CLASSIFICATION OF ROTA-BAXTER OPERATORS ON THE 3-LIE ALGEBRA A_4

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Classification of Rota-Baxter operators of weight zero on the simple 3-Lie algebra A_4 is given in this paper. As one application, from these Rota-Baxter operators, we give the induced skew-symmetric solutions of the 3-Lie classical Yang-Baxter equation in the semi-direct product 3-Lie algebra $A_4 \ltimes_{\mathrm{ad}^*} A_4^*$ and hence the corresponding (8-dimensional) local cocycle 3-Lie bialgebras. We also give the induced 3-pre-Lie algebras and their sub-adjacent 3-Lie algebras as another application.

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1. INTRODUCTION

Rota-Baxter operators on associative algebras originated from Baxter's probability study [3] and were populated by the work of Atkinson, Cartier and Rota [2, 10, 22, 23]. This concept is applied widely in mathematics and physics such as combinatorics, number theory, operads, boundary value problems, Hopf algebras and quantum field theory [16, 17, 4, 1, 14, 11, 13]. In particular, Rota-Baxter operators on Lie algebras have been widely studied. Semenov-Tian-Shansky's work [24] shows that a Rota-Baxter operator of weight zero on a Lie algebra is exactly the operator form of classical Yang-Baxter equation which was regarded as a "classical limit" of the quantum Yang-Baxter equation [9]. Moreover, Rota-Baxter operators have close relationship with Lie bialgebras and pre-Lie algebras [18, 21]. One can use coboundary theory to construct Lie bialgebras by solutions of classical Yang-Baxter equation [12], and pre-Lie algebras can be regarded as the algebraic structure behind both Rota-Baxter operators and the classical Yang-Baxter equation in Lie algebras [5].

Rota-Baxter operators on 3-Lie algebras [7] are also important both in theory and application. Rota-Baxter operators of weight zero on 3-Lie algebras are $\mathcal{O}$-operators on 3-Lie algebras associated to the adjoint representations

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and have applications in the solutions of 3-Lie classical Yang-Baxter equation, local cocycle 3-Lie bialgebras and 3-pre-Lie algebras [6]. Therefore, explicit classification of Rota-Baxter operators of weight zero on 3-Lie algebras is necessary since many applications in the related fields depend strongly on the explicit expression. On the one hand, a Rota-Baxter operator of weight zero gives a skew-symmetric solution of 3-Lie classical Yang-Baxter equation in the semi-direct product 3-Lie algebra $A \ltimes_{\mathrm{ad}^*} A^*$ from the dual representation of the adjoint representation of 3-Lie algebra A, whereas the latter gives a local cocycle 3-Lie bialgebra. Hence from the classification results, the induced skew-symmetric solutions of 3-Lie classical Yang-Baxter equation in the semi-direct product 3-Lie algebra and the corresponding local cocycle 3-Lie bialgebras can be obtained. On the other hand, a Rota-Baxter operator of weight zero can induce a 3-pre-Lie algebra. Hence from the classification results, the induced 3-pre-Lie algebras as well as their sub-adjacent 3-Lie algebras can be obtained.

Unfortunately, it is not easy to get Rota-Baxter operators of weight zero on arbitrary 3-Lie algebras. Thus, in this paper, we will focus on Rota-Baxter operators of weight zero on the 4-dimensional simple complex 3-Lie algebra A_4 and give its classification. By [20] and [8], A_4 is the unique finite dimensional simple 3-Lie algebra over the complex field, and the multiplication in a basis $\{e_1, e_2, e_3, e_4\}$ is

$$(1) \quad [e_1, e_2, e_3] = e_3, \ \ [e_1, e_2, e_4] = -e_4, \ \ [e_1, e_3, e_4] = e_1, \ \ [e_2, e_3, e_4] = -e_2.$$

As one application, from these Rota-Baxter operators of weight zero, we give the induced skew-symmetric solutions of the 3-Lie classical Yang-Baxter equation in the semi-direct product 3-Lie algebra $A_4 \ltimes_{\mathrm{ad}^*} A_4^*$ and hence the corresponding (8-dimensional) local cocycle 3-Lie bialgebras. We also give the induced 3-pre-Lie algebras and their sub-adjacent 3-Lie algebras as another application.

In Section 2, we give some basic notions. In Section 3, we discuss the classification of Rota-Baxter operators of weight zero on A_4. In Section 4, we give the induced skew-symmetric solutions of 3-Lie classical Yang-Baxter equation in the semi-direct product 3-Lie algebra $A_4 \ltimes_{\mathrm{ad}^*} A_4^*$ and 8-dimensional local cocycle 3-Lie bialgebras by means of Rota-Baxter operators obtained in Section 3. In Section 5, we give the induced 3-pre-Lie algebras and the sub-adjacent 3-Lie algebras.

From now on, algebras and vector spaces are over the field $\mathbb{C}$ of complex numbers, and $\mathbb{Z}$ denotes the set of integers.

2. PRELIMINARY

A 3-*Lie algebra* [15] is a vector space A over $\mathbb{C}$ endowed with a 3-ary multilinear skew-symmetric multiplication $[,,]$ satisfying the following identity, for all $x_1, x_2, x_3, y_2, y_3 \in A$,

$$[[x_1, x_2, x_3], y_2, y_3] = [[x_1, y_2, y_3], x_2, x_3]+[x_1, [x_2, y_2, y_3], x_3]+[x_1, x_2, [x_3, y_2, y_3]].$$

A *Rota-Baxter 3-Lie algebra* (A, P) *of weight* λ ([7]) is a 3-Lie algebra A and a $\mathbb{C}$-linear map P: $A \to A$ satisfying, for all $x_1, x_2, x_3 \in A$,

$$\begin{aligned}[P(x_1), P(x_2), P(x_3)] = P([P(x_1), P(x_2), x_3] &+ [P(x_1), x_2, P(x_3)] \\ &+ [x_1, P(x_2), P(x_3)] + \lambda[P(x_1), x_2, x_3] + \lambda[x_1, P(x_2), x_3] \\ &+ \lambda[x_1, x_2, P(x_3)] + \lambda^2[x_1, x_2, x_3]),\end{aligned}$$

where $\lambda \in \mathbb{C}$, and P is called a Rota-Baxter operator of weight λ on the 3-Lie algebra A.

If $\lambda = 0$, then for all $x_1, x_2, x_3 \in A$,

$$\begin{aligned}[P(x_1), P(x_2), P(x_3)] = P([P(x_1), P(x_2), x_3] \\ + [P(x_1), x_2, P(x_3)] + [x_1, P(x_2), P(x_3)]).\end{aligned} \tag{2}$$

The notion of a representation of an n-Lie algebra was introduced in [19]. See [8] for more information. Let A be a 3-Lie algebra, V be a vector space and $\rho : \wedge^2 A \to \mathfrak{gl}(V)$ be a linear map. The pair (V, ρ) is called a *representation* (or A-module) of A on V if ρ satisfies, for all $x_1, x_2, x_3, y_1, y_2 \in A$,

$$[\rho(x_1, x_2), \rho(y_1, y_2)] = \rho([x_1, x_2, y_1], y_2) + \rho(y_1, [x_1, x_2, y_2]),$$

$$\rho([x_1, x_2, x_3], y_1) = \rho(x_2, x_3)\rho(x_1, y_1) - \rho(x_1, x_3)\rho(x_2, y_1) + \rho(x_1, x_2)\rho(x_3, y_1).$$

If (V, ρ) is a representation of the 3-Lie algebra A, V^* is the dual space of V and for all $f \in V^*, x \in V$, $\langle f, x\rangle = f(x) \in \mathbb{C}$. Then (V^*, ρ^*) is also a representation of A, which is called the *dual representation* of (V, ρ), where $\rho^* : \wedge^2 A \to \mathfrak{gl}(V^*)$,

$$\langle \rho^*(x_1, x_2)f, x_3\rangle = -\langle f, \rho(x_1, x_2)x_3\rangle, \quad \forall\, x_1, x_2, x_3 \in A,\ f \in V^*.$$

A *1-cocyle* [6] on a 3-Lie algebra A associated to a representation (V, ρ) is a linear map $g : A \to V$ satisfying

$$g([x_1, x_2, x_3])=\rho(x_1, x_2)g(x_3)+\rho(x_2, x_3)g(x_1)+\rho(x_3, x_1)g(x_2),\ \forall\, x_1, x_2, x_3 \in A.$$

For a 3-Lie algebra A, the *adjoint representation* (A, ad) is defined by

$$\mathrm{ad} : \wedge^2 A \to \mathfrak{gl}(A),\ \ \mathrm{ad}_{x_1,x_2}x_3 = [x_1, x_2, x_3], \quad \forall\, x_1, x_2, x_3 \in A.$$

Then we obtain the dual representation (A, ad^*), where $\mathrm{ad}^* : \wedge^2 A \to \mathfrak{gl}(A^*)$,

$$\langle \mathrm{ad}^*_{x_1,x_2}f, x_3\rangle = -\langle f, \mathrm{ad}_{x_1,x_2}x_3\rangle = -\langle f, [x_1, x_2, x_3]\rangle, \quad \forall\, x_1, x_2, x_3 \in A,\ f \in A^*.$$

LEMMA 2.1. [8] *Let A be a 3-Lie algebra over $\mathbb{C}$, V be a vector space and $\rho : \wedge^2 A \to \mathfrak{gl}(V)$ be a $\mathbb{C}$-linear map. Then (V, ρ) is a representation of A if and only if $(A \oplus V, [, ,]_*)$ is a 3-Lie algebra, where for all $x_i \in A,\ v_i \in V,\ 1 \leq i \leq 3$,*

$$[x_1 + v_1, x_2 + v_2, x_3 + v_3]_* = [x_1, x_2, x_3] + \rho(x_1, x_2)v_3 + \rho(x_3, x_1)v_2 + \rho(x_2, x_3)v_1.$$

The 3-Lie algebra $(A \oplus V, [, ,]_*)$ is called the semi-direct product 3-Lie algebra, and is denoted by $A \ltimes_\rho V$.

lt is straightforward to obtain $(A \oplus A^*, [, ,]_*) = A \ltimes_{\mathrm{ad}^*} A^*$ is a 3-Lie algebra, for all $x_i \in A,\ y_i^* \in A^*,\ 1 \leq i \leq 3$,

(3) $$[x_1 + y_1^*, x_2 + y_2^*, x_3 + y_3^*]_* = [x_1, x_2, x_3] + \mathrm{ad}^*_{x_1,x_2} y_3^* + \mathrm{ad}^*_{x_3,x_1} y_2^* + \mathrm{ad}^*_{x_2,x_3} y_1^*.$$

A 3-pre-Lie algebra [6] is a vector space A over $\mathbb{C}$ endowed with a $\mathbb{C}$-linear multiplication $\{, , \} : A^{\otimes 3} \to A$ satisfying, for all $x_i \in A,\ 1 \leq i \leq 5$,

$$\begin{aligned}
\{x_1, x_2, x_3\} &= -\{x_2, x_1, x_3\},\\
\{x_1, x_2, \{x_3, x_4, x_5\}\} &= \{[x_1, x_2, x_3]_C, x_4, x_5\} + \{x_3, [x_1, x_2, x_4]_C, x_5\}\\
&\quad + \{x_3, x_4, \{x_1, x_2, x_5\}\},\\
\{[x_1, x_2, x_3]_C, x_4, x_5\} &= \{x_1, x_2, \{x_3, x_4, x_5\}\} + \{x_2, x_3, \{x_1, x_4, x_5\}\}\\
&\quad + \{x_3, x_1, \{x_2, x_4, x_5\}\},
\end{aligned}$$

where

(4) $$[x_1, x_2, x_3]_C = \{x_1, x_2, x_3\} + \{x_2, x_3, x_1\} + \{x_3, x_2, x_1\}.$$

Thanks to Proposition 3.21 in [6], the pair $(A, [, ,]_C)$ is a 3-Lie algebra, which is called the *sub-adjacent 3-Lie algebra* of the 3-pre-Lie algebra $(A, \{, , \})$.

Let $\wedge^3 A$ denote the 3-th exterior power of 3-Lie algebra A. For all $x_i, x_j, x_k \in A$,

$$x_i \wedge x_j \wedge x_k = \sum_{\sigma \in S_3} sgn(\sigma) x_{\sigma(i)} \otimes x_{\sigma(j)} \otimes x_{\sigma(k)},$$

and if $(i_1, i_2, i_3) = (i, j, k)$, then $x_{\sigma(i)} \otimes x_{\sigma(j)} \otimes x_{\sigma(k)} = x_{i_{\sigma(1)}} \otimes x_{i_{\sigma(2)}} \otimes x_{i_{\sigma(3)}}$.

Let A be a 3-Lie algebra, A^* be the dual space of A, and $\Delta : A \to \wedge^3 A$ be a linear map. The dual map of Δ is a linear map $\Delta^* : \wedge^3 A^* \to A^*$ defined by

(5) $$\langle \Delta^*(v_1, v_2, v_3), x\rangle = \langle v_1 \wedge v_2 \wedge v_3, \Delta(x)\rangle, \quad \forall\ v_1, v_2, v_3 \in A^*,\ x \in A.$$

A local cocycle 3-Lie bialgebra over $\mathbb{C}$ [6] is a pair (A, Δ), where A is a 3-Lie algebra, $\Delta = \Delta_1 + \Delta_2 + \Delta_3 : A \to \wedge^3 A$ is a $\mathbb{C}$-linear map and the following conditions are satisfied:

- (A^*, Δ^*) is a 3-Lie algebra;

- Δ_1 is a 1-cocycle associated to the representation $(A\otimes A\otimes A, \mathrm{ad}\otimes 1\otimes 1)$;
- Δ_2 is a 1-cocycle associated to the representation $(A\otimes A\otimes A, 1\otimes \mathrm{ad}\otimes 1)$;
- Δ_3 is a 1-cocycle associated to the representation $(A\otimes A\otimes A, 1\otimes 1\otimes \mathrm{ad})$.

The map $\phi_{pq}:\ A^{\otimes m}\to A^{\otimes m}$ defined by, for all $\sum x_1\otimes x_2\otimes\cdots\otimes x_p\otimes\cdots\otimes x_q\otimes\cdots\otimes x_m\in A^{\otimes m}$,

$$\begin{aligned}\phi_{pq}(\sum x_1\otimes\cdots\otimes x_p\otimes\cdots\otimes x_q\otimes\cdots\otimes x_m) = \\ \sum x_1\otimes\cdots\otimes x_q\otimes\cdots\otimes x_p\otimes\cdots\otimes x_m,\end{aligned}\tag{6}$$

therefore, ϕ_{pq} exchanges the position of x_p and x_q.

For any $r=\sum_i x_i\otimes y_i\in A\otimes A$, set $\Delta_1,\Delta_2,\Delta_3: A\to\wedge^3 A$, for all $x\in A$,

$$\begin{aligned}\Delta_1(x)=\sum_{i,j}[x,x_i,x_j]\otimes y_j\otimes y_i;\ \Delta_2(x)=\sum_{i,j} y_i\otimes[x,x_i,x_j]\otimes y_j;\Delta_3(x)\\ =\sum_{i,j} y_j\otimes y_i\otimes[x,x_i,x_j].\end{aligned}\tag{7}$$

Thanks to (6), $\Delta_2(x)=\phi_{13}\phi_{12}\Delta_1(x)$, $\Delta_3(x)=\phi_{12}\phi_{13}\Delta_1(x)$. Let $\Delta=\Delta_1+\Delta_2+\Delta_3$, then

$$\Delta(x)=\Delta_1(x)+\phi_{13}\phi_{12}\Delta_1(x)+\phi_{12}\phi_{13}\Delta_1(x), \forall x\in A.\tag{8}$$

Let A be a 3-Lie algebra, $p,q\in\mathbb{Z}$, $1\le p\ne q\le 4$. Define maps τ_{pq} and $[[,,]]: A^{\otimes^2}\to A^{\otimes^4}$ by, for any $r=\sum_i x_i\otimes y_i\in A\otimes A$,

$$\tau_{pq}(r)=r_{pq}=\sum_i z_{i1}\otimes z_{i2}\otimes z_{i3}\otimes z_{i4},\ \text{where } z_{ij}=\begin{cases} x_i, & j=p,\\ y_i, & j=q,\\ 1, & i\ne p,q,\end{cases}$$

that is, r_{pq} puts x_i at the p-th position, y_i at the q-th position and 1 elsewhere in the 4-tensor, 1 is a symbol playing a similar role of the unit, and

$$\begin{aligned}[[r,r,r]]=\sum_{i,j,k}\big([x_i,x_j,x_k]\otimes y_i\otimes y_j\otimes y_k+x_i\otimes[y_i,x_j,x_k]\otimes y_j\otimes y_k\\ +x_i\otimes x_j\otimes[y_i,y_j,x_k]\otimes y_k+x_i\otimes x_j\otimes x_k\otimes[y_i,y_j,y_k]\big).\end{aligned}$$

The equation

$$[[r,r,r]]=0\tag{9}$$

is called the *3-Lie classical Yang-Baxter equation* [6] in the 3-Lie algebra A, and is abbreviated by 3-Lie CYBE.

LEMMA 2.2. [6] *Let A be a 3-Lie algebra, $P : A \to A$ be a Rota-Baxter operator of weight zero on A, $\bar{P} \in A^* \otimes A$ be the corresponding tensor, that is, for all $x \in A,\ f \in A^*$, $\langle \bar{P}, x \otimes f\rangle = \langle f, P(x)\rangle$. Then*

$$r = \bar{P} - \phi_{12}\bar{P}$$

is a skew-symmetric solution of the 3-Lie CYBE in the semi-direct product 3-Lie algebra $A \ltimes_{\mathrm{ad}^*} A^*$.

Let A be a 3-Lie algebra with a basis $\{e_1, \cdots, e_n\}$, and $\{e_1^*, \cdots, e_n^*\}$ be the dual basis of A, that is, $\{e_1^*, \cdots, e_n^*\}$ is a basis of A^* satisfying

$$\langle e_i^*, e_j\rangle = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \quad 1 \leq i, j \leq n. \end{cases} \tag{10}$$

If $P : A \to A$ is a Rota-Baxter operator of weight zero on A, by Lemma 2.2, we have

$$\bar{P} = \sum_{i=1}^{n} e_i^* \otimes P(e_i) \in A^* \otimes A \subseteq (A^* \ltimes A)^{\otimes^2},$$

and

$$r = \bar{P} - \phi_{12}\bar{P} = \sum_{i=1}^{n}(e_i^* \otimes P(e_i) - P(e_i) \otimes e_i^*)$$

is a skew-symmetric solution of 3-Lie CYBE.

Suppose

$$[e_i, e_j, e_k] = \sum_{s=1}^{n} C_{ijk}^s e_s, \quad C_{ijk}^s \in \mathbb{C}, \quad 1 \leq s, i, j, k \leq n, \tag{11}$$

$$P(e_i) = \sum_{j=1}^{n} a_{ij} e_j, \quad a_{ij} \in \mathbb{C}, \quad 1 \leq i, j \leq n, \tag{12}$$

$$A_{ij}^{ks} = a_{ik}a_{js} - a_{is}a_{jk}, \quad 1 \leq s, i, j, k \leq n. \tag{13}$$

Then for all $l, m, q \in \mathbb{Z},\ 1 \leq l, m, q \leq n$,

$$\begin{aligned}[P(e_l), P(e_m), P(e_q)] &= \sum_{i,j,k}^{n} a_{li}a_{mj}a_{qk}[e_i, e_j, e_k] \\ &= \sum_{i<j<k}^{n} \sum_{s=1}^{n} \sum_{\sigma \in S_3} a_{l\sigma(i)}a_{m\sigma(j)}a_{q\sigma(k)} sgn(\sigma) C_{ijk}^s e_s,\end{aligned}$$

$P([P(e_l), P(e_m), e_q] + [P(e_l), e_m, P(e_q)] + [e_l, P(e_m), P(e_q)])$

$$= \sum_{r=1}^{n}\sum_{s=1}^{n}\Big(\sum_{i<j}^{n}(a_{li}a_{mj}-a_{lj}a_{mi})C_{ijq}^{s}+\sum_{i<k}^{n}(a_{li}a_{qk}-a_{lk}a_{qi})C_{imk}^{s}$$
$$+\sum_{m<k}^{n}\sum_{s=1}^{n}(a_{mj}a_{qk}-a_{mk}a_{qj})C_{ljk}^{s}\Big)a_{sr}e_{r}.$$

By the above notations, we have the following result.

THEOREM 2.3. *Let A be a 3-Lie algebra with the multiplication* (11) *in the basis $\{e_1,\cdots,e_n\}$, $P: A\to A$ be a $\mathbb{C}$-linear map defined as* (12). *Then P is a Rota-Baxter operator of weight zero on A if and only if P satisfies, for all $l,m,q\in\mathbb{Z}_{>0}$,*

$$\sum_{i<j<k}^{n}\sum_{s=1}^{n}\sum_{\sigma\in S_3} a_{l\sigma(i)}a_{m\sigma(j)}a_{q\sigma(k)}sgn(\sigma)C_{ijk}^{s}e_s$$
$$=\sum_{r=1}^{n}\sum_{s=1}^{n}\Big(\sum_{i<j}^{n}A_{lm}^{ij}C_{ijq}^{s}+\sum_{i<k}^{n}A_{lq}^{ik}C_{imk}^{s}+\sum_{m<k}^{n}A_{mq}^{jk}C_{ljk}^{s}\Big)a_{sr}e_r.$$

Proof. Apply (2) and (11), with a direct computation. □

Theorem 2.3 gives us sufficient and necessary conditions for the existence of Rota-Baxter operators of weight zero on a 3-Lie algebra A with the multiplication (11).

LEMMA 2.4. [6] *Let A be a 3-Lie algebra and $r=\sum_i x_i\otimes y_i\in A\otimes A$ be a skew-symmetric solution of 3-Lie CYBE in A. Then r induces a local cocycle 3-Lie bialgebra (A,Δ), where $\Delta=\Delta_1+\Delta_2+\Delta_3$ is defined by* (7).

THEOREM 2.5. *Let A be a 3-Lie algebra with a basis $\{e_1,\cdots,e_n\}$, $P: A\to A$ be a Rota-Baxter operator of weight zero on A defined as* (12), *and $\{e_1^*,\cdots,e_n^*\}$ be the dual basis of A satisfying* (10). *Then $r=\bar{P}-\phi_{12}\bar{P}$ induces a local cocycle 3-Lie bialgebra $(A\ltimes_{\mathrm{ad}^*}A^*,\Delta)$, where for all $x\in A\ltimes_{\mathrm{ad}^*}A^*$,*

$$\Delta(x)=\sum_{i,j,k,l=1}^{n}a_{jk}a_{il}[x,e_i^*,e_k]\wedge e_j^*\wedge e_l+\sum_{i<j}^{n}\sum_{k<l}^{n}A_{ij}^{lk}[x,e_l,e_k]\wedge e_j^*\wedge e_i^*. \tag{14}$$

Furthermore, if the multiplication of A is given by (11), *then*

$$\Delta(e_s)=\sum_{i,j,k,l,t=1}^{n}a_{jk}a_{il}C_{skt}^{i}e_t^*\wedge e_j^*\wedge e_l+\sum_{i<j}^{n}\sum_{k<l}^{n}\sum_{t=1}^{n}A_{ij}^{lk}C_{slk}^{t}e_t\wedge e_j^*\wedge e_i^*, \tag{15}$$

$$\Delta(e_s^*)=\sum_{i<j}^{n}\sum_{k<l}^{n}\sum_{t=1}^{n}A_{ij}^{lk}C_{klt}^{s}e_t^*\wedge e_j^*\wedge e_i^*,\quad 1\le s\le n. \tag{16}$$

Proof. By Lemma 2.2 and 2.4, the skew-symmetric solution $r = \bar{P} - \phi_{12}\bar{P}$ induces a local cocycle 3-Lie bialgebra $(A \ltimes_{\mathrm{ad}^*} A^*, \Delta = \Delta_1 + \Delta_2 + \Delta_3)$, where $\Delta_1, \Delta_2, \Delta_3$ are defined as (7), $\bar{P} \in (A^* \ltimes A)^{\otimes^2}$, and

$$r = \sum_{i,j=1}^{n} a_{ij}(e_i^* \otimes e_j - e_j \otimes e_i^*) = \sum_{i=1}^{2n} x_i \otimes y_i \in (A^* \ltimes A)^{\otimes^2},$$

where $x_i = \begin{cases} e_i^*, & i \le n, \\ -P(e_{i-n}), & i > n, \end{cases}$ $\quad y_i = \begin{cases} P(e_i), & i \le n, \\ e_{i-n}^*, & i > n. \end{cases}$

By (12),

$$x_i = \begin{cases} e_i^*, & i \le n, \\ -\sum\limits_{k=1}^{n} a_{(i-n)k} e_k, & i > n, \end{cases} \qquad y_i = \begin{cases} \sum\limits_{l=1}^{n} a_{il} e_l, & i \le n, \\ e_{i-n}^*, & i > n. \end{cases}$$

Thanks to (7) and (8),

$$\begin{aligned}
\Delta_1(x) &= \sum_{i,j=1}^{n} [x, x_i, x_j] \otimes y_j \otimes y_i \\
&= \sum_{i,j=1}^{n} [x, e_i^*, e_j^*] \otimes \sum_{k=1}^{n} a_{jk} e_k \otimes \sum_{l=1}^{n} a_{il} e_l \\
&+ \sum_{i=1}^{n} \sum_{j=n+1}^{2n} [x, e_i^*, -\sum_{k=1}^{n} a_{(j-n)k} e_k] \otimes e_{j-n}^* \otimes \sum_{l=1}^{n} a_{il} e_l \\
&+ \sum_{i=n+1}^{2n} \sum_{j=1}^{n} [x, -\sum_{l=1}^{n} a_{(i-n)l} e_l, e_j^*] \otimes \sum_{k=1}^{n} a_{jk} e_k \otimes e_{i-n}^* \\
&+ \sum_{i=n+1}^{2n} \sum_{j=n+1}^{2n} [x, -\sum_{l=1}^{n} a_{(i-n)l} e_l, -\sum_{k=1}^{n} a_{(j-n)k} e_k] \otimes e_{j-n}^* \otimes e_{i-n}^* \\
&= \sum_{i,j,k,l=1}^{n} a_{jk} a_{il} [x, e_k, e_i^*] \otimes (e_j^* \wedge e_l) \\
&+ \sum_{i<j}^{n} \sum_{l<k}^{n} (a_{il} a_{jk} - a_{ik} a_{jl}) [x, e_l, e_k] \otimes (e_j^* \wedge e_i^*),
\end{aligned}$$

$$\Delta(x) = \Delta_1(x) + \phi_{13}\phi_{12}\Delta_1(x) + \phi_{12}\phi_{13}\Delta_1(x) = \sum_{i,j,k,l=1}^{n} a_{jk} a_{il} [x, e_k, e_i^*] \wedge e_j^* \wedge e_l$$

$$+ \sum_{i<j}^{n} \sum_{l<k}^{n} (a_{il} a_{jk} - a_{ik} a_{jl}) [x, e_l, e_k] \wedge e_j^* \wedge e_i^*.$$

It follows that (14) holds.

Set $\mathrm{ad}^*_{e_l,e_k} e^*_s = \sum\limits_{t=1}^{n} a^t_{lks} e^*_t,\ a^t_{lks} \in \mathbb{C},\ 1 \le l,k,s \le n$. By (10) and (11), for all $r \in n$,

$\langle \mathrm{ad}^*_{e_l,e_k} e^*_s, e_r\rangle = \langle \sum\limits_{t=1}^{n} a^t_{lks} e^*_t, e_r\rangle = a^r_{lks}$,

$\langle \mathrm{ad}^*_{e_l,e_k} e^*_s, e_r\rangle = -\langle e^*_s, \sum\limits_{m=1}^{n} C^m_{lkr} e_m\rangle = -C^s_{lkr}$.

Therefore, $a^r_{lks} = -C^s_{lkr}$, $1 \le l,k,s,r \le n$, and

$$[e^*_s, e_l, e_k] = -\sum_{t=1}^{n} C^s_{lkt} e^*_t, \quad 1 \le l,k,s \le n. \tag{17}$$

Thanks to (11), (14), (17) and $C^s_{klt} = -C^s_{lkt}$, (15) and (16) hold. □

3. CLASSIFICATION OF ROTA-BAXTER OPERATORS OF WEIGHT ZERO ON A_4

In this section, we suppose that $\{e_1, e_2, e_3, e_4\}$ is a basis of A_4, $\{e^*_1, e^*_2, e^*_3, e^*_4\}$ is the dual basis of A_4 satisfying (10), and the multiplication of A_4 in the basis is (1).

Let $P : A_4 \to A_4$ be a linear map defined as (12), and the matrix of P in the basis $\{e_1, e_2, e_3, e_4\}$ be $M(P) = (a_{ij})$, $1 \le i,j \le 4$, then

$$(P(e_1),\ P(e_2),\ P(e_3),\ P(e_4))^T = M(P)(e_1,\ e_2,\ e_3,\ e_4)^T.$$

Let M_{ij} be the complement minor of the element a_{ij} in the matrix $M(P)$, and A_{ij} be the algebraic cofactor of a_{ij}, that is, $A_{ij} = (-1)^{i+j} M_{ij}$, $1 \le i,j \le 4$.

Thanks to Theorem 2.3, if l,m,q,s satisfy $1 \le l,m,q,s \le 4$ and $(s-q)(s-l)(s-m) \ne 0$, then

$$\begin{aligned}[P(e_l), P(e_m), P(e_q)] &= M_{s1}[e_2,e_3,e_4] + M_{s2}[e_1,e_3,e_4] + M_{s3}[e_1,e_2,e_4] \\ &\quad + M_{s4}[e_1,e_2,e_3] = -M_{s1}e_2 + M_{s2}e_1 - M_{s3}e_4 + M_{s4}e_3.\end{aligned} \tag{18}$$

For convenience, denote

$$\begin{cases} H^{123}_1 = -A^{14}_{12} + A^{34}_{23},\ H^{123}_2 = A^{34}_{13} + A^{24}_{12},\ H^{123}_3 = A^{12}_{12} + A^{13}_{13} + A^{23}_{23}, \\ H^{123}_4 = -A^{14}_{13} - A^{24}_{23},\ H^{124}_1 = A^{13}_{12} + A^{34}_{24},\ H^{124}_2 = A^{34}_{14} - A^{23}_{12}, \\ H^{124}_3 = A^{13}_{14} + A^{23}_{24},\quad H^{124}_4 = -A^{12}_{12} - A^{14}_{14} - A^{24}_{24}, \\ H^{134}_2 = -A^{23}_{13} - A^{24}_{14},\ H^{134}_3 = A^{23}_{34} - A^{12}_{14},\ H^{134}_4 = -A^{12}_{13} - A^{24}_{34}, \\ H^{234}_1 = A^{13}_{23} + A^{14}_{24},\quad H^{234}_2 = -A^{23}_{23} - A^{24}_{24} - A^{34}_{34},\ H^{234}_3 = -A^{12}_{24} - A^{13}_{34}, \\ H^{234}_4 = A^{14}_{34} - A^{12}_{23},\quad H^{134}_1 = A^{13}_{13} + A^{14}_{14} + A^{34}_{34}, \end{cases} \tag{19}$$

where A_{ij}^{kl} is defined as (13). Then for all $1 \leq l, m, q \leq 4$,

$$P([P(e_l), P(e_m), e_q] + [P(e_l), e_m, P(e_q)] + [e_l, P(e_m), P(e_q)]) \tag{20}$$
$$= \sum_{i,j=1}^{4} a_{ij} H_i^{lmq} e_j.$$

THEOREM 3.1. *Any linear map $P : A_4 \to A_4$ with $Rank(P) = 1$ is a Rota-Baxter operator of weight zero on A_4.*

Proof. The result follows from a direct computation according to (18) and (20). □

THEOREM 3.2. *Let P be a linear map of A_4 with $Rank(P) = 2$ defined as (12), $P(e_1)$ and $P(e_2)$ be linearly independent. Then P is a Rota-Baxter operator of weight zero if and only if P satisfies*

$$\begin{pmatrix} H_3^{123} & 0 & H_4^{123} & 0 \\ 0 & H_3^{123} & 0 & H_4^{123} \\ H_3^{124} & 0 & H_4^{124} & 0 \\ 0 & H_3^{124} & 0 & H_4^{124} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \mu_1 \\ \mu_2 \end{pmatrix} = - \begin{pmatrix} H_1^{123} \\ H_2^{123} \\ H_1^{124} \\ H_2^{124} \end{pmatrix},$$

where $P(e_3) = \lambda_1 P(e_1) + \lambda_2 P(e_2)$, $P(e_4) = \mu_1 P(e_1) + \mu_2 P(e_2)$, $\lambda_1, \mu_1, \lambda_2, \mu_2 \in \mathbb{C}$, H_s^{123} and H_s^{124} are defined as (21) below, $1 \leq s \leq 4$.

Proof. Since $Rank(P) = 2$, $P(e_1)$ and $P(e_2)$ are linearly independent. Then there are λ_1, μ_1, λ_2, $\mu_2 \in \mathbb{C}$ such that $P(e_3) = \lambda_1 P(e_1) + \lambda_2 P(e_2)$, $P(e_4) = \mu_1 P(e_1) + \mu_2 P(e_2)$, and

$$[P(e_1), P(e_2), P(e_3)] = [P(e_1), P(e_2), P(e_4)] = 0.$$

Thanks to (2), (18) and (20),

$$\begin{cases} H_1^{123} = -A_{12}^{14} - \lambda_1 A_{12}^{34}, \ H_2^{123} = A_{12}^{24} + \lambda_2 A_{12}^{34}, \ H_3^{123} = A_{12}^{12} - \lambda_1 A_{12}^{23} + \lambda_2 A_{12}^{13}, \\ H_4^{123} = \lambda_1 A_{12}^{24} - \lambda_2 A_{12}^{14}, \ H_1^{124} = A_{12}^{13} - \mu_1 A_{12}^{34}, \ H_2^{124} = -A_{12}^{23} + \mu_2 A_{12}^{34}, \\ H_3^{124} = -\mu_1 A_{12}^{23} + \mu_2 A_{12}^{13}, \ H_4^{124} = -A_{12}^{12} + \mu_1 A_{12}^{24} - \mu_2 A_{12}^{14}, \\ H_1^{134} = \lambda_2 A_{12}^{13} + \mu_2 A_{12}^{14} + (\lambda_1\mu_2 - \lambda_2\mu_1) A_{12}^{34}, \ H_2^{134} = -\lambda_2 A_{12}^{23} - \mu_2 A_{12}^{24}, \\ H_1^{234} = -\lambda_1 A_{12}^{13} - \mu_1 A_{12}^{14}, \ H_3^{134} = -\mu_2 A_{12}^{12} + (\lambda_1\mu_2 - \lambda_2\mu_1) A_{12}^{23}, \\ H_4^{134} = -\lambda_2 A_{12}^{12} - (\lambda_1\mu_2 - \lambda_2\mu_1) A_{12}^{24}, \\ H_2^{234} = \lambda_1 A_{12}^{23} + \mu_1 A_{12}^{24} - (\lambda_1\mu_2 - \lambda_2\mu_1) A_{12}^{34}, \\ H_3^{234} = \mu_1 A_{12}^{12} - (\lambda_1\mu_2 - \lambda_2\mu_1) A_{12}^{13}, \quad H_4^{234} = \lambda_1 A_{12}^{12} + (\lambda_1\mu_2 - \lambda_2\mu_1) A_{12}^{14}, \end{cases} \tag{21}$$

and P is a Rota-Baxter operator of weight zero on A_4 if and only if P has properties that

$$\begin{aligned}
[P(e_1), P(e_2), P(e_3)] &= (H_1^{123} + \lambda_1 H_3^{123} + \mu_1 H_4^{122})P(e_1) \\
&\quad + (H_2^{123} + \lambda_2 H_3^{123} + \mu_2 H_4^{123})P(e_2) = 0, \\
[P(e_1), P(e_2), P(e_4)] &= (H_1^{124} + \lambda_1 H_3^{124} + \mu_1 H_4^{124})P(e_1) \\
&\quad + (H_2^{124} + \lambda_2 H_3^{124} + \mu_2 H_4^{124})P(e_2) = 0, \\
[P(e_1), P(e_3), P(e_4)] &= (H_1^{134} + \lambda_1 H_3^{134} + \mu_1 H_4^{134})P(e_1) \\
&\quad + (H_2^{134} + \lambda_2 H_3^{134} + \mu_2 H_4^{134})P(e_2) = 0, \\
[P(e_2), P(e_3), P(e_4)] &= (H_1^{234} + \lambda_1 H_3^{234} + \mu_1 H_4^{234})P(e_1) \\
&\quad + (H_2^{234} + \lambda_2 H_3^{234} + \mu_2 H_4^{234})P(e_2) = 0.
\end{aligned}$$

By the linear independence of $P(e_1)$ and $P(e_2)$, the result is obtained. □

COROLLARY 3.3. *If $Rank(P) = 2$, $P(e_3) = P(e_4) = 0$, then P is a Rota-Baxter operator of weight zero on A_4 if and only if $A_{12}^{14} = A_{12}^{24} = A_{12}^{13} = A_{12}^{23} = 0$, and $A_{12}^{12} \neq 0$ or $A_{12}^{34} \neq 0$.*

Proof. If $P(e_3) = P(e_4) = 0$, then by Theorem 3.2, P is a Rota-Baxter operator of weight zero on A_4 if and only if P satisfies

$$H_1^{123} = H_2^{123} = H_1^{124} = H_2^{124} = 0.$$

By (21), we have $H_1^{123} = -A_{12}^{14}$, $H_2^{123} = A_{12}^{24}$, $H_1^{124} = A_{12}^{13}$, $H_2^{124} = -A_{12}^{23}$. Thanks to $Rank(P) = 2$, $A_{12}^{12} \neq 0$ or $A_{12}^{34} \neq 0$. The result follows. □

THEOREM 3.4. *Let P be a linear map of A_4 defined as (12) with $Rank(P) = 3$, $P(e_1)$, $P(e_2)$ and $P(e_3)$ be linearly independent, $P(e_4) = s_1P(e_1) + s_2P(e_2) + s_3P(e_3), s_1, s_2, s_3 \in \mathbb{C}$. Then P is a Rota-Baxter operator of weight zero on A_4 if and only if P satisfies*

$$M(P)^T \alpha_i = s_i \beta, \quad 1 \leq i \leq 4, \quad \textit{and} \quad s_4 = 1, \tag{22}$$

where

$\alpha_1 = (H_1^{234}, H_2^{234}, H_3^{234}, H_4^{234})^T$, $\alpha_2 = (H_1^{134}, H_2^{134}, H_3^{134}, H_4^{134})^T$,
$\alpha_3 = (H_1^{124}, H_2^{124}, H_3^{124}, H_4^{124})^T$, $\alpha_4 = (H_1^{123}, H_2^{123}, H_3^{123}, H_4^{123})^T$,
$\beta = (A_{42}, A_{41}, A_{44}, A_{43})^T$, *and* H_s^{123},H_s^{124}, H_s^{134}, H_s^{234}
are given in (23) *below,* $1 \leq s \leq 4$.

Proof. Since $P(e_1), P(e_2), P(e_3)$ are linearly independent,

$$P(e_4) = s_1P(e_1) + s_2P(e_2) + s_3P(e_3), \ s_1, s_2, s_3 \in \mathbb{C}.$$

Therefore, (19) can be written as

(23)

$$\begin{cases} H_1^{123}=-A_{12}^{14}+A_{23}^{34},\ H_2^{123}=A_{13}^{34}+A_{12}^{24},\ H_3^{123}=A_{12}^{12}+A_{13}^{13}+A_{23}^{23},\\ H_4^{123}=-A_{13}^{14}-A_{23}^{24},\ H_2^{134}=-A_{13}^{23}-s_2A_{12}^{24}-s_3A_{13}^{24},\\ H_1^{124}=A_{12}^{13}-s_1A_{12}^{34}+s_3A_{23}^{34},\ H_2^{124}=s_2A_{12}^{34}+s_3A_{13}^{34}-A_{12}^{23},\\ H_3^{124}=s_2A_{12}^{13}+s_3A_{13}^{13}-s_1A_{12}^{23}+s_3A_{23}^{23},\ H_1^{234}=A_{23}^{13}-s_1A_{12}^{14}+s_3A_{23}^{14},\\ H_3^{134}=-s_1A_{13}^{23}-s_2A_{23}^{23}-s_2A_{12}^{12}-s_3A_{13}^{12},\ H_4^{134}=-A_{13}^{12}+s_1A_{13}^{24}+s_2A_{23}^{24},\\ H_3^{234}=s_1A_{12}^{12}-s_3A_{23}^{12}+s_1A_{13}^{13}+s_2A_{23}^{13},\ H_4^{234}=-s_1A_{13}^{14}-s_2A_{23}^{14}-A_{23}^{12},\\ H_4^{124}=-A_{12}^{12}-s_2A_{12}^{14}-s_3A_{13}^{14}+s_1A_{12}^{24}-s_3A_{23}^{24},\\ H_1^{134}=A_{13}^{13}+s_2A_{12}^{14}+s_3A_{13}^{14}-s_1A_{13}^{34}-s_2A_{23}^{34},\\ H_2^{234}=-A_{23}^{23}+s_1A_{12}^{24}-s_3A_{23}^{24}+s_1A_{13}^{34}+s_2A_{23}^{34}. \end{cases}$$

By (2), (18), (20) and (23), and a direct computation, it can be verified that P is a Rota-Baxter operator of weight zero if and only if P satisfies

$$G\begin{pmatrix} H_1^{123}\\ H_2^{123}\\ H_3^{123}\\ H_4^{123}\end{pmatrix}=Y,\quad G\begin{pmatrix} H_1^{123}\\ H_2^{123}\\ H_3^{123}\\ H_4^{123}\end{pmatrix}=s_3Y,$$

$$G\begin{pmatrix} H_1^{123}\\ H_2^{123}\\ H_3^{123}\\ H_4^{123}\end{pmatrix}=s_2Y,\quad G\begin{pmatrix} H_1^{123}\\ H_2^{123}\\ H_3^{123}\\ H_4^{123}\end{pmatrix}=s_1Y,$$

where $G=\begin{pmatrix} a_{11} & a_{21} & a_{31} & s_1a_{11}+s_2a_{21}+s_3a_{31}\\ a_{12} & a_{22} & a_{32} & s_1a_{12}+s_2a_{22}+s_3a_{32}\\ a_{13} & a_{23} & a_{33} & s_1a_{13}+s_2a_{23}+s_3a_{33}\\ a_{14} & a_{24} & a_{34} & s_1a_{14}+s_2a_{24}+s_3a_{34}\end{pmatrix}, Y=\begin{pmatrix} M_{42}\\ -M_{41}\\ M_{44}\\ -M_{43}\end{pmatrix}.$

Therefore, (22) holds, and

$$\alpha_1=(H_1^{234},H_2^{234},H_3^{234},H_4^{234})^T,\ \alpha_2=(H_1^{134},H_2^{134},H_3^{134},H_4^{134})^T,$$
$$\beta=(A_{42},A_{41},A_{44},A_{43})^T,$$
$$\alpha_3=(H_1^{124},H_2^{124},H_3^{124},H_4^{124})^T,\ \alpha_4=(H_1^{123},H_2^{123},H_3^{123},H_4^{123})^T.$$

The proof is complete. □

Theorem 3.5. *Let P be a linear map of A_4 defined as (12) with $Rank(P)=3$ and $P(e_4)=0$. Then P is a Rota-Baxter operator of weight* *zero on A_4 if and only if $M(P)$ is one of the following matrices:*

(24)
$$M(P_1) = \begin{pmatrix} \lambda a_{21} & \lambda^2 a_{21} & 0 & a_{14} \\ a_{21} & \lambda a_{21} & 0 & a_{24} \\ a_{31} & a_{32} & 0 & a_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\lambda a_{21} \neq 0, \begin{vmatrix} \lambda & a_{14} \\ 1 & a_{24} \end{vmatrix} = \begin{vmatrix} 1 & a_{31} \\ \lambda & a_{32} \end{vmatrix} \neq 0, \ \lambda \in \mathbb{C};$$

(25)
$$M(P_2) = \begin{pmatrix} 0 & 0 & 0 & a_{14} \\ a_{21} & 0 & 0 & a_{24} \\ a_{31} & -a_{14} & 0 & a_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad a_{21}a_{14} \neq 0;$$

(26)
$$M(P_3) = \begin{pmatrix} 0 & a_{12} & 0 & a_{14} \\ 0 & 0 & 0 & a_{24} \\ -a_{24} & a_{32} & 0 & a_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ a_{12}a_{24} \neq 0, \ \textit{where } a_{ij} \in \mathbb{C}, 1 \leq i,j \leq 4.$$

Proof. Since $P(e_4) = 0$, then $M(P) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

By (19) and Theorem 3.4, P is a Rota-Baxter operator of weight zero on A_4 if and only if $M(P)$ satisfies

$$M(P)^T \alpha_i = (0,0,0,A_{43})^T, \quad 1 \leq i \leq 4,$$

where

$$\alpha_1 = \begin{pmatrix} A_{23}^{13} \\ -A_{23}^{23} \\ 0 \\ -A_{23}^{12} \end{pmatrix}, \ \alpha_2 = \begin{pmatrix} A_{13}^{13} \\ -A_{13}^{23} \\ 0 \\ -A_{13}^{12} \end{pmatrix},$$

$$\alpha_3 = \begin{pmatrix} A_{12}^{13} \\ -A_{12}^{23} \\ 0 \\ -A_{12}^{12} \end{pmatrix}, \ \alpha_4 = \begin{pmatrix} -A_{12}^{14} + A_{23}^{34} \\ A_{13}^{34} + A_{12}^{24} \\ A_{12}^{12} + A_{13}^{13} + A_{23}^{23} \\ -A_{13}^{14} - A_{23}^{24} \end{pmatrix}.$$

Thanks to (13), $A_{12}^{13} = A_{12}^{23} = A_{13}^{13} = A_{13}^{23} = A_{23}^{13} = A_{23}^{23} = 0, \ a_{13} = a_{23} = a_{33} = 0$, and

$$\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{14} & a_{24} & a_{34} \end{pmatrix} \begin{pmatrix} -A_{12}^{14} \\ A_{12}^{24} \\ A_{12}^{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -M_{43} \end{pmatrix}.$$

Therefore,

$$A_{12}^{14} = A_{23}^{12}, \quad A_{12}^{24} = A_{13}^{12}, \quad A_{12}^{12} = 0. \tag{27}$$

• If $(a_{11}, a_{21}) \neq (0,0)$, then by $A_{12}^{12} = 0$, there is $\lambda \in \mathbb{C}$ such that $a_{12} = \lambda a_{11}, \quad a_{22} = \lambda a_{21}$.

Thanks to (27), $\begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{14} & a_{24} & a_{34} \end{vmatrix} = -\begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \begin{vmatrix} 1 & \lambda \\ a_{31} & a_{32} \end{vmatrix} \neq 0$, and

$$\begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} = a_{21} \begin{vmatrix} 1 & \lambda \\ a_{31} & a_{32} \end{vmatrix}, \quad \lambda \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} = a_{11} \begin{vmatrix} 1 & \lambda \\ a_{31} & a_{32} \end{vmatrix}.$$

If $\lambda = 0$, then from $\begin{vmatrix} 1 & \lambda \\ a_{31} & a_{32} \end{vmatrix} \neq 0$, we have $a_{11} = a_{12} = a_{22} = 0$, $a_{32} = -a_{14}$, $a_{21}a_{14} \neq 0$. We get $M(P_2)$.

If $\lambda \neq 0$, then $a_{11} = \lambda a_{21}$, $\begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} = a_{21} \begin{vmatrix} \lambda & a_{14} \\ 1 & a_{24} \end{vmatrix} = a_{21} \begin{vmatrix} 1 & \lambda \\ a_{31} & a_{32} \end{vmatrix}$.
Therefore, $a_{21} \neq 0$, $\begin{vmatrix} \lambda & a_{14} \\ 1 & a_{24} \end{vmatrix} = \begin{vmatrix} 1 & \lambda \\ a_{31} & a_{32} \end{vmatrix} \neq 0$, we get $M(P_1)$.

•• If $(a_{11}, a_{21}) = (0,0)$, then $a_{13} = a_{23} = a_{33} = 0$ and $(a_{12}, a_{22}) \neq (0,0)$. By a completely similar discussion to the above, $a_{13} = -a_{24}$ and $a_{12}a_{24} \neq 0$, we get $M(P_3)$.

It is clear that if $M(P)$ is the one of (24), (25) and (26), then P satisfies (2). The proof is complete. □

Example 3.6. If $Rank(P) = 3$, $P(e_4) = 0$, then by Theorem 3.5, P_4 and P_5 are Rota-Baxter operators of weight zero on A_4, where

$$M(P_4) = \begin{pmatrix} 0 & a_{12} & 0 & 0 \\ 0 & 0 & 0 & a_{24} \\ -a_{24} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M(P_5) = \begin{pmatrix} 0 & 0 & 0 & a_{14} \\ a_{21} & 0 & 0 & 0 \\ 0 & -a_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and $a_{12}a_{24} \neq 0, \ a_{14}a_{21} \neq 0$.

THEOREM 3.7. *Let P be a linear map with $Rank(P) = 4$ defined as* (12). *Then P is a Rota-Baxter operator of weight zero on A_4 if and only if*

$$M(P) = \begin{pmatrix} a_{11} & 0 & a_{13} & a_{14} \\ 0 & -a_{11} & a_{23} & a_{24} \\ -a_{24} & -a_{14} & a_{33} & 0 \\ -a_{23} & -a_{13} & 0 & -a_{33} \end{pmatrix}, \quad \textit{where} \quad a_{11}a_{33} + A_{12}^{34} \neq 0. \tag{28}$$

Proof. Let D be a linear map of A_4 with $Rank(D) = 4$, and $D(e_i) = \sum_{j=1}^{4} b_{ij}e_j,\ b_{ij} \in \mathbb{C},\ 1 \leq i, j \leq 4$. Thanks to (1), D is a derivation of A if and only if

$$M(D) = \begin{pmatrix} b_{11} & 0 & b_{13} & b_{14} \\ 0 & -b_{11} & b_{23} & b_{24} \\ -b_{24} & -b_{14} & b_{33} & 0 \\ -b_{23} & -b_{13} & 0 & -b_{33} \end{pmatrix}.$$

By Theorem 2.2 in [7], P is a Rota-Baxter operator of weight zero on A_4 if and only if there is a derivation D of A_4 such that $P = D^{-1}$, then

$$M(P) = M(D^{-1}) = \frac{1}{b_{11}b_{33}+b_{13}b_{24}-b_{14}b_{23}} \begin{pmatrix} b_{33} & 0 & -b_{13} & b_{14} \\ 0 & -b_{33} & b_{23} & -b_{24} \\ b_{24} & -b_{14} & b_{11} & 0 \\ -b_{23} & b_{13} & 0 & -b_{11} \end{pmatrix}.$$

It follows the result. □

4. SKEW-SYMMETRIC SOLUTIONS OF 3-LIE CYBE AND 8-DIMENSIONAL LOCAL COCYCLE 3-LIE BIALGEBRAS

In this section, we study skew-symmetric solutions of 3-Lie CYBE in the semi-direct product 3-Lie algebra $A_4 \ltimes_{\mathrm{ad}^*} A_4^*$, and construct 8-dimensional local cocycle 3-Lie bialgebras by means of the Rota-Baxter operators obtained in Section 3, where A_4 is the 4-dimensional simple 3-Lie algebra with the multiplication (1) in the basis $\{e_1, e_2, e_3, e_4\}$, and $\{e_1^*, e_2^*, e_3^*, e_4^*\}$ is the dual basis of A_4 satisfying (10).

THEOREM 4.1. *Let P be a Rota-Baxter operator of weight zero on A_4 defined as* (12). *Then the skew-symmetric solutions of 3-Lie CYBE in the semi-direct product 3-Lie algebra $A_4 \ltimes_{\mathrm{ad}^*} A_4^*$ induced by P are as follows:*

1) *If $Rank(P) = 1$ and $P(e_1) = \sum_{l=1}^{4} a_{1l}e_l \neq 0$, then*

$$r_1 = \sum_{l=1}^{4} a_{1l}(e_1^* + k_1e_2^* + k_2e_3^* + k_3e_4^*) \wedge e_l, \quad k_1, k_2, k_3 \in \mathbb{C}.$$

2) *If $Rank(P) = 2$, $P(e_1) = a_{11}e_1 + a_{12}e_2$, $P(e_2) = a_{21}e_1 + a_{22}e_2$, and $P(e_3) = P(e_4) = 0$, then*

$$r_2 = e_1^* \wedge (a_{11}e_1 + a_{12}e_2) + e_2^* \wedge (a_{21}e_1 + a_{22}e_2);$$

If $P(e_1) = a_{13}e_3 + a_{14}e_4$, $P(e_2) = a_{23}e_3 + a_{24}e_4$, and $P(e_3) = P(e_4) = 0$, then

$$r_3 = e_1^* \wedge (a_{13}e_3 + a_{14}e_4) + e_2^* \wedge (a_{23}e_3 + a_{24}e_4).$$

3) *If* $Rank(P)=3$, $P(e_4)=0$, $M(P_1)$, $M(P_2)$ *and* $M(P_3)$ *are defined as* (24), (25) *and* (26), *then*

$$r_4 = e_1^*\wedge(\lambda a_{21}e_1+\lambda^2 a_{21}e_2+a_{14}e_4)+e_2^*\wedge(a_{21}e_1+\lambda a_{21}e_2+a_{24}e_4)$$
$$+e_3^*\wedge(a_{31}e_1+a_{32}e_2+a_{34}e_4),\quad \lambda a_{21}\neq 0,\quad \begin{vmatrix}\lambda & a_{14}\\ 1 & a_{24}\end{vmatrix}=\begin{vmatrix}1 & a_{31}\\ \lambda & a_{32}\end{vmatrix}\neq 0;$$
$$r_5 = a_{14}e_1^*\wedge e_4+e_2^*\wedge(a_{21}e_1+a_{24}e_4)+e_3^*\wedge(a_{31}e_1-a_{14}e_2+a_{34}e_4),$$
$$a_{21}a_{14}\neq 0;$$
$$r_6 = e_1^*\wedge(a_{12}e_2+a_{14}e_4)+a_{24}e_2^*\wedge e_4+e_3^*\wedge(-a_{24}e_1+a_{32}e_2+a_{34}e_4),$$
$$a_{12}a_{24}\neq 0.$$

4) *If* $Rank(P)=4$, *then*

$$r_7 = e_1^*\wedge(a_{11}e_1+a_{13}e_3+a_{14}e_4)+e_2^*\wedge(-a_{11}e_2+a_{23}e_3+a_{24}e_4)$$
$$+e_3^*\wedge(-a_{24}e_1-a_{14}e_2+a_{33}e_3)+e_4^*\wedge(-a_{23}e_1-a_{13}e_2-a_{33}e_4),$$

where $a_{ij}\in\mathbb{C},\ 1\leq i,j\leq 4$.

Proof. The result can be verified by a direct computation according to Lemma 2.2 and Theorem 3.1, 3.2, 3.4, 3.5 and 3.7. We omit the discussion process. □

By (1) and (11),

$$C_{123}^3=1,\ C_{124}^4=-1,\ C_{134}^1=1,\ C_{234}^2=-1.$$

Therefore, the multiplication of $A_4\ltimes_{\mathrm{ad}^*}A_4^*$ in the basis $\{e_1,e_2,e_3,e_4,e_1^*,e_2^*,e_3^*,e_4^*\}$ is given by

$$(29)\quad \begin{cases}[e_1^*,e_1,e_3]=-e_4^*,\ [e_1^*,e_1,e_4]=e_3^*,\ [e_1^*,e_3,e_4]=-e_1^*,\ [e_2^*,e_2,e_3]=e_4^*,\\ [e_2^*,e_2,e_4]=-e_3^*,\ [e_2^*,e_3,e_4]=e_2^*,\ [e_3^*,e_1,e_2]=-e_3^*,\ [e_3^*,e_1,e_3]=e_2^*,\\ [e_3^*,e_2,e_3]=-e_1^*,\ [e_4^*,e_1,e_2]=e_4^*,\ [e_4^*,e_1,e_4]=-e_2^*,\ [e_4^*,e_2,e_4]=e_1^*.\end{cases}$$

THEOREM 4.2. *Let* P *be a Rota-Baxter operator of weight zero on* A_4 *defined as* (12). *Then there are 8-dimensional local cocycle 3-Lie bialgebras* $(A_4\ltimes_{\mathrm{ad}^*}A_4^*,\Delta^t)$ *and 3-Lie algebras* $(A_4\oplus A_4^*,\Delta^{t*})$ *induced by* P, $1\leq t\leq 6$, *where*

$$(30)\quad \begin{cases}\Delta^1(e_1)=\sum_{l=1}^{4}(a_{1l}a_{13}e_4^*\wedge e_l\wedge e_1^*+a_{1l}a_{14}e_3^*\wedge e_1^*\wedge e_l),\\ \Delta^1(e_4)=\sum_{l=1}^{4}a_{1l}a_{11}e_3^*\wedge e_l^*\wedge e_1^*,\Delta^1(e_3)=\sum_{l=1}^{4}a_{1l}a_{11}e_4^*\wedge e_1^*\wedge e_l,\\ \Delta^1(e_2)=\Delta^1(e_s^*)=0,\quad 1\leq s\leq 4;\end{cases}$$

$$(31)\quad \begin{cases} \Delta^2(e_3^*) = (a_{12}a_{21}-a_{11}a_{22})e_3^*\wedge e_1^*\wedge e_2^*, \Delta^2(e_4^*) \\ \qquad = (a_{11}a_{22}-a_{12}a_{21})e_4^*\wedge e_1^*\wedge e_2^*, \\ \Delta^2(e_3) = (a_{11}^2-a_{12}a_{21})e_4^*\wedge e_1^*\wedge e_1 + (a_{11}a_{12}-a_{12}a_{22})e_4^*\wedge e_1^*\wedge e_2 \\ \qquad + (a_{21}a_{11}-a_{22}a_{21})e_4^*\wedge e_2^*\wedge e_1 + (a_{21}a_{12}-a_{22}^2)e_4^*\wedge e_2^*\wedge e_2 \\ \qquad + (a_{11}a_{22}-a_{12}a_{21})e_3\wedge e_1^*\wedge e_2^*, \\ \Delta^2(e_4) = (a_{12}a_{21}-a_{11}^2)e_3^*\wedge e_1^*\wedge e_1 + (a_{12}a_{22}-a_{11}a_{12})e_3^*\wedge e_1^*\wedge e_2 \\ \qquad + (a_{22}a_{21}-a_{21}a_{11})e_3^*\wedge e_2^*\wedge e_1 + (a_{22}^2-a_{21}a_{12})e_3^*\wedge e_2^*\wedge e_2 \\ \qquad + (a_{12}a_{21}-a_{11}a_{22})e_4\wedge e_1^*\wedge e_2^*, \\ \Delta^2(e_1^*) = \Delta^2(e_2^*) = \Delta^2(e_1) = \Delta^2(e_2) = 0; \end{cases}$$

(32)

$$\begin{cases} \Delta^3(e_1^*) = \Delta^3(e_2^*) = \Delta^3(e_3^*) = \Delta^3(e_4^*) = 0, \\ \Delta^3(e_1) = a_{13}^2e_4^*\wedge e_3\wedge e_1^* + a_{13}a_{14}e_4^*\wedge e_4\wedge e_1^* + a_{13}a_{14}e_4^*\wedge e_3\wedge e_2^* \\ \quad + a_{14}^2e_4^*\wedge e_4\wedge e_2^* + a_{24}a_{13}e_3^*\wedge e_2^*\wedge e_3 + a_{24}a_{14}e_3^*\wedge e_2^*\wedge e_4 \\ \quad + a_{14}a_{13}e_3^*\wedge e_1^*\wedge e_3 + a_{14}^2e_3^*\wedge e_1^*\wedge e_4 + (a_{13}a_{24}-a_{14}a_{23})e_1\wedge e_1^*\wedge e_2^*, \\ \Delta^3(e_2) = a_{13}a_{23}e_4^*\wedge e_1^*\wedge e_3 + a_{13}a_{24}e_4^*\wedge e_1^*\wedge e_4 + a_{23}^2e_4^*\wedge e_2^*\wedge e_3 \\ \quad + a_{23}a_{24}e_4^*\wedge e_2^*\wedge e_4 + a_{14}a_{23}e_3^*\wedge e_3\wedge e_1^* + a_{24}a_{14}e_3^*\wedge e_4\wedge e_1^* \\ \quad + a_{24}a_{23}e_3^*\wedge e_3\wedge e_2^* + a_{24}^2e_3^*\wedge e_4\wedge e_2^* + (a_{14}a_{23}-a_{13}a_{24})e_2\wedge e_1^*\wedge e_2^*, \\ \Delta^3(e_3) = a_{14}a_{23}e_2^*\wedge e_1^*\wedge e_3 + a_{14}a_{24}e_2^*\wedge e_1^*\wedge e_4, \\ \Delta^3(e_4) = 2a_{13}a_{23}e_1^*\wedge e_2^*\wedge e_3 + (a_{23}a_{14}+a_{13}a_{24})e_1^*\wedge e_2^*\wedge e_4; \end{cases}$$

$$(33)\quad \begin{cases} \Delta^4(e_2^*) = -a_{12}a_{24}e_3^*\wedge e_1^*\wedge e_2^*, \Delta^4(e_1^*) = \Delta^4(e_3^*) = \Delta^4(e_4^*) = 0, \\ \Delta^4(e_1) = a_{12}a_{24}e_3^*\wedge e_2^*\wedge e_2 - a_{12}a_{24}e_3^*\wedge e_1\wedge e_1^* - a_{12}a_{24}e_4\wedge e_2^*\wedge e_3^*, \\ \Delta^4(e_2) = -a_{12}a_{24}e_3\wedge e_1^*\wedge e_2^* + a_{24}^2e_4\wedge e_2^*\wedge e_3^*, \\ \Delta^4(e_3) = a_{12}a_{24}e_4^*\wedge e_4\wedge e_1^* + a_{12}a_{24}e_2^*\wedge e_1^*\wedge e_2^* - a_{24}^2e_1\wedge e_2^*\wedge e_3^*, \\ \Delta^4(e_4) = a_{12}a_{24}e_3^*\wedge e_1^*\wedge e_4 + a_{12}a_{24}e_1\wedge e_1^*\wedge e_2^*; \end{cases}$$

$$(34)\quad \begin{cases} \Delta^5(e_1^*) = -a_{14}a_{21}e_3^*\wedge e_1^*\wedge e_2^*, \Delta^5(e_4^*) = -a_{14}a_{21}e_4^*\wedge e_2^*\wedge e_3^*, \\ \Delta^5(e_1) = -a_{14}^2e_4\wedge e_1^*\wedge e_3^*, \quad \Delta^5(e_4) = 2a_{14}a_{21}e_3^*\wedge e_4\wedge e_2^*, \\ \Delta^5(e_2) = a_{14}a_{21}e_3^*\wedge e_1\wedge e_1^* - a_{14}a_{21}e_3^*\wedge e_2^*\wedge e_2 - a_{14}a_{21}e_4\wedge e_1^*\wedge e_2^*, \\ \Delta^5(e_3) = -a_{14}a_{21}e_3\wedge e_2^*\wedge e_3^* + a_{14}^2e_2\wedge e_1^*\wedge e_3^*, \Delta^5(e_2^*) = \Delta^5(e_3^*) = 0; \end{cases}$$

(35)

$$\begin{cases}
\Delta^6(e_1^*) = -a_{11}a_{23}e_4^* \wedge e_1^* \wedge e_2^* - (a_{33}a_{11} + a_{13}a_{24})e_4^* \wedge e_1^* \wedge e_3^* \\
\quad - a_{23}a_{24}e_4^* \wedge e_2^* \wedge e_3^* + a_{11}a_{24}e_3^* \wedge e_1^* \wedge e_2^* + (a_{14}a_{23} - a_{33}a_{11})e_3^* \wedge e_1^* \wedge e_4^* \\
\quad + a_{23}a_{24}e_3^* \wedge e_2^* \wedge e_4^* + a_{33}a_{24}e_1^* \wedge e_2^* \wedge e_3^* + a_{33}a_{23}e_1^* \wedge e_2^* \wedge e_4^* \\
\quad + a_{33}^2 e_1^* \wedge e_3^* \wedge e_4^*, \\
\Delta^6(e_1) = -2a_{14}a_{23}e_1^* \wedge e_2^* \wedge e_1 - (a_{14}a_{11} + a_{14}a_{33})e_1^* \wedge e_3^* \wedge e_1 \\
\quad + (a_{13}a_{11} - a_{13}a_{33})e_1^* \wedge e_4^* \wedge e_1 - 2a_{33}a_{23}e_2^* \wedge e_4^* \wedge e_1 \\
\quad + (-a_{33}^2 + a_{14}a_{23} - a_{24}a_{13})e_3^* \wedge e_4^* \wedge e_1 + (a_{14}a_{11} + a_{14}a_{33})e_2^* \wedge e_3^* \wedge e_2 \\
\quad + (-a_{13}a_{11} - a_{13}a_{33})e_2^* \wedge e_4^* \wedge e_2 - 2a_{13}a_{14}e_3^* \wedge e_4^* \wedge e_2 \\
\quad + (-a_{13}a_{33} + a_{13}a_{11})e_1^* \wedge e_2^* \wedge e_3 + 2a_{13}^2 e_1^* \wedge e_4^* \wedge e_3 \\
\quad - (a_{33}^2 + 2a_{33}a_{11})e_2^* \wedge e_3^* \wedge e_3 + 2a_{13}a_{23}e_2^* \wedge e_4^* \wedge e_3 \\
\quad + 2a_{33}a_{13}e_3^* \wedge e_4^* \wedge e_3 - (a_{14}a_{33} + a_{14}a_{11})e_1^* \wedge e_2^* \wedge e_4 \\
\quad - 2a_{14}^2 e_1^* \wedge e_3^* \wedge e_4 + 2a_{13}a_{14}e_1^* \wedge e_4^* \wedge e_4 - 2a_{14}a_{24}e_2^* \wedge e_3^* \wedge e_4 \\
\quad + (-a_{33}^2 + a_{14}a_{23} - 2a_{33}a_{11} - a_{24}a_{13})e_2^* \wedge e_4^* \wedge e_4 - 2a_{14}a_{33}e_3^* \wedge e_4^* \wedge e_4, \\
\Delta^6(e_2) = (a_{11}a_{24} + a_{33}a_{24})e_1^* \wedge e_3^* \wedge e_1 + (-a_{23}a_{11} + a_{23}a_{33})e_1^* \wedge e_4^* \wedge e_1 \\
\quad + 2a_{23}a_{24}e_3^* \wedge e_4^* \wedge e_1 + 2a_{33}a_{14}e_1^* \wedge e_3^* \wedge e_2 - 2a_{33}a_{11}e_1^* \wedge e_4^* \wedge e_2 \\
\quad + (-a_{24}a_{11} + a_{24}a_{33})e_2^* \wedge e_3^* \wedge e_2 + (a_{33}a_{11} - a_{14}a_{33})e_2^* \wedge e_4^* \wedge e_2 \\
\quad + (a_{14}a_{23} + a_{24}a_{13} + a_{33}^2)e_3^* \wedge e_4^* \wedge e_2 - (a_{23}a_{33} - a_{23}a_{11})e_1^* \wedge e_2^* \wedge e_3 \\
\quad + (-a_{33}^2 + a_{14}a_{23} - 2a_{33}a_{11} - a_{13}a_{24})e_1^* \wedge e_3^* \wedge e_3 - 2a_{13}a_{23}e_1^* \wedge e_4^* \wedge e_3 \\
\quad - 2a_{23}^2 e_2^* \wedge e_4^* \wedge e_3 - 2a_{33}a_{23}e_3^* \wedge e_4^* \wedge e_3 + (-a_{24}a_{33} + a_{24}a_{11})e_1^* \wedge e_2^* \wedge e_4 \\
\quad + 2a_{14}a_{24}e_1^* \wedge e_3^* \wedge e_4 + (a_{33}^2 - a_{13}a_{24} - 2a_{33}a_{11} + a_{14}a_{23})e_1^* \wedge e_4^* \wedge e_4 \\
\quad + 2a_{24}^2 e_2^* \wedge e_3^* \wedge e_4, \\
\Delta^6(e_3) = -2a_{24}a_{11}e_1^* \wedge e_2^* \wedge e_1 - (-a_{11}^2 + 2a_{33}a_{11} + a_{13}a_{24} - a_{14}a_{23})e_1^* \wedge e_4^* \wedge e_1 \\
\quad - 2a_{23}a_{24}e_2^* \wedge e_4^* \wedge e_1 + (2a_{24}a_{11} - a_{33}a_{24})e_3^* \wedge e_4^* \wedge e_1 + 2a_{14}a_{11}e_1^* \wedge e_2^* \wedge e_2 \\
\quad + 2a_{14}^2 e_1^* \wedge e_3^* \wedge e_2 + 2a_{13}a_{14}e_1^* \wedge e_4^* \wedge e_2 + (a_{11}^2 + 2a_{33}a_{11})e_2^* \wedge e_4^* \wedge e_2 \\
\quad + (a_{14}a_{11} + a_{14}a_{33})e_3^* \wedge e_4^* \wedge e_2 - (a_{11}^2 + a_{14}a_{23} + a_{13}a_{24})e_1^* \wedge e_2^* \wedge e_3 \\
\quad - (a_{14}a_{33} - a_{14}a_{11})e_1^* \wedge e_3^* \wedge e_3 - 2a_{13}a_{11}e_1^* \wedge e_4^* \wedge e_3 + (a_{33}a_{24} \\
\quad - a_{11}a_{24})e_2^* \wedge e_3^* \wedge e_3 + (-2a_{11}a_{23} - a_{13}a_{11})e_2^* \wedge e_4^* \wedge e_3 \\
\quad + (2a_{13}a_{24} - 2a_{14}a_{23})e_3^* \wedge e_4^* \wedge e_3 - 2a_{14}a_{24}e_1^* \wedge e_2^* \wedge e_4 \\
\quad + (a_{11}a_{14} + a_{33}a_{14})e_1^* \wedge e_4^* \wedge e_4 + (-a_{11}a_{24} - a_{33}a_{24})e_2^* \wedge e_4^* \wedge e_4,
\end{cases}$$

$$\begin{cases}\Delta^6(e_4) = 2a_{23}a_{11}e_1^*\wedge e_2^*\wedge e_1 + (a_{11}^2+2a_{33}a_{11}-a_{14}a_{23}+a_{13}a_{24})e_1^*\wedge e_3^*\wedge e_1\\ \quad +2a_{23}a_{24}e_2^*\wedge e_3^*\wedge e_1 + 2a_{23}^2e_2^*\wedge e_4^*\wedge e_1 + (a_{23}a_{11}+a_{23}a_{33})e_3^*\wedge e_4^*\wedge e_1\\ \quad -2a_{13}a_{11}e_1^*\wedge e_2^*\wedge e_2 - 2 - a_{13}^2e_1^*\wedge e_4^*\wedge e_2 + (-a_{11}^2+2a_{33}a_{11})e_2^*\wedge e_3^*\wedge e_2\\ \quad +(a_{13}a_{11}-a_{33}a_{13})e_3^*\wedge e_4^*\wedge e_2 + 2a_{13}a_{23}e_1^*\wedge e_2^*\wedge e_3\\ \quad +(a_{13}a_{11}+a_{13}a_{33})e_1^*\wedge e_3^*\wedge e_3 + (a_{11}a_{23}-a_{33}a_{23})e_2^*\wedge e_3^*\wedge e_3\\ \quad +(a_{11}^2+a_{13}a_{24}+a_{23}a_{14})e_1^*\wedge e_2^*\wedge e_4 + 2a_{11}a_{14}e_1^*\wedge e_3^*\wedge e_4\\ \quad +(-a_{33}a_{13}+a_{11}a_{13})e_1^*\wedge e_4^*\wedge e_4 + 2a_{11}a_{24}e_2^*\wedge e_3^*\wedge e_4\\ \quad +(a_{23}a_{33}+a_{11}a_{23})e_2^*\wedge e_4^*\wedge e_4 + (2a_{23}a_{14}-2a_{13}a_{24})e_3^*\wedge e_4^*\wedge e_4,\\ \Delta^6(e_2^*) = a_{13}a_{11}e_4^*\wedge e_1^*\wedge e_2^* + a_{13}a_{14}e_4^*\wedge e_1^*\wedge e_3^* + (a_{23}a_{14}-a_{33}b_{33})e_4^*\wedge e_2^*\wedge e_3^*\\ \quad -a_{14}a_{11}e_3^*\wedge e_1^*\wedge e_2^* + a_{14} - a_{13}e_3^*\wedge e_1^*\wedge e_4^* - (a_{33}a_{11}+a_{13}a_{24})e_3^*\wedge e_2^*\wedge e_4^*\\ \quad -a_{33}a_{14}e_2^*\wedge e_1^*\wedge e_3^* - a_{33}a_{13}e_2^*\wedge e_1^*\wedge e_4^* - a_{33}^2e_2^*\wedge e_3^*\wedge e_4^*,\\ \Delta^6(e_3^*) = a_{11}^2e_3^*\wedge e_1^*\wedge e_2^* + a_{13}a_{11}e_3^*\wedge e_1^*\wedge e_4^* + a_{11}a_{23}e_3^*\wedge e_2^*\wedge e_4^*\\ \quad +(a_{33}a_{11}+a_{13}a_{24})e_2^*\wedge e_1^*\wedge e_3^* + a_{13}a_{23}e_2^*\wedge e_1^*\wedge e_4^* + a_{33}a_{23}e_2^*\wedge e_3^*\wedge e_4^*\\ \quad -(a_{23}a_{14}-a_{33}a_{11})e_1^*\wedge e_2^*\wedge e_3^* - a_{13}a_{23}e_1^*\wedge e_2^*\wedge e_4^* - a_{33}a_{13}e_1^*\wedge e_3^*\wedge e_4^*,\\ \Delta^6(e_4^*) = -a_{11}^2e_4^*\wedge e_1^*\wedge e_2^* - a_{14}a_{11}e_4^*\wedge e_1^*\wedge e_3^* - a_{11}a_{24}e_4^*\wedge e_2^*\wedge e_3^*\\ \quad +a_{14}a_{24}e_2^*\wedge e_1^*\wedge e_3^* - a_{33}a_{24}e_2^*\wedge e_3^*\wedge e_4^* + a_{24}a_{14}e_1^*\wedge e_2^*\wedge e_3^*\\ \quad -(a_{23}a_{14}-a_{33}a_{11})e_2^*\wedge e_1^*\wedge e_4^* + (a_{33}a_{11}+a_{13}a_{24})e_1^*\wedge e_2^*\wedge e_4^*\\ \quad +a_{33}a_{14}e_1^*\wedge e_3^*\wedge e_4^*.\end{cases}$$

$$\Delta^{1*}(e_4,e_1,e_l^*) = -a_{1l}a_{13}e_1^* + a_{1l}a_{11}e_3^*,\ \Delta^{1*}(e_3,e_1,e_l^*) = a_{1l}a_{14}e_1^* - a_{1l}a_{11}e_4^*,\quad 1\le l\le 4;$$

$$\begin{cases}\Delta^{2*}(e_4,e_1,e_1^*) = (a_{11}^2-a_{12}a_{21})e_3^*, \qquad \Delta^{2*}(e_4,e_1,e_2^*) = (a_{11}a_{12}-a_{12}a_{22})e_3^*,\\ \Delta^{2*}(e_4,e_2,e_1^*) = (a_{21}a_{11}-a_{22}a_{21})e_3^*,\\ \Delta^{2*}(e_4,e_2,e_2^*) = (a_{21}a_{12}-a_{22}^2)e_3^*,\\ \Delta^{2*}(e_3,e_1,e_1^*) = (a_{12}a_{21}-a_{11}^2)e_4^*,\\ \Delta^{2*}(e_3,e_1,e_2^*) = (a_{12}a_{22}-a_{11}a_{12})e_4^*,\\ \Delta^{2*}(e_3,e_2,e_1^*) = (a_{22}a_{21}-a_{21}a_{11})e_4^*,\\ \Delta^{2*}(e_3,e_2,e_2^*) = (a_{22}^2-a_{21}a_{12})e_4^*,\\ \Delta^{2*}(e_3^*,e_1,e_2) = (a_{11}a_{22}-a_{12}a_{21})e_3^*,\\ \Delta^{2*}(e_4^*,e_1,e_2) = (a_{12}a_{21}-a_{11}a_{22})e_4^*,\\ \Delta^{2*}(e_3,e_1,e_2) = (a_{12}a_{21}-a_{11}a_{22})e_3,\\ \Delta^{2*}(e_4,e_1,e_2) = (a_{11}a_{22}-a_{12}a_{21})e_4;\end{cases}$$

$$\begin{cases} \Delta^{3*}(e_4, e_1, e_3^*) = -a_{13}^2 e_1^* + a_{13}a_{23}e_2^*, & \Delta^{3*}(e_4, e_1, e_4^*) = -a_{13}a_{14}e_1^* + a_{13}a_{24}e_2^*, \\ \Delta^{3*}(e_4, e_2, e_3^*) = -a_{14}a_{13}e_1^* + a_{23}^2 e_2^*, & \Delta^{3*}(e_4, e_2, e_4^*) = -a_{14}^2 e_1^* + a_{23}a_{24}e_2^*, \\ \Delta^{3*}(e_3, e_2, e_3^*) = a_{13}a_{24}e_1^* - a_{23}a_{24}e_2^*, & \Delta^{3*}(e_3, e_2, e_4^*) = a_{14}a_{24}e_1^* - a_{24}^2 e_2^*, \\ \Delta^{3*}(e_3, e_1, e_3^*) = a_{13}a_{14}e_1^* - a_{14}a_{23}e_2^*, & \Delta^{3*}(e_3, e_1, e_4^*) = a_{14}^2 e_1^* - a_{14}a_{24}e_2^*, \\ \Delta^{3*}(e_1^*, e_1, e_2) = (a_{13}a_{24} - a_{14}a_{23})e_1^*, & \Delta^{3*}(e_2^*, e_1, e_2) = (a_{14}a_{23} - a_{13}a_{24})e_2^*, \\ \Delta^{3*}(e_2, e_1, e_3^*) = a_{14}a_{23}e_3^* - 2a_{13}a_{23}e_4^*, & \\ \Delta^{3*}(e_2, e_1, e_4^*) = a_{14}a_{24}e_3^* - (a_{23}a_{14} + a_{13}a_{24})e_4^*; & \end{cases}$$

$$\begin{cases} \Delta^{4*}(e_3, e_1, e_2) = -a_{12}a_{24}e_2, & \Delta^{4*}(e_3, e_2, e_2^*) = a_{12}a_{24}e_1^*, \\ \Delta^{4*}(e_3, e_1, e_1^*) = a_{12}a_{24}e_1^*, & \Delta^{4*}(e_4^*, e_1, e_2) = -a_{12}a_{24}e_1^*, \\ \Delta^{4*}(e_3^*, e_1, e_2) = -a_{12}a_{24}e_1^*, & \Delta^{4*}(e_4^*, e_2, e_3) = a_{24}^2 e_2^*, \\ \Delta^{4*}(e_4, e_4^*, e_1) = a_{12}a_{24}e_3^*, & \Delta^{4*}(e_2^*, e_1, e_2) = a_{12}a_{24}e_3^*, \\ \Delta^{4*}(e_1^*, e_2, e_3) = -a_{24}^2 e_3^*, & \Delta^{4*}(e_3, e_1, e_4^*) = -a_{12}a_{24}e_4^*, \\ \Delta^{4*}(e_1^*, e_1, e_2) = a_{12}a_{24}e_4^*; & \end{cases}$$

$$\begin{cases} \Delta^{5*}(e_3, e_1, e_2) = -a_{14}a_{21}e_1, & \Delta^{5*}(e_4, e_2, e_3) = -a_{14}a_{21}e_4, \\ \Delta^{5*}(e_4^*, e_1, e_3) = -a_{14}^2 e_1^*, & \Delta^{5*}(e_3, e_1^*, e_1) = a_{14}a_{21}e_2^*, \\ \Delta^{5*}(e_3, e_2, e_2^*) = -a_{14}a_{21}e_2^*, & \Delta^{5*}(e_4^*, e_1, e_2) = -a_{14}a_{21}e_2^*, \\ \Delta^{5*}(e_3^*, e_2, e_3) = -a_{14}a_{21}e_3^*, & \Delta^{5*}(e_2^*, e_1, e_3) = a_{14}^2 e_3^*, \\ \Delta^{5*}(e_3, e_4^*, e_2) = 2a_{14}a_{21}e_4^*; & \end{cases}$$

$$\begin{cases} \Delta^{6*}(e_1, e_2, e_3) = a_{24}(a_{11} + a_{33})e_1 - a_{14}(a_{33} + a_{11})e_2 + 2a_{14}a_{24}e_4 \\ \qquad - (-a_{11}^2 + a_{13}a_{24} + a_{14}a_{23})e_3, \\ \Delta^{6*}(e_1, e_2, e_4) = -a_{23}(a_{33} - a_{11})e_1 + -a_{13}(a_{11} + a_{33})e_2 - 2a_{13}a_{23}e_3 \\ \qquad + (-a_{11}^2 + a_{14}a_{23} + a_{13}a_{24})e_4, \\ \Delta^{6*}(e_1, e_3, e_4) = -(-a_{33}^2 + a_{13}a_{24} + a_{23}a_{14})e_1 + 2a_{13}a_{14}e_2 \\ \qquad + (-a_{33}a_{13} - a_{13}a_{11})e_3 - (a_{14}a_{11} - a_{33}a_{14})e_4, \\ \Delta^{6*}(e_2, e_3, e_4) = -2a_{23}a_{24}e_1 + (a_{14}a_{23} + a_{13}a_{24})e_2 - (a_{23}a_{11} - a_{33}a_{23})e_3 \\ \qquad - (a_{24}a_{11} + a_{33}a_{24})e_4, \\ \Delta^{6*}(e_1, e_2, e_1^*) = -2a_{14}a_{23}e_1 - 2a_{24}a_{11}e_3 + 2a_{23}a_{11}e_4, \\ \Delta^{6*}(e_1, e_3, e_1^*) = (-a_{14}a_{11} - a_{14}a_{33})e_1 + (a_{24}a_{11} + a_{33}a_{24})e_2 \\ \qquad + (a_{11}^2 + 2a_{33}a_{11} - a_{14}a_{23} + a_{13}a_{24})e_4, \\ \Delta^{6*}(e_1, e_4, e_1^*) = (a_{13}a_{11} - a_{13}a_{33})e_1 + (-a_{23}a_{11} + a_{23}a_{33})e_2 \\ \qquad + (-a_{11}^2 + 2a_{33}a_{11} + a_{13}a_{24} - a_{14}a_{23})e_3, \\ \Delta^{6*}(e_2, e_4, e_1^*) = -2a_{33}a_{23}e_1 - 2a_{23}a_{24}e_3 + 2a_{23}^2 e_4, \end{cases}$$

$$
\begin{cases}
\Delta^{6*}(e_3,e_4,e_1^*) = (-a_{33}^2+a_{14}a_{23}-a_{24}a_{13})e_1+2a_{23}a_{24}e_2\\
\qquad\qquad\qquad + (a_{24}a_{11}-a_{33}a_{24})e_3+(a_{23}a_{11}+a_{23}a_{33})e_4,\\
\Delta^{6*}(e_1,e_2,e_2^*) = 2a_{14}a_{11}e_3-2a_{13}a_{11}e_4,\\
\Delta^{6*}(e_2,e_3,e_1^*) = 2a_{23}a_{24}e_4,\\
\Delta^{6*}(e_1,e_3,e_2^*) = 2a_{33}a_{14}e_2+2a_{14}^2e_3,\\
\Delta^{6*}(e_1,e_4,e_2^*) = -2a_{33}a_{11}e_2+2a_{13}a_{14}e_3+2a_{13}^2e_4,\\
\Delta^{6*}(e_2,e_3,e_2^*) = (a_{14}a_{11}+a_{14}a_{33})e_1+(-a_{24}a_{11}\\
\qquad\qquad\qquad + a_{24}a_{33})e_2+(-a_{11}^2+2a_{33}a_{11})e_4,\\
\Delta^{6*}(e_2,e_4,e_2^*) = (-a_{13}a_{11}-a_{13}a_{33})e_1+(a_{33}a_{11}-a_{14}a_{33})e_2\\
\qquad\qquad\qquad + (a_{11}^2+2a_{33}a_{11})e_3,\\
\Delta^{6*}(e_3,e_4,e_2^*) = -2a_{13}a_{14}e_1+(a_{33}^2+a_{14}a_{23}+a_{24}a_{13})e_2\\
\qquad\qquad\qquad + (a_{14}a_{11}+a_{14}a_{33})e_3+(a_{13}a_{11}-a_{13}a_{33})e_4,\\
\Delta^{6*}(e_1,e_2,e_3^*) = (-a_{13}a_{33}+a_{13}a_{11})e_1-(a_{23}a_{33}+a_{23}a_{11})e_2\\
\qquad\qquad\qquad + (-a_{11}^2-a_{14}a_{23}-a_{13}a_{24})e_3+2a_{13}a_{23}e_4,\\
\Delta^{6*}(e_1,e_3,e_3^*) = (-a_{33}^2+a_{14}a_{23}-2a_{33}a_{11}-a_{13}a_{24})e_2\\
\qquad\qquad\qquad - (a_{14}a_{33}+a_{14}a_{11})e_3+(a_{13}a_{11}+a_{13}a_{33})e_4,\\
\Delta^{6*}(e_1,e_4,e_3^*) = 2a_{13}^2e_1+2a_{23}a_{23}e_2+2-a_{13}a_{11}e_3,\\
\Delta^{6*}(e_2,e_3,e_3^*) = (-a_{33}^2-2a_{33}a_{11})e_1+(a_{33}a_{24}-a_{11}a_{24})e_3\\
\qquad\qquad\qquad + (a_{11}a_{23}-a_{33}a_{23})e_4,\\
\Delta^{6*}(e_2,e_4,e_3^*) = 2a_{13}a_{23}e_1-2a_{23}^2e_2-(2a_{11}a_{23}+a_{11}a_{13})e_3,\\
\Delta^{6*}(e_3,e_4,e_3^*) = 2a_{33}a_{13}e_1-2a_{33}a_{23}e_2+(2a_{13}a_{24}-2a_{14}a_{23})e_3,\\
\Delta^{6*}(e_1,e_2,e_4^*) = -(a_{14}a_{33}+a_{14}a_{11})e_1+(-a_{24}a_{33}+a_{24}a_{11})e_2-2a_{14}a_{24}e_3\\
\qquad\qquad\qquad + (a_{11}^2+a_{13}a_{24}+a_{23}a_{14})e_4,\\
\Delta^{6*}(e_1,e_3,e_4^*) = -2a_{14}^2e_1+2a_{14}a_{24}e_2+2a_{14}a_{11}e_4,\\
\Delta^{6*}(e_1,e_4,e_4^*) = 2a_{13}a_{14}e_1+(a_{33}^2-a_{13}a_{24}-2a_{33}a_{11}+a_{14}a_{23})e_2\\
\qquad\qquad\qquad + (a_{11}a_{14}+a_{33}a_{14})e_3+(-a_{33}a_{13}+a_{11}a_{13})e_4,\\
\Delta^{6*}(e_2,e_3,e_4^*) = -2a_{14}a_{24}e_1+2a_{24}^2e_2+2a_{11}a_{24}e_4,\\
\Delta^{6*}(e_2,e_4,e_4^*) = (-a_{33}^2-a_{14}a_{23}-2a_{33}a_{11}-a_{24}a_{13})e_1-(a_{11}a_{24}+a_{33}a_{24})e_3\\
\qquad\qquad\qquad + (a_{23}a_{33}+a_{11}a_{23})e_4\\
\Delta^{6*}(e_3,e_4,e_4^*) = -2a_{14}a_{33}e_1+(2a_{23}a_{14}-2a_{13}a_{24})e_4.
\end{cases}
$$

Proof. According to (1), (29) and Theorem 2.5, for any Rota-Baxter op-

erator $P : A_4 \to A_4$ of weight zero defined as (12), we have
(36)

$$\begin{cases}
\Delta(e_1^*) = A_{12}^{13}(-e_4^*) \wedge e_1^* \wedge e_2^* + A_{13}^{13}(-e_4^*) \wedge e_1^* \wedge e_3^* + A_{24}^{14} e_3^* \wedge e_2^* \wedge e_4^* \\
\quad + A_{23}^{13}(-e_4^*) \wedge e_2^* \wedge e_3^* + A_{12}^{14} e_3^* \wedge e_1^* \wedge e_2^* + A_{14}^{14} e_3^* \wedge e_1^* \wedge e_4^* \\
\quad + A_{23}^{34}(-e_1^*) \wedge e_2^* \wedge e_3^* + A_{24}^{34}(-e_1^*) \wedge e_2^* \wedge e_4^* + A_{34}^{34}(-e_1^*) \wedge e_3^* \wedge e_4^*, \\
\Delta(e_2^*) = A_{12}^{23} e_4^* \wedge e_1^* \wedge e_2^* + A_{13}^{23} e_4^* \wedge e_1^* \wedge e_3^* + A_{23}^{23} e_4^* \wedge e_2^* \wedge e_3^* \\
\quad + A_{12}^{24}(-e_3^*) \wedge e_1^* \wedge e_2^* + A_{14}^{24}(-e_3^*) \wedge e_1^* \wedge e_4^* + A_{24}^{24}(-e_3)^* \wedge e_2^* \wedge e_4^* \\
\quad + A_{13}^{34} e_2^* \wedge e_1^* \wedge e_3^* + A_{14}^{34} e_2^* \wedge e_1^* \wedge e_4^* + A_{34}^{34} e_2^* \wedge e_3^* \wedge e_4^*, \\
\Delta(e_3^*) = A_{12}^{12}(-e_3^*) \wedge e_1^* \wedge e_2^* + A_{14}^{12} \wedge e_1^* \wedge e_4^* + A_{24}^{12}(-e_3^*) \wedge e_2^* \wedge e_4^* \\
\quad + A_{13}^{13} e_2^* \wedge e_1^* \wedge e_3^* + A_{14}^{13} e_2^* \wedge e_1^* \wedge e_4^* + A_{34}^{13} e_2^* \wedge e_3^* \wedge e_4^* \\
\quad + A_{23}^{23}(-e_1^*) \wedge e_2^* \wedge e_3^* + A_{24}^{23}(-e_1^*) \wedge e_2^* \wedge e_4^* + A_{34}^{23}(-e_1^*) \wedge e_3^* \wedge e_4^*, \\
\Delta(e_4^*) = A_{12}^{12} e_4^* \wedge e_1^* \wedge e_2^* + A_{13}^{12} e_4^* \wedge e_1^* \wedge e_3^* + A_{23}^{12} e_4^* \wedge e_2^* \wedge e_3^* \\
\quad + A_{13}^{14}(-e_2^*) \wedge e_1^* \wedge e_3^* + A_{14}^{14}(-e_2^*) \wedge e_1^* \wedge e_4^* + A_{34}^{14}(-e_2)^* \wedge e_3^* \wedge e_4^* \\
\quad + A_{23}^{24} e_1^* \wedge e_2^* \wedge e_3^* + A_{24}^{24} e_1^* \wedge e_2^* \wedge e_4^* + A_{34}^{24} e_1^* \wedge e_3^* \wedge e_4^*, \\
\Delta(e_1) = \sum_{k,i} (a_{i3} a_{1k} e_4^* \wedge e_k \wedge e_i^* + a_{i4} a_{1k} e_3^* \wedge e_i^* \wedge e_k + a_{i2} a_{3k} e_3^* \wedge e_k \wedge e_i^* \\
\quad + a_{i3} a_{3k} e_2^* \wedge e_i^* \wedge e_k + a_{i2} a_{4k} e_4^* \wedge e_i^* \wedge e_k + a_{i4} a_{4k} e_2^* \wedge e_k \wedge e_i^*) \\
\quad + A_{12}^{23} e_3 \wedge e_1^* \wedge e_2^* + A_{13}^{23} e_3 \wedge e_1^* \wedge e_3^* + A_{14}^{23} e_3 \wedge e_1^* \wedge e_4^* + A_{23}^{23} e_3 \wedge e_2^* \wedge e_3^* \\
\quad + A_{24}^{23} e_3 \wedge e_2^* \wedge e_4^* + A_{34}^{23} e_3 \wedge e_3^* \wedge e_4^* - A_{12}^{24} e_4 \wedge e_1^* \wedge e_2^* - A_{13}^{24} e_4 \wedge e_1^* \wedge e_3^* \\
\quad - A_{14}^{24} e_4 \wedge e_1^* \wedge e_4^* - A_{23}^{24} e_4 \wedge e_2^* \wedge e_3^* - A_{24}^{24} e_4 \wedge e_2^* \wedge e_4^* - A_{34}^{24} e_4 \wedge e_3^* \wedge e_4^* \\
\quad + A_{12}^{34} e_1 \wedge e_1^* \wedge e_2^* + A_{13}^{34} e_1 \wedge e_1^* \wedge e_3^* + A_{14}^{34} e_1 \wedge e_1^* \wedge e_4^* + A_{23}^{34} e_1 \wedge e_2^* \wedge e_3^* \\
\quad + A_{24}^{34} e_1 \wedge e_2^* \wedge e_4^* + A_{34}^{34} e_1 \wedge e_3^* \wedge e_4^*, \\
\Delta(e_2) = \sum_{j,l} (a_{j3} a_{2l} e_4^* \wedge e_j^* \wedge e_l + a_{j4} a_{2l} e_3^* \wedge e_l \wedge e_j^* + a_{j1} a_{3l} e_3^* \wedge e_j^* \wedge e_l \\
\quad + a_{j3} a_{3l} e_1^* \wedge e_l \wedge e_j^* + a_{j1} a_{4l} e_4^* \wedge e_l \wedge e_j^* + a_{j4} a_{4l} e_1^* \wedge e_j^* \wedge e_l) \\
\quad - A_{12}^{13} e_3 \wedge e_1^* \wedge e_2^* - A_{13}^{13} e_3 \wedge e_1^* \wedge e_3^* - A_{14}^{13} e_3 \wedge e_1^* \wedge e_4^* - A_{23}^{13} e_3 \wedge e_2^* \wedge e_3^* \\
\quad - A_{24}^{13} e_3 \wedge e_2^* \wedge e_4^* - A_{34}^{13} e_3 \wedge e_3^* \wedge e_4^* + A_{12}^{14} e_4 \wedge e_1^* \wedge e_2^* + A_{13}^{34} e_4 \wedge e_1^* \wedge e_3^* \\
\quad + A_{14}^{14} e_4 \wedge e_1^* \wedge e_4^* + A_{23}^{14} e_4 \wedge e_2^* \wedge e_3^* + A_{14}^{14} e_4 \wedge e_2^* \wedge e_4^* + A_{34}^{14} e_4 \wedge e_3^* \wedge e_4^* \\
\quad - A_{12}^{34} e_2 \wedge e_1^* \wedge e_2^* - A_{13}^{34} e_2 \wedge e_1^* \wedge e_3^* - A_{14}^{34} e_2 \wedge e_1^* \wedge e_4^* - A_{23}^{34} e_2 \wedge e_2^* \wedge e_3^* \\
\quad - A_{24}^{34} e_2 \wedge e_2^* \wedge e_4^* - A_{34}^{34} e_2 \wedge e_3^* \wedge e_4^*,
\end{cases}$$

$$\begin{cases}\Delta(e_3)=\sum\limits_{j,l}(a_{j1}a_{1l}e_4^*\wedge e_j^*\wedge e_l+a_{j4}a_{1l}e_1^*\wedge e_l\wedge e_j^*+a_{j2}a_{2l}e_4^*\wedge e_l\wedge e_j^*\\ \quad +a_{j4}a_{2l}e_2^*\wedge e_j^*\wedge e_l+a_{j1}a_{3l}e_2^*\wedge e_l\wedge e_j^*+a_{j2}a_{3l}e_1^*\wedge e_j^*\wedge e_l)\\ \quad +A_{12}^{12}e_3\wedge e_1^*\wedge e_2^*+A_{13}^{12}e_3\wedge e_1^*\wedge e_3^*+A_{14}^{12}e_3\wedge e_1^*\wedge e_4^*+A_{23}^{12}e_3\wedge e_2^*\wedge e_3^*\\ \quad +A_{24}^{12}e_3\wedge e_2^*\wedge e_4^*+A_{34}^{12}e_3\wedge e_3^*\wedge e_4^*-A_{12}^{14}e_1\wedge e_1^*\wedge e_2^*-A_{13}^{14}e_1\wedge e_1^*\wedge e_3^*\\ \quad -A_{14}^{14}e_1\wedge e_1^*\wedge e_4^*-A_{23}^{14}e_1\wedge e_2^*\wedge e_3^*-A_{24}^{14}e_1\wedge e_2^*\wedge e_4^*-A_{34}^{14}e_1\wedge e_3^*\wedge e_4^*\\ \quad +A_{12}^{24}e_2\wedge e_1^*\wedge e_2^*+A_{13}^{24}e_2\wedge e_1^*\wedge e_3^*+A_{14}^{24}e_2\wedge e_1^*\wedge e_4^*+A_{23}^{24}e_2\wedge e_2^*\wedge e_3^*\\ \quad +A_{24}^{24}e_2\wedge e_2^*\wedge e_4^*+A_{34}^{24}e_2\wedge e_3^*\wedge e_4^*,\\ \Delta(e_4)=\sum\limits_{j,l}(a_{j1}a_{1l}e_3^*\wedge e_l\wedge e_j^*+a_{j3}a_{1l}e_1^*\wedge e_j^*\wedge e_l+a_{j2}a_{2l}e_3^*\wedge e_j^*\wedge e_l\\ \quad +a_{j3}a_{2l}e_2^*\wedge e_l\wedge e_j^*+a_{j1}a_{4l}e_2^*\wedge e_j^*\wedge e_l+a_{j2}a_{4l}e_1^*\wedge e_l\wedge e_j^*)\\ \quad -A_{12}^{12}e_4\wedge e_1^*\wedge e_2^*-A_{13}^{12}e_4\wedge e_1^*\wedge e_3^*-A_{14}^{12}e_4\wedge e_1^*\wedge e_4^*-A_{23}^{12}e_4\wedge e_2^*\wedge e_3^*\\ \quad -A_{24}^{12}e_4\wedge e_2^*\wedge e_4^*-A_{34}^{12}e_4\wedge e_3^*\wedge e_4^*+A_{12}^{13}e_1\wedge e_1^*\wedge e_2^*+A_{13}^{13}e_1\wedge e_1^*\wedge e_3^*\\ \quad +A_{14}^{13}e_1\wedge e_1^*\wedge e_4^*+A_{23}^{13}e_1\wedge e_2^*\wedge e_3^*+A_{14}^{13}e_1\wedge e_2^*\wedge e_4^*+A_{34}^{13}e_1\wedge e_3^*\wedge e_4^*\\ \quad -A_{12}^{23}e_2\wedge e_1^*\wedge e_2^*-A_{13}^{23}e_2\wedge e_1^*\wedge e_3^*-A_{14}^{23}e_2\wedge e_1^*\wedge e_4^*-A_{23}^{23}e_2\wedge e_2^*\wedge e_3^*\\ \quad -A_{24}^{23}e_2\wedge e_2^*\wedge e_4^*-A_{34}^{23}e_2\wedge e_3^*\wedge e_4^*.\end{cases}$$

Therefore, we get that

(1) If $Rank(P)=1$, $P(e_1)=\sum\limits_{l=1}^{4}a_{1l}e_1\neq 0$, and $P(e_2)=P(e_3)=P(e_4)=0$, then by submitting $(a_{11},a_{12},a_{13},a_{14})\neq(0,0,0,0)$ and $a_{ij}=0$ into (36), $2\leq i\leq 4$, $1\leq j\leq 4$, we obtain the local cocycle 3-Lie bialgebra $(A_4\ltimes_{\mathrm{ad}^*}A_4^*,\Delta^1)$, which satisfies (30).

(2) If $Rank(P)=2$, $P(e_1)=a_{11}e_1+a_{12}e_2, P(e_2)=a_{21}e_1+a_{22}e_2$, and $P(e_3)=P(e_4)=0$, then the local cocycle 3-Lie bialgebra $(A_4\ltimes_{\mathrm{ad}^*}A_4^*,\Delta^2)$ satisfies (31).

If $Rank(P)=2$, $P(e_1)=a_{13}e_3+a_{14}e_4,\ P(e_2)=a_{23}e_3+a_{24}e_4$, $P(e_3)=P(e_4)=0$, and $a_{13}a_{24}-a_{14}a_{23}\neq 0$, then $(A_4\ltimes_{\mathrm{ad}^*}A^*,\Delta^3)$ satisfies (32).

(3) If $Rank(P)=3$, $P(e_1)=a_{12}e_2$, $P(e_2)=a_{24}e_4$, $P(e_3)=-a_{24}e_1$, $P(e_4)=0$, and $a_{12}a_{24}\neq 0$, then $(A_4\ltimes_{\mathrm{ad}^*}A_4^*,\Delta^4)$ satisfies (33).

If $P(e_1)=a_{14}e_4$, $P(e_2)=a_{21}e_1$, $P(e_3)=-a_{14}e_2$, $P(e_4)=0$, and $a_{21}a_{14}\neq 0$, then the local cocycle 3-Lie bialgebra $(A_4\ltimes_{\mathrm{ad}^*}A_4^*,\Delta^5)$ satisfies (34).

(4) If $Rank(P)=4$, then $(A_4\ltimes_{\mathrm{ad}^*}A_4^*,\Delta^6)$ satisfies (35).

Furthermore, set

$$\Delta^{1*}(e_i, e_j, e_k^*) = \sum_{i=1}^{4}(\lambda_i e_i + \mu_i e_i^*), \quad \forall\ \lambda_i, \mu_i \in \mathbb{C}. \tag{37}$$

By (5) and (37),

$$\langle \Delta^{1*}(e_4, e_1, e_l^*), e_1\rangle = \langle e_4 \wedge e_1 \wedge e_l^*, \Delta e_1\rangle = -a_{1l}a_{13} = \langle \sum_{i=1}^{n}(\lambda_i e_i + \mu_i e_i^*), e_1\rangle = \mu_1,$$

$$\langle \Delta^{1*}(e_4, e_1, e_l^*), e_3\rangle = \langle e_4 \wedge e_1 \wedge e_l^*, \Delta e_3\rangle = a_{1l}a_{11} = \langle \sum_{i=1}^{n}(\lambda_i e_i + \mu_i e_i^*), e_1\rangle = \mu_3,$$

$$\langle \Delta^{1*}(e_4, e_1, e_l^*), e_2\rangle = \langle \Delta^{*}(e_4, e_1, e_l^*), e_4\rangle = 0 = \langle \sum_{i=1}^{n}(\lambda_i e_i + \mu_i e_i^*), e_1\rangle = \mu_4,$$

$$\langle \Delta^{1*}(e_4, e_1, e_l^*), e_s^*\rangle = 0 = \langle \sum_{i=1}^{n}(\lambda_i e_i + \mu_i e_i^*), e_1\rangle = \lambda_s, \quad 1 \le s \le 4.$$

Therefore, $\mu_1 = -a_{1l}a_{13}$, $\mu_3 = a_{1l}a_{11}$, $\mu_2 = \mu_4 = \lambda_s = 0$, $1 \le s \le 4$, $\Delta^{1*}(e_4, e_1, e_l^*) = -a_{1l}a_{13}e_1^* + a_{1l}a_{11}e_3^*$, $\Delta^{1*}(e_3, e_1, e_l^*) = a_{1l}a_{14}e_1^* - a_{1l}a_{11}e_4^*$, and others are zero.

By a completely similar discussion to the above, if $Rank(P) \ge 2$, we get $(A_4 \oplus A_4^*, \Delta^{k*})$ for $2 \le k \le 6$. The proof is complete. $\square$

5. 3-PRE-LIE ALGEBRAS INDUCED BY ROTA-BAXTER OPERATORS

A Rota-Baxter operator of weight zero on a 3-Lie algebra can induce a 3-pre-Lie algebra. Then in this section we will construct 3-pre-Lie algebras by the Rota-Baxter operators of weight zero on A_4 obtained in Section 3.

LEMMA 5.1. *Let A be a 3-Lie algebra with a basis $\{e_1, \cdots, e_n\}$, and P be a Rota-Baxter operator of weight zero on A defined as* (12). *Then P induces a 3-pre-Lie algebra $(A, \{,,\})$, where*

$$\{e_l, e_m, e_q\} = \sum_{j<k}^{n} A_{lm}^{jk}[e_j, e_k, e_q], \qquad 1 \le l, m, q \le n, \tag{38}$$

and the sub-adjacent 3-Lie algebra $(A, [,,]_C)$ of $(A, \{,,\})$ is given by

$$[e_l, e_m, e_q]_C = \sum_{j<k}^{n} A_{lm}^{jk}[e_j, e_k, e_q] + \sum_{k<i}^{n} A_{mq}^{ki}[e_k, e_i, e_l] + \sum_{i<j}^{n} A_{ql}^{ij}[e_i, e_j, e_m], \tag{39}$$

where A_{lm}^{jk}, $1 \le l, m, j, k \le n$, are defined as (13).

Proof. By Proposition 3.25 in [6], for all $(l,m,q)\in\mathbb{Z}^{\otimes 3}$, $1\le l,m,q\le n$, we have

$$\{e_l,e_m,e_q\}=[P(e_l),P(e_m),e_q]=\sum_{j<k}^{n}\begin{vmatrix} a_{lj} & a_{lk}\\ a_{mj} & a_{mk}\end{vmatrix}[e_j,e_k,e_q],$$

it follows (38). By (4),

$$\begin{aligned}[e_l,e_m,e_q]_C&=[P(e_l),P(e_m),e_q]+[P(e_m),P(e_q),e_l]+[P(e_q),P(e_l),e_m]\\&=\sum_{j<k}^{n}\begin{vmatrix} a_{lj} & a_{lk}\\ a_{mj} & a_{mk}\end{vmatrix}[e_j,e_k,e_q]+\sum_{k<i}^{n}\begin{vmatrix} a_{mk} & a_{mi}\\ a_{qk} & a_{qi}\end{vmatrix}[e_k,e_i,e_l]\\&+\sum_{i<j}^{n}\begin{vmatrix} a_{qi} & a_{qj}\\ a_{li} & a_{lj}\end{vmatrix}[e_i,e_j,e_m],\end{aligned}$$

then (39) holds. □

THEOREM 5.2. *Let A_4 be the 4-dimensional 3-Lie algebra with the multiplication* (1) *in the basis $\{e_1,e_2,e_3,e_4\}$. Then Rota-Baxter operators of weight zero on A_4 defined by* (12) *induce 3-pre-Lie algebras $(A_4,\{,,\}^t)$, $1\le t\le 6$, and the sub-adjacent 3-Lie algebras $(A_4,[,,]^t_C)$, where*

$(A_4,\{,,\}^t)$, $1\le t\le 5$	$(A_4,[,,]^t_C)$, $1\le t\le 5$
$\{e_l,e_m,e_q\}^1=0,\ 1\le l,m,q\le 4$	$(A_4,[\cdot,\cdot,\cdot]^1_C)$ is abelian
$\{e_1,e_2,e_3\}^2=(a_{11}a_{22}-a_{12}a_{21})e_3$, $\{e_1,e_2,e_4\}^2=-(a_{11}a_{22}-a_{12}a_{21})e_4$	$[e_1,e_2,e_3]^2_C$ $=(a_{11}a_{22}-a_{12}a_{21})e_3$, $[e_1,e_2,e_4]^2_C$ $=-(a_{11}a_{22}-a_{12}a_{21})e_4$
$\{e_1,e_2,e_1\}^3=(a_{13}a_{24}-a_{14}a_{23})e_1$, $\{e_1,e_2,e_2\}^3=-(a_{13}a_{24}-a_{14}a_{23})e_2$	$(A_4,[\cdot,\cdot,\cdot]^3_C)$ is abelian
$\{e_1,e_2,e_3\}^4=a_{12}a_{24}e_2$, $\{e_1,e_2,e_1\}^4=-a_{12}a_{24}e_4$, $\{e_1,e_3,e_3\}^4=a_{12}a_{24}e_3$, $\{e_1,e_3,e_4\}^4=-a_{12}a_{24}e_4$, $\{e_2,e_3,e_2\}^4=a_{24}^2e_4$, $\{e_2,e_3,e_3\}^4=a_{24}^2e_2$	$[e_1,e_2,e_3]^4_C=a_{12}a_{24}e_2$, $[e_1,e_3,e_4]^4_C=-a_{12}a_{24}e_4$
$\{e_1,e_4,e_2\}^5=-a_{14}a_{21}e_2$, $\{e_1,e_4,e_3\}^5=-a_{12}a_{24}e_4$, $\{e_2,e_4,e_1\}^5=a_{12}a_{24}e_3$, $\{e_2,e_4,e_3\}^5=-a_{12}a_{24}e_4$, $\{e_1,e_2,e_3\}^5=a_{24}^2e_4$, $\{e_1,e_2,e_4\}^5=a_{24}^2e_2$	$[e_1,e_2,e_3]^5_C=a_{14}a_{21}e_1$, $[e_2,e_3,e_4]^5_C=a_{14}a_{21}e_4$

$$(40)\quad \begin{cases} \{e_1,e_2,e_1\}^6 = (a_{13}a_{24}-a_{14}a_{23})e_1 + a_{13}a_{11}e_3 - a_{14}a_{11}e_4, \\ \{e_1,e_2,e_2\}^6 = -(a_{13}a_{24}-a_{14}a_{23})e_2 - a_{23}a_{11}e_3 + a_{24}a_{11}e_4, \\ \{e_1,e_3,e_2\}^6 = a_{33}a_{14}e_2 - (a_{33}a_{11}+a_{13}a_{24})e_3 + a_{14}a_{24}e_4, \\ \{e_1,e_3,e_4\}^6 = (a_{33}a_{11}+a_{13}a_{24})e_1 - a_{13}a_{14}e_2 + a_{14}a_{11}e_4, \\ \{e_1,e_4,e_2\}^6 = a_{33}a_{13}e_2 - a_{13}a_{23}e_3 + (-a_{33}a_{11}+a_{14}a_{23})e_4, \\ \{e_1,e_4,e_3\}^6 = (a_{14}a_{23}-a_{33}a_{11})e_1 + a_{13}a_{14}e_2 - a_{13}a_{11}e_3, \\ \{e_2,e_3,e_1\}^6 = -a_{33}a_{24}e_1 + (a_{23}a_{14}-a_{33}a_{11})e_3 - a_{14}a_{24}e_4, \\ \{e_2,e_3,e_4\}^6 = a_{23}a_{24}e_1 + (a_{33}a_{11}-a_{14}a_{23})e_2 + a_{24}a_{11}e_4, \\ \{e_2,e_4,e_1\}^6 = -a_{33}a_{23}e_1 + a_{13}a_{23}e_3 - (a_{33}a_{11}+a_{13}a_{24})e_4, \\ \{e_2,e_4,e_3\}^6 = -a_{23}a_{24}e_1 + (a_{33}a_{11}+a_{13}a_{24})e_3 - a_{23}a_{11}e_3, \\ \{e_3,e_4,e_3\}^6 = -a_{33}a_{24}e_1 - a_{33}a_{14}e_2 + (a_{13}a_{24}-a_{14}a_{23})e_3, \\ \{e_3,e_4,e_4\}^6 = a_{33}a_{23}e_1 - a_{33}a_{13}e_2 + (a_{14}a_{23}-a_{13}a_{24})e_4, \\ \{e_1,e_2,e_3\}^6 = -a_{11}a_{24}e_1 + a_{14}a_{11}e_2 - a_{11}^2e_3, \\ \{e_1,e_2,e_4\}^6 = a_{23}a_{11}e_1 - a_{13}a_{11}e_3 + a_{11}^2e_4, \\ \{e_2,e_4,e_2\}^6 = a_{33}a_{23}e_2 - a_{23}^2e_3 + a_{23}a_{24}e_4, \\ \{e_2,e_4,e_4\}^6 = a_{23}^2e_1 - a_{13}a_{23}e_2 + a_{23}a_{11}e_4, \\ \{e_1,e_3,e_1\}^6 = -a_{33}a_{14}e_1 + a_{13}a_{14}e_3 - a_{14}^2e_4, \\ \{e_1,e_3,e_3\}^6 = -a_{14}a_{24}e_1 + a_{14}^2e_2 - a_{14}a_{11}e_3, \\ \{e_1,e_4,e_1\}^6 = -a_{33}a_{13}e_1 - a_{13}^2e_3 - a_{13}a_{14}e_4, \\ \{e_1,e_4,e_4\}^6 = a_{13}a_{23}e_1 - a_{13}^2e_2 + a_{13}a_{11}e_4, \\ \{e_2,e_3,e_2\}^6 = a_{33}a_{24}e_2 - a_{23}a_{24}e_3 + a_{24}^2e_4, \\ \{e_2,e_3,e_3\}^6 = -a_{24}^2e_1 + a_{14}a_{24}e_2 - a_{24}a_{11}e_3, \\ \{e_3,e_4,e_1\}^6 = -a_{33}^2e_1 + a_{33}a_{13}e_3 - a_{33}a_{14}e_4, \\ \{e_3,e_4,e_2\}^6 = a_{33}^2e_2 - a_{33}a_{23}e_3 + a_{33}a_{24}e_4; \end{cases}$$

$$(41)\quad \begin{cases} [e_1,e_2,e_3]_C^6 = (-a_{24}a_{11}-a_{33}a_{24})e_1 + (a_{33}a_{14}+a_{14}a_{11})e_2 - 2a_{14}a_{24}e_4 \\ \qquad\qquad + (-a_{11}^2+a_{13}a_{24}+a_{14}a_{23})e_3, \\ [e_1,e_2,e_4]_C^6 = (a_{33}a_{23}-a_{23}a_{11})e_1 + (-a_{13}a_{11}-a_{33}a_{13})e_2 + 2a_{13}a_{23}e_3 \\ \qquad\qquad - (-a_{11}^2+a_{14}a_{23}+a_{13}a_{24})e_4, \\ [e_1,e_3,e_4]_C^6 = (-a_{33}^2+a_{13}a_{24}+a_{23}a_{14})e_1 - 2a_{13}a_{14}e_2 \\ \qquad\qquad + (a_{33}a_{13}+a_{13}a_{11})e_3 + (a_{14}a_{11}-a_{33}a_{14})e_4, \\ [e_2,e_3,e_4]_C^6 = 2a_{23}a_{24}e_1 - (a_{14}a_{23}+a_{13}a_{24})e_2 + (a_{23}a_{11}-a_{33}a_{23})e_3 \\ \qquad\qquad + (a_{24}a_{11}+a_{33}a_{24})e_4; \end{cases}$$

Proof. If $Rank(P) = 1$, then by (38) and Theorem 3.1, $(A_4, \{,,\}^1)$ is a

3-pre-Lie algebra satisfies $\{e_l, e_m, e_q\}^1 = 0$, and the sub-adjacent 3-Lie algebra $(A_4, [, ,]^1_C)$ is abelian.

If $Rank(P) = 2$ and $P(e_1) = a_{11}e_1+a_{12}e_2,\ P(e_2) = a_{21}e_1+a_{22}e_2,\ P(e_3) = P(e_4) = 0$, then by (38) and Theorem 3.2, we get 3-pre-Lie algebra $(A_4, \{, , \}^2)$ and the sub-adjacent 3-Lie algebra $(A, [, ,]^2_C)$.

If $Rank(P) = 3$, then by (38) and Theorem 3.5, we get 3-pre-Lie algebras $(A_4, \{, , \}^3)$, $(A_4, \{, , \}^4)$, $(A_4, \{, , \}^5)$ and the sub-adjacent 3-Lie algebras $(A_4, [, ,]^3_C)$, $(A_4, [, ,]^4_C)$, and $(A_4, [, ,]^5_C)$, respectively.

If $Rank(P) = 4$, then by (38) and Theorem 3.7, we get 3-pre-Lie algebra $(A_4, \{, , \}^6)$ and the sub-adjacent 3-Lie algebra $(A_4, [, ,]^6_C)$ which satisfy (40) and (41), respectively. □

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