c,d) = \sum_{\ell=0}^{\infty}\sum_{i=0}^{r}\binom{r}{s}\left\{{s \atop \ell}\right\}(\ln d)^s\mathcal{B}_{r-s}(\lambda;\mathbf{a},b,c)(x)_\ell. \tag{19}$$

where

$$(x)_m = x(x-1)\cdots(x-m+1) \quad (m \geq 1) \quad \textit{with} \quad (x)_0 = 1.$$

In particular,

$$\mathcal{B}_r(x;\mathbf{a},b,c,d) = \sum_{\ell=0}^{\infty}\sum_{i=0}^{r}\binom{r}{s}\left\{{s \atop \ell}\right\}(\ln d)^i\mathcal{B}_{r-s}(\mathbf{a},b,c)(x)_\ell.$$

Proof. From (15) and (6), and by the relation

$$\binom{x}{s} = \frac{(x)_s}{s!}.$$

The following identities hold:

$$\sum_{r=0}^{\infty}\mathcal{B}_r(x;\lambda;\mathbf{a},b,c,d)\frac{z^r}{r!} = \prod_{j=1}^{N}\left(\frac{z}{\lambda b^{a_j z} - c^z}\right)e^{xz\ln d}$$
$$= \prod_{j=1}^{N}\left(\frac{z}{\lambda b^{a_j z} - c^z}\right)((e^{z\ln d} - 1) + 1)^x$$
$$= \prod_{j=1}^{N}\left(\frac{z}{\lambda b^{a_j z} - c^z}\right)\sum_{\ell=0}^{\infty}\binom{x}{\ell}(e^{z\ln d} - 1)^\ell$$

$$
\begin{aligned}
&= \sum_{\ell=0}^{\infty} \frac{(x)_\ell}{\ell!}(e^{z\ln d}-1)^\ell \prod_{j=1}^{N}\left(\frac{z}{\lambda b^{a_j z}-c^z}\right)\\
&= \sum_{\ell=0}^{\infty}(x)_\ell\left(\sum_{r=0}^{\infty}\left\{ {r \atop \ell} \right\}(\ln d)^r\frac{z^r}{r!}\right)\left(\sum_{r=0}^{\infty}\mathcal{B}_r(\lambda;\mathbf{a},b,c)\frac{z^r}{r!}\right)\\
&= \sum_{\ell=0}^{\infty}(x)_\ell\sum_{r=0}^{\infty}\left(\sum_{s=0}^{r}\binom{r}{s}\left\{ {s \atop \ell} \right\}(\ln d)^s\mathcal{B}_{r-s}(\lambda;\mathbf{a},b,c)\frac{z^r}{r!}\right)\\
&= \sum_{r=0}^{\infty}\left(\sum_{\ell=0}^{\infty}\sum_{s=0}^{r}\binom{r}{s}\left\{ {s \atop \ell} \right\}(\ln d)^s\mathcal{B}_{r-s}(\lambda;\mathbf{a},b,c)\frac{z^r}{r!}(x)_\ell\right).
\end{aligned}
$$

Comparing the coefficients on both sides, we have (19). □

If $\lambda = 1 = c$ and $b = e = d$ we obtain the following relation.

Corollary 16. *The Bernoulli-Barnes polynomials satisfy the relation*

$$
\mathcal{B}_r(x;\mathbf{a}) = \sum_{\ell=0}^{\infty}\sum_{i=0}^{r}\binom{r}{s}\left\{ {s \atop \ell} \right\}\mathcal{B}_{r-s}(\mathbf{a})(x)_\ell, \quad r \geq 0.
$$

Next, we will show a relation between the Nørlund polynomials $B_n^{(s)}(x)$ and the Apostol-Bernoulli-Barnes polynomials.

Theorem 17. *For any positive integer s, the Apostol-Bernoulli-Barnes polynomials satisfy the relation:*

(20)

$$
\mathcal{B}_r(x;\lambda;\mathbf{a},b,c,d) = \sum_{\ell=0}^{r}\binom{r}{\ell}\left\{ {\ell+s \atop s} \right\}\sum_{i=0}^{r-\ell}\frac{\binom{r-\ell}{i}}{\binom{\ell+s}{s}}(\ln d)^{\ell+i}B_i^{(s)}(x)\mathcal{B}_{r-\ell-i}(\lambda;\mathbf{a},b,c).
$$

In particular,

$$
\mathcal{B}_r(x;\mathbf{a},b,c,d) = \sum_{\ell=0}^{r}\binom{r}{\ell}\left\{ {\ell+s \atop s} \right\}\sum_{i=0}^{r-\ell}\frac{\binom{r-\ell}{i}}{\binom{\ell+s}{s}}(\ln d)^{\ell+i}B_i^{(s)}(x)\mathcal{B}_{r-\ell-i}(\mathbf{a},b,c).
$$

Proof. From (15) and (5) we get

$$
\begin{aligned}
\sum_{r=0}^{\infty}\mathcal{B}_r(x;\lambda;\mathbf{a},b,c,d)\frac{z^r}{r!} &= \prod_{j=1}^{N}\left(\frac{z}{\lambda b^{a_j z}-c^z}\right)e^{xz\ln d}\\
&= \frac{(e^{z\ln d}-1)^s}{s!}\left(\frac{z\ln d}{e^{z\ln d}-1}\right)^s e^{xz\ln d}\left(\sum_{r=0}^{\infty}\mathcal{B}_r(\lambda;\mathbf{a},b,c)\frac{z^r}{r!}\right)\frac{s!}{(z\ln d)^s}
\end{aligned}
$$

$$= \left(\sum_{r=0}^{\infty} \left\{ {r+s \atop s} \right\} \frac{(z\ln d)^{r+s}}{(r+s)!}\right) \left(\sum_{r=0}^{\infty} B_r^{(s)}(x) \frac{(z\ln d)^r}{r!}\right)$$
$$\times \left(\sum_{r=0}^{\infty} \mathcal{B}_r(\lambda;\mathbf{a},b,c)\frac{z^r}{r!}\right) \frac{s!}{(z\ln d)^s}$$
$$= \left(\sum_{r=0}^{\infty} \left\{ {r+s \atop s} \right\} \frac{(z\ln d)^{r+s}}{(r+s)!}\right)$$
$$\times \left(\sum_{r=0}^{\infty} \left(\sum_{i=0}^{r} \binom{r}{i} (\ln d)^i B_i^{(s)}(x) \mathcal{B}_{r-i}(\lambda;\mathbf{a},b,c)\right) \frac{z^r}{r!}\right) \frac{s!}{(z\ln d)^s}$$
$$= \sum_{r=0}^{\infty} \left(\sum_{\ell=0}^{r} \left\{ {\ell+s \atop s} \right\} \frac{(\ln d)^{\ell+s} z^{\ell+s}}{(\ell+s)!}\right.$$
$$\left. \times \sum_{i=0}^{r-\ell} \binom{r-\ell}{i} (\ln d)^i B_i^{(s)}(x) \mathcal{B}_{r-\ell-i}(\lambda;\mathbf{a},b,c) \frac{z^{r-\ell}}{(r-\ell)!}\right) \frac{s!}{(z\ln d)^s}$$
$$= \sum_{r=0}^{\infty} \left(\sum_{\ell=0}^{r} \binom{r}{\ell} \left\{ {\ell+s \atop s} \right\} \sum_{i=0}^{r-\ell} \frac{\binom{r-\ell}{i}}{\binom{\ell+s}{s}} (\ln d)^{\ell+i} B_i^{(s)}(x) \mathcal{B}_{r-\ell-i}(\lambda;\mathbf{a},b,c)\right) \frac{z^r}{r!}.$$

Comparing the coefficients on both sides, we have (20). □

If $\lambda = 1 = c$ and $b = e = d$ we obtain the following relation.

Corollary 18. *The Bernoulli-Barnes polynomials satisfy the relation*

$$\mathcal{B}_r(x;\mathbf{a}) = \sum_{\ell=0}^{r} \binom{r}{\ell} \left\{ {\ell+s \atop s} \right\} \sum_{i=0}^{r-\ell} \frac{\binom{r-\ell}{i}}{\binom{\ell+s}{s}} B_i^{(s)}(x) \mathcal{B}_{r-\ell-i}(\mathbf{a}).$$

The Frobenius-Euler functions $H_n^{(s)}(x;u)$ are defined by the generating function (cf. [25]):

$$\left(\frac{1-u}{e^t-u}\right)^s e^{xz} = \sum_{n=0}^{\infty} H_n^{(s)}(x;u)\frac{z^n}{n!}. \tag{21}$$

Theorem 19. *The Apostol-Bernoulli-Barnes polynomials satisfy the relation*

$$\mathcal{B}_r(x;\lambda;\mathbf{a},b,c,d) = \frac{1}{(1-u)^s} \sum_{\ell=0}^{r} \binom{r}{\ell} \sum_{i=0}^{s} \binom{s}{i} (-u)^{s-i} H_\ell^{(s)}(x;u) \times (\ln d)^\ell \mathcal{B}_{r-\ell}(i;\lambda;\mathbf{a},b,c,d). \tag{22}$$

In particular,

$$\mathcal{B}_r(x;\mathbf{a},b,c,d) = \frac{1}{(1-u)^s}\sum_{\ell=0}^{r}\binom{r}{\ell}\sum_{i=0}^{s}\binom{s}{i}(-u)^{s-i}H_\ell^{(s)}(x;u) \times (\ln d)^\ell \mathcal{B}_{r-\ell}(i;\mathbf{a},b,c,d).$$

Proof. From (15) and (21) we get

$$\sum_{r=0}^{\infty}\mathcal{B}_r(x;\lambda;\mathbf{a},b,c,d)\frac{z^r}{r!} = \prod_{j=1}^{N}\left(\frac{z}{\lambda b^{a_j z}-c^z}\right)e^{xz\ln d}$$

$$= \frac{(1-u)^s}{(e^{z\ln d}-u)^s}e^{xz\ln d}\frac{(e^{z\ln d}-u)^s}{(1-u)^s}\prod_{j=1}^{N}\left(\frac{z}{\lambda b^{a_j z}-c^z}\right)$$

$$= \frac{1}{(1-u)^s}\left(\sum_{r=0}^{\infty}H_r^{(s)}(x;u)\frac{(z\ln d)^r}{r!}\right)\sum_{i=0}^{s}\binom{s}{i}e^{iz\ln d}(-u)^{s-i}\prod_{j=1}^{N}\left(\frac{z}{\lambda b^{a_j z}-c^z}\right)$$

$$= \frac{1}{(1-u)^s}\left(\sum_{r=0}^{\infty}H_r^{(s)}(x;u)(\ln d)^r\frac{z^r}{r!}\right)\sum_{i=0}^{s}\binom{s}{i}(-u)^{s-i}\sum_{r=0}^{\infty}\mathcal{B}_r(i;\lambda;\mathbf{a},b,c,d)\frac{z^r}{r!}$$

$$= \frac{1}{(1-u)^s}\sum_{i=0}^{s}\binom{s}{i}(-u)^{s-i}\sum_{r=0}^{\infty}\left(\sum_{\ell=0}^{r}\binom{r}{\ell}H_\ell^{(s)}(x;u) \times (\ln d)^\ell\mathcal{B}_{r-\ell}(i;\lambda;\mathbf{a},b,c,d)\right)\frac{z^r}{r!}$$

$$= \sum_{r=0}^{\infty}\left(\frac{1}{(1-u)^s}\sum_{\ell=0}^{r}\binom{r}{\ell}\sum_{i=0}^{s}\binom{s}{i}(-u)^{s-i}H_\ell^{(s)}(x;u) \times (\ln d)^\ell\mathcal{B}_{r-\ell}(i;\lambda;\mathbf{a},b,c,d)\right)\frac{z^r}{r!}.$$

Comparing the coefficients on both sides, we have (22).

□

If $\lambda = 1 = c$ and $b = e = d$ we obtain the following relation.

COROLLARY 20. *The Bernoulli-Barnes polynomials satisfy the relation*

$$\mathcal{B}_r(x;\mathbf{a}) = \frac{1}{(1-u)^s}\sum_{\ell=0}^{r}\binom{r}{\ell}\sum_{i=0}^{s}\binom{s}{i}(-u)^{s-i}H_\ell^{(s)}(x;u)\mathcal{B}_{r-\ell}(i;\mathbf{a}).$$

It is worthwhile to mention that the above results strongly depend on the use of the umbral calculus derived from exponential generating functions (cf. [23]).

3.3. The Barnes-type ζ-function with parameters

The connection between the Barnes ζ-function and the Bernoulli-Barnes polynomials $\mathcal{B}_r(x;\mathbf{a})$ can be generalized by using the following ζ-function. The Barnes ζ-function with parameters $\overline{\mathbf{a}}, b, c, d$ is defined by (cf. [3])

$$\zeta_N(s,t;\overline{\mathbf{a}},b,c,d) := \sum_{m_1,m_2,\ldots,m_N=0}^{\infty} \frac{1}{(t\ln d + N\ln c + m_1\overline{a}_1 + \cdots + m_N\overline{a}_N)^s},$$

where $\overline{a}_i = a_i \ln b + \ln c$. It is clear that if $b = e = d$ and $c = 1$ we recover the Barnes ζ-function $\zeta_N(s,t;\mathbf{a})$. Moreover, the following relation is clear.

$$\zeta_N(s,t;\overline{\mathbf{a}},b,c,d) = \zeta_N(s, t\ln d + N\ln c;\overline{\mathbf{a}}).$$

Since

$$\mathcal{B}_r(x\ln d + \ln c;\overline{\mathbf{a}}) = \mathcal{B}_r(x;\mathbf{a},b,c,d)$$

then we have the following relation

$$\frac{(-1)^N k!}{(N+k)!}\mathcal{B}_{N+k}(x;\mathbf{a},b,c,d) = \zeta_N(-k, x\ln d + N\ln c;\overline{\mathbf{a}}) = \zeta_N(-k,x;\overline{\mathbf{a}},b,c,d).$$

Additionally, the Barnes ζ-function with parameters $\overline{\mathbf{a}}, b, c, d$ has the following integral representation:

$$\zeta_N(s,t;\overline{\mathbf{a}},b,c,d) = \frac{1}{\Gamma(s)}\int_0^{\infty} z^{s-1}\frac{d^{-tz}}{(c^{-z}-b^{-a_1 z})(c^{-z}-b^{-a_2 z})\cdots(c^{-z}-b^{-a_N z})}\mathrm{d}z.$$

For $b = d = e$ and $c = 1$ we obtain the integral representation given in [1, pp. 210]. Note that the above integral representation is a kind of Laplace-Mellin transform.

In Figure 1, we show the phase plot for complex functions (cf. [27]) of the classical Zeta function (left) and of the particular Barnes ζ-function

$$\zeta_2(s,1;(1,1)) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{(1+i+j)^s}.$$

3.4. Some connection formulas for the generalized Apostol-type Bernoulli-Barnes polynomials

From Theorem 10, it is possible to deduce some interesting algebraic relations connecting the generalized Apostol-type Bernoulli-Barnes polynomials with other families of polynomials such as Jacobi polynomials, generalized Bernoulli polynomials of level m, Genocchi polynomials and Apostol-Euler polynomials.

Recall the connection between the n-th monomial x^n and the following families of polynomials:

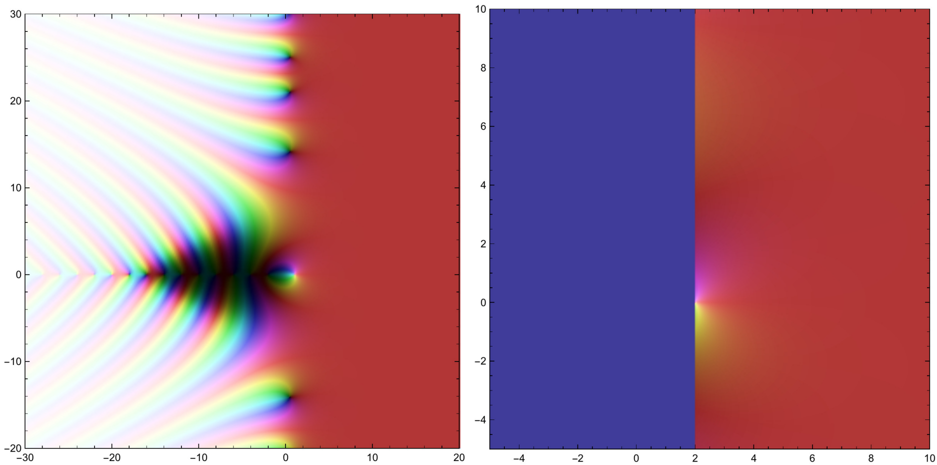

Figure 1 – Phase plot for Zeta functions ζ and $\zeta_2(s,1;(1,1))$ (Re(s)> 2).

- Jacobi polynomials [22, Equation (2), p. 262]:

$$x^n = n! \sum_{k=0}^{n} \binom{n+\kappa}{n-k} (-1)^k \frac{(1+\kappa+\beta+2k)}{(1+\kappa+\beta+k)_{n+1}} P_k^{(\kappa,\beta)}(1-2x),$$

where $P_n^{(\kappa,\beta)}(x)$ are de Jacobi polynomials defined by

$$P_n^{(\kappa,\beta)}(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{\kappa+n}{k} \binom{\beta+n}{n-k} (x-1)^{n-k}(x+1)^k,$$

with $\kappa, \beta > -1$.

- Generalized Bernoulli polynomials of level m [17, Equation (2.6)]:

$$x^n = \sum_{k=0}^{n} \binom{n}{k} \frac{k!}{(k+m)!} B_{n-k}^{[m-1]}(x),$$

where $B_n^{[m-1]}(x)$ are defined by the generating function

$$\sum_{n=0}^{\infty} B_n^{[m-1]}(x) \frac{z^n}{n!} = \frac{z^m e^{xz}}{e^z - \sum_{i=0}^{m-1} \frac{z^i}{i!}}.$$

- Genocchi polynomials:

$$x^n = \frac{1}{2(n+1)} \left[\sum_{k=0}^{n+1} \binom{n+1}{k} G_k(x) + G_{n+1}(x) \right],$$

where $G_n(x)$ are defined by the generating function

$$\sum_{n=0}^{\infty} G_n(x)\frac{z^n}{n!} = \frac{2ze^{xz}}{e^z+1}.$$

- Apostol-Euler polynomials [16, Equation (32)]:

$$x^n = \frac{1}{2}\left[\lambda \sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k(x;\lambda) + \mathcal{E}_n(x;\lambda)\right],$$

where $\mathcal{E}_n(x;\lambda)$ are defined by the generating function

$$\sum_{n=0}^{\infty} \mathcal{E}_n(x;\lambda)\frac{z^n}{n!} = \frac{2e^{xz}}{\lambda e^z+1}.$$

THEOREM 21. *The Apostol-Bernoulli-Barnes polynomials are related with the Jacobi polynomials, generalized Bernoulli polynomials of level m, Genocchi polynomials and Apostol-Euler polynomials by means of the identities:*

(a) $\mathcal{B}_r(x;\lambda;\mathbf{a},b,c,e)$

$$= \sum_{i=0}^{r}\sum_{k=0}^{r-i}\frac{r!}{i!}\binom{r-i+k}{r-i-k}(-1)^k\frac{(1+\kappa+\beta+2k)}{(1+\kappa+\beta+k)_{r-i+1}} \times \mathcal{B}_i(x;\lambda;\mathbf{a},b,c,e)P_k^{(\kappa,\beta)}(1-2x).$$

(b) $$\mathcal{B}_r(x;\lambda;\mathbf{a},b,c,e) = \sum_{i=0}^{r}\sum_{k=0}^{r-i}\binom{r}{k}\binom{r-k}{r-i-k}\frac{k!}{(k+m)!} \times \mathcal{B}_i(x;\lambda;\mathbf{a},b,c,e)B_{r-i-k}^{[m-1]}(x).$$

(c) $$\mathcal{B}_r(x;\lambda;\mathbf{a},b,c,e) = \sum_{i=0}^{r}\frac{\mathcal{B}_i(x;\lambda;\mathbf{a},b,c,e)}{2} \times \left[\sum_{k=0}^{r-i+1}\binom{r}{k}\binom{r-i}{r-i-k}\frac{G_k(y)}{r-i+1-k} + \binom{r}{i}G_{r-i+1}(y)\right].$$

(d) $$\mathcal{B}_r(x;\lambda;\mathbf{a},b,c,e) = \frac{1}{2}\sum_{i=0}^{r}\mathcal{B}_i(x;\lambda;\mathbf{a},b,c,e) \times \left[\lambda\sum_{k=0}^{r-i}\binom{r}{k}\binom{r-k}{r-k-i}\mathcal{E}_k(x;\lambda) + \binom{r}{i}\mathcal{E}_{r-i}(x;\lambda)\right].$$

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Yamilet Quintana
Universidad Simón Bolívar,
Departamento de Matemáticas,
Apartado 89000, Caracas 1080-A, Venezuela
and
Universidad Carlos III de Madrid,
Departamento de Matemáticas,
Avenida de la Universidad 30,
28911 Leganés, Madrid, Spain
yquintana@usb.ve, yaquinta@math.uc3m.es

José L. Ramírez
Universidad Nacional de Colombia,
Departamento de Matemáticas,
Bogotá, Colombia
jlramirezr@unal.edu.co

Víctor F. Sirvent
Universidad Católica del Norte,
Departamento de Matemáticas,
Casilla 1280, Antofagasta, Chile
victor.sirvent@ucn.cl