NEW COMPACT EMBEDDING THEOREM AND FRACTIONAL HAMILTONIAN SYSTEMS

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In this paper, we introduce a new condition to obtain a new compact embedding theorem. Moreover, under this theorem, we study the existence and multiplicity of nontrivial solutions for the following fractional Hamiltonian systems:

$$\begin{cases} -{}_tD^\alpha_\infty({}_{-\infty}D^\alpha_t x(t)) - L(t).x(t) + \nabla W(t,x(t)) = 0, \\ x \in H^\alpha(\mathbb{R},\mathbb{R}^N), \end{cases}$$

where $\alpha \in \left(\frac{1}{2},1\right]$, $t \in \mathbb{R}, x \in \mathbb{R}^N$, ${}_{-\infty}D^\alpha_t$ and ${}_tD^\alpha_\infty$ are left and right Liouville-Weyl fractional derivatives of order α on the whole axis $\mathbb{R}$, respectively.

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1. INTRODUCTION AND MAIN RESULTS

Fractional differential equations including both ordinary and partial ones are applied in mathematical modeling of some processes in physics, mechanics, chemistry, economics and bioengineering; see [1, 17, 22] and the references therein. Indeed, the associated fractional-order differential operators of these equations admit the characteristic of nonlocal behavior, which can provide a more realistic and practical description of these processes than the usual integer-order differential operators. Therefore, the theory of fractional differential equations is an area intensively developed during the last decades.

In recent years, fractional differential equations including both left and right fractional derivatives are also gradually investigated. Apart from their possible applications, the research of these equations is a relatively new and interesting field in the theory of fractional differential equations. Some early works on this topic can be found in papers [4, 7] and their references.

In 2012, Jiao and Zhou [20] showed the existence of solutions for the fractional boundary value problem

$${}_tD^\alpha_T({}_0D^\alpha_t x(t)) = \nabla W(t,x(t)), \quad \text{a.e.} t \in [0,T],$$

$$x(0) = x(T) = 0,$$

where ${}_tD_T^\alpha$ and ${}_0D_t^\alpha$ are the right and left Riemann-Liouville fractional derivatives of order $\alpha \in [\frac{1}{2}, 1]$. Inspired by this work, in [27], Torres considered the fractional Hamiltonian system

$$(1.1) \qquad \begin{gathered} {}_tD_\infty^\alpha({}_{-\infty}D_t^\alpha x(t)) + L(t)x(t) = \nabla W(t, x(t)), \\ x \in H^\alpha(\mathbb{R}, \mathbb{R}^N), \end{gathered}$$

where ${}_tD_\infty^\alpha$ and ${}_{-\infty}D_t^\alpha$ are the Liouville-Weyl fractional derivatives of order $\frac{1}{2} < \alpha < 1$, $L \in C(\mathbb{R}, \mathbb{R}^{N\times N})$ is a symmetric matrix-valued function, $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, and $\nabla W(t, u)$ denotes the gradient of $W(t, u)$ with respect to u. To be more precise, he showed that the fractional Hamiltonian system (1.1) possesses at least one nontrivial solution under the following assumptions:

(A1) There exists an $l \in C(\mathbb{R}, (0, \infty))$ such that $l(t) \to +\infty$ as $t \to \infty$ and

$$(L(t)x, x) \geq l(t)|x|^2, \quad \forall t \in \mathbb{R},\ x \in \mathbb{R}^N.$$

(A2) There exists a constant $\mu > 2$ such that

$$0 < \mu W(t, x) \leq (\nabla W(t, x), x), \quad \forall t \in \mathbb{R},\ x \in \mathbb{R}^N \backslash \{0\}.$$

(A3) $|\nabla W(t, x)| = o(|x|)$ as $x \to 0$ uniformly with respect to $t \in \mathbb{R}$.

(A4) There exists $\overline{W} \in C(\mathbb{R}^N, \mathbb{R})$ such that

$$|W(t, x)| + |\nabla W(t, x)| \leq |\overline{W}(x)|, \quad \forall t \in \mathbb{R},\ u \in \mathbb{R}^N.$$

Subsequently, the existence and multiplicity of solutions for the fractional Hamiltonian system (1.1) have been extensively investigated in many papers, see [8, 10, 11, 12, 15, 19, 28, 30] and the references therein. However, we note that in almost all, L is required to satisfy either the coercivity condition (A1) or the uniform positive-definiteness condition

(A5) there exists $b_0 > 0$ such that

$$(L(t)x, x) \geq b_0|x|^2, \quad \forall t \in \mathbb{R},\ x \in \mathbb{R}^N.$$

Besides, some of them (see [31]) dealt with the case where W satisfies the well-known Ambrosetti-Rabinowitz condition (A2), which is more restrictive than the following weaker superquadratic condition

(A6) $\lim_{|x|\to\infty} W(t, x)/|x|^2 = +\infty$ uniformly with respect to $t \in \mathbb{R}$.

Then, more papers were devoted to the case where W satisfies the weaker superquadratic growth condition (A6) and various additional technical conditions, see [8, 10, 11, 12, 15, 19, 28, 30]. For example in [11], the author studied system (1.1) and proved it has at least one ground state solution, under (A3),(A6) and the following assumptions:

(A7) $L(t)$ is $T-$periodic in t, and it is a symmetric and positive definite matrix for all $t \in \mathbb{R}$;

(A8) $W(t,x)$ is $T-$periodic and, there exist constants $C > 0$ and $p > 2$ such that

$$|\nabla W(t,x)| \leq C(|x| + |x|^{p-1}), \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N;$$

(W9) $s \longmapsto s^{-1}\nabla W(t,sx).x$ is strictly increasing of $s > 0$ for all $x \neq 0$ and $t \in \mathbb{R}$.

In the recent paper [12], the author proved that system (1.1) possesses a sequence of solutions (x_k) satisfying

$$\|x_k\|_{L^\infty} \longrightarrow 0 \text{ as } k \longrightarrow \infty,$$

under the following conditions:

(A10) there exists a constant $a_0 > 0$ such that for

$$l(t) := \min_{x \in \mathbb{R}^N, |x|=1} L(t)x.x,$$

we have $l(t) + a_0 \geq 1,\ \forall t \in \mathbb{R}$ and $\int_{\mathbb{R}} \frac{1}{l(t)+a_0} \mathrm{d}t < \infty$;

(A11) $W(t,0) \equiv 0,\ W \in C^1(\mathbb{R} \times B_\rho(0), \mathbb{R})$ is even, and there exists a constant $c > 0$ such that

$$|\nabla W(t,x)| \leq c,\ \forall (t,x) \in \mathbb{R} \times B_\rho(0),$$

where $B_\rho(0)$ is the open ball in $\mathbb{R}^N$ centered at 0 with radius ρ.

(A12) there exist a constant $\delta > 0$, a closed interval $\Lambda \subset \mathbb{R}$, and two sequences of positives numbers $\sigma_n \longrightarrow 0$, $M_n \longrightarrow \infty$ as $n \longrightarrow \infty$ such that

$$W(t,x) \geq -\delta |x|^2, \forall\ (t,x) \in \Lambda \times B_\rho(0)$$

and

$$W(t,x) \geq \sigma_n^2 M_n, \forall\ t \in \Lambda \text{ and } |x| = \sigma_n.$$

However, each result in the above papers implies that

(A13) There exists $\kappa > 0$ such that $\sup_{t \in \mathbb{R}, |x|=\kappa} W(t,x) < +\infty$.

Clearly, (A13) holds when $W(t,x)$ is periodic in t. If there is no periodic assumption, (A13) is an important requirement in many papers.

Obviously, if we take $W(t,x)$ of the following form

$$W(t,x) = a(t)H(x), \tag{1.2}$$

where $a : \mathbb{R} \to \mathbb{R}^+$ is continuous and $H \in C^1(\mathbb{R}^N, \mathbb{R})$. Then condition (A13) does not hold since $a(t)$ can go to infinity as $|t| \to \infty$. On the other hand, the assumptions on $L(t)$ in the previous works do not cover examples like $L(t) = (|t|\sin^2(t) - 1)I_N$.

Motivated by the works mentioned above and by this previous discussion, in this paper we study the existence and infinite solutions for systems (1.1) under new conditions which are more general than in previous works and covers examples like the previous ones. Precisely, we introduce the following conditions:

(H1) $W(t,x) = a(t)H(x)$, where $a : \mathbb{R} \to \mathbb{R}^+$ is continuous, and $H \in C^1(\mathbb{R}^N, \mathbb{R})$, $H(0) = 0$ and $\nabla H(x) = o(|x|)$ as $|x| \longrightarrow 0$;

(H2) There exists a constants $\theta \geq 1$ such that $\theta\widetilde{H}(x) \geq \widetilde{H}(sx)$ for all $x \in \mathbb{R}^N$ and $s \in [0,1]$, where $\widetilde{H}(x) = \nabla H(x).x - 2H(x)$.

(H3) $\frac{H(x)}{|x|^2} \to +\infty$ as $|x| \to \infty$.

(H4) There are constants $\zeta > 2$ and $d_1 > 0$ such that

$$|H(x)| \leq d_1(|x|^2 + |x|^\zeta) \text{ for all } x \in \mathbb{R}^N;$$

(H5) There exists $A > 0$ such that $a(t) \leq Al(t)$ for all $t \in \mathbb{R}$;

(L1) There exists a constant $r_0 > 0$ such that

$$\lim_{|s|\to\infty} meas(\{t \in (s - r_0, s + r_0) : \frac{l(t)}{a(t)} \ngeq M\}) = 0, \quad \forall M > 0,$$

where *meas* denotes the Lebesgue measure in $\mathbb{R}$.

Now we state our main results.

THEOREM 1.1. *Suppose that* (H1)–(H5), *(L1) and (A5) are satisfied. Then the fractional Hamiltonian system* (1.1) *possesses at least one nontrivial solution.*

THEOREM 1.2. *Suppose that (H1), (H3), (H5), (L1), (A5),* $\widetilde{H}(x) \geq 0$ *for any* $x \in \mathbb{R}^N$, *and the following condition hold:*

(H4') $\frac{\widetilde{H}(x)}{H(x)}|x|^2 \to +\infty$ *as* $|x| \to \infty$.

Then the fractional Hamiltonian system (1.1) *possesses at least one nontrivial solution.*

If $H(x)$ is even in x, we can obtain the following multiplicity results.

THEOREM 1.3. *Suppose that* (H1)–(H5), *(L1) and (A5) are satisfied and* $H(-x) = H(x)$. *Then the fractional Hamiltonian system* (1.1) *possesses infinitely many solutions.*

THEOREM 1.4. *Suppose that (H1), (H3), (H4'), (H5), (L1), (A5),* $\widetilde{H}(x) \geq 0$ *for any* $x \in \mathbb{R}^N$, *are satisfied and* $H(-x) = H(x)$. *Then the fractional Hamiltonian system* (1.1) *possesses infinitely many solutions.*

Here and in the following, $x.y$ denotes the inner product of $x, y \in \mathbb{R}^N$ and $|.|$ denotes the associated norm. Throughout the paper, we denote by c, c_i the various positive constants which may vary from line to line and are not essential to the problem.

The remaining part of this paper is organized as follows. Some preliminary results are presented in Section 2 and Section 3 is devoted to the proof of Theorem 1.1.

2. PRELIMINARY RESULTS

2.1. Liouville-Weyl Fractional Calculus

Definition 2.1. The left and right Liouville-Weyl fractional integrals of order $0 < \alpha < 1$ on the whole axis $\mathbb{R}$ are defined by

$$_{-\infty}I_t^\alpha x(t) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-\xi)^{\alpha-1} x(\xi) \mathrm{d}\xi,$$

$$_tI_\infty^\alpha x(t) := \frac{1}{\Gamma(\alpha)} \int_{t}^{\infty} (\xi-t)^{\alpha-1} x(\xi) \mathrm{d}\xi,$$

respectively, where $t \in \mathbb{R}$.

Definition 2.2. The left and right Liouville-Weyl fractional derivatives of order $0 < \alpha < 1$ on the whole axis $\mathbb{R}$ are defined by

$$_{-\infty}D_t^\alpha x(t) := \frac{d}{dt} {}_{-\infty}I_t^{1-\alpha} x(t), \tag{2.1}$$

$$_tD_\infty^\alpha x(t) := -\frac{d}{dt} {}_tI_\infty^{1-\alpha} x(t), \tag{2.2}$$

respectively, where $t \in \mathbb{R}$.

Remark 2.1. The definitions (2.1) and (2.2) may be written in an alternative form:

$$ {}_{-\infty}D_t^{\alpha}x(t) = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^{\infty}\frac{x(t)-x(t-\xi)}{\xi^{\alpha+1}}\mathrm{d}\xi, $$

$$ {}_tD_{\infty}^{\alpha}x(t) = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^{\infty}\frac{x(t)-x(t+\xi)}{\xi^{\alpha+1}}\mathrm{d}\xi\,. $$

Recall that the Fourier transform $\widehat{x}(z)$ of $x(t)$ is defined by

$$ \widehat{x}(z) = \int_{-\infty}^{\infty} e^{-itz}x(t)\mathrm{d}t. $$

We establish the Fourier transform properties of the fractional integral and fractional operators as follows:

$$ \widehat{{}_{-\infty}I_t^{\alpha}x(t)}(z) := (iz)^{-\alpha}\widehat{x}(z), $$
$$ \widehat{{}_tI_{\infty}^{\alpha}x(t)}(z) := (-iz)^{-\alpha}\widehat{x}(z), $$
$$ \widehat{{}_{-\infty}D_t^{\alpha}x(t)}(z) := (iz)^{\alpha}\widehat{x}(z), $$
$$ \widehat{{}_tD_{\infty}^{\alpha}x(t)}(z) := (-iz)^{\alpha}\widehat{x}(z). $$

2.2. **Fractional derivative spaces**

Let us recall that for any $\alpha > 0$, the semi-norm

$$ |x|_{I_{-\infty}^{\alpha}} := \|{}_{-\infty}D_t^{\alpha}x\|_{L^2}\,, $$

and the norm

$$ \|x\|_{I_{-\infty}^{\alpha}} := \left(\|x\|_{L^2}^2 + |x|_{I_{-\infty}^{\alpha}}^2\right)^{1/2}, $$

and let the space $I_{-\infty}^{\alpha}(\mathbb{R})$ denote the completion of $C_0^{\infty}(\mathbb{R})$ with respect to the norm $\|.\|_{I_{-\infty}^{\alpha}}$, i.e.,

$$ I_{-\infty}^{\alpha}(\mathbb{R}) = \overline{C_0^{\infty}(\mathbb{R})}^{\|\cdot\|_{I_{-\infty}^{\alpha}}}. $$

Next, we define the fractional Sobolev space $H^{\alpha}(\mathbb{R})$ in terms of the Fourier transform. For $0 < \alpha < 1$, define the semi-norm

$$ |x|_{\alpha} = \||z|^{\alpha}\widehat{x}\|_{L^2}\,, $$

and the norm

$$ \|x\|_{\alpha} = \left(\|x\|_{L^2}^2 + |x|_{\alpha}^2\right)^{1/2}, $$

and let

$$ H^{\alpha}(\mathbb{R}) := \overline{C_0^{\infty}(\mathbb{R})}^{\|\cdot\|_{\alpha}}. $$

We note that a function $x \in L^2(\mathbb{R})$ belongs to $I_{-\infty}^{\alpha}(\mathbb{R})$ if and only if

$$ |z|^{\alpha}\widehat{x} \in L^2(\mathbb{R}). $$

In particular, $|x|_{I^\alpha_{-\infty}} = \||z|^\alpha \widehat{x}\|_{L^2(\mathbb{R})}$. Therefore $H^\alpha(\mathbb{R})$ and $I^\alpha_{-\infty}(\mathbb{R})$ are equivalent, with equivalent semi-norm and norm (see [27]).
Analogous to $I^\alpha_{-\infty}(\mathbb{R})$, we introduce $I^\alpha_{\infty}(\mathbb{R})$. Let the semi-norm

$$|x|_{I^\alpha_\infty} := \|{}_tD^\alpha_\infty\|_{L^2(\mathbb{R})},$$

and norm

$$\|x\|_{I^\alpha_\infty} := \left(\|x\|^2_{L^2} + |x|^2_{I^\alpha_\infty}\right)^{1/2},$$

and let

$$I^\alpha_{-\infty}(\mathbb{R}) = \overline{C^\infty_0(\mathbb{R})}^{\|\cdot\|_{I^\alpha_{-\infty}}}.$$

Moreover, $I^\alpha_\infty(\mathbb{R})$ and $I^\alpha_{-\infty}(\mathbb{R})$ are equivalent, with equivalent semi-norm and norm.

Lemma 2.1 ([27]). *If $\alpha > 1/2$, then $H^\alpha(\mathbb{R}) \subset C(\mathbb{R})$ and there is a constant $c = c_\alpha$ such that*

(2.3) $$\|x\|_{L^\infty} = \sup_{t\in\mathbb{R}} |x(t)| \le c\,\|x\|_\alpha,$$

where $C(\mathbb{R})$ denote the space of continuous functions from $\mathbb{R}$.

Remark 2.2. If $x \in H^\alpha(\mathbb{R})$, then $x \in L^q(\mathbb{R})$ for all $q \in [2,\infty]$, since

(2.4) $$\int_\mathbb{R} |x(t)|^q \mathrm{d}t \le \|x\|^{q-2}_{L^\infty}\,\|x\|^2_{L^2}.$$

Now we introduce a new fractional space. Set

$$X^\alpha = \left\{x \in H^\alpha(\mathbb{R},\mathbb{R}^N) : \int_\mathbb{R} |{}_{-\infty}D^\alpha_t x(t)|^2 + L(t)x(t).x(t)\mathrm{d}t < \infty\right\}.$$

The space X^α is a Hilbert space with the inner product

$$(x,y)_{X^\alpha} = \int_\mathbb{R} \left(({}_{-\infty}D^\alpha_t x(t).{}_{-\infty}D^\alpha_t y(t)) + L(t)x(t).y(t)\right)\mathrm{d}t,$$

and the corresponding norm

$$\|x\|_{X^\alpha} = \sqrt{(x,x)_{X^\alpha}}.$$

Let $L^2_a(\mathbb{R},\mathbb{R}^N)$ be the weighted space of measurable functions $x : \mathbb{R} \to \mathbb{R}^N$ under the norm

$$||x||_{L^2_a} = \left(\int_\mathbb{R} a(t)|x(t)|^2\mathrm{d}t\right)^{\frac{1}{2}}.$$

Note that under condition (L1)

$$X^\alpha \subset H^\alpha \subset L^p$$

for any $p \in [2,+\infty]$ with the embedding being continuous, which implies that there is a positive constant $C_\infty > 0$ such that the following inequality holds:

(2.5) $$\max\{||x||_2, ||x||_\infty\} \le C_\infty\|x\|_{X^\alpha}.$$

First, we show a compact embedding theorem.

LEMMA 2.2. *If L satisfies (L1), (A5) and (H5) then, the embedding $X^\alpha \hookrightarrow L^2_a(\mathbb{R},\mathbb{R}^N)$ is continuous and compact.*

Proof. Suppose that $\{x_n\}$ is a bounded sequence in X^α. Then there exists $M_0>0$ such that $\|x_n\|_{X^\alpha}\le M_0$. Hence there exists a weak convergent subsequence, still denoted by $\{x_n\}$, such that $x_n \rightharpoonup x_0$ in X^α. Assuming $y_n = x_n - x_0$, we obtain that $\{y_n\}$ is a bounded sequence in X^α. and $y_n \rightharpoonup 0$ in X^α. Next, we show that $y_n \to 0$ in L^2_a. It follows from the Sobolev compact embedding theorem that $y_n \to 0$ in $L^1(B_R(0),\mathbb{R}^N)$ for any $R>0$, where $B_R(0)=\{t\in\mathbb{R}: 0-R\le t\le 0+R\}$. Choose $\{s_i\}\subset\mathbb{R}$ such that $\mathbb{R}\subset \cup_{i=1}^{\infty} B_{r_0}(s_i)$ and each $t\in\mathbb{R}$ is contained by two such intervals at most. Set $A(M,R)=\left\{t\in B^c_{R(0)} : \frac{l(t)}{a(t)} \ngeq M\right\}$ and $B(M,R)=\left\{t\in B^c_R(0) : \frac{l(t)}{a(t)} \ge M\right\}$.

On the one hand, we have

$$\int_{B(M,R)} a(t)|y_n(t)|^2\mathrm{d}t \le \frac{1}{M}\int_{B(M,R)} l(t)|y_n(t)|^2\mathrm{d}t \le \frac{||y_n||_{X^\alpha}}{M} \le \frac{2M_0}{M}. \tag{2.6}$$

On the other hand, let $\varepsilon_R = \sup_{s_i}(meas(A(M,R)\cap B_{r_0}(s_i)))$, we obtain

$$\begin{aligned}
\int_{A(M,R)} a(t)|y_n(t)|^2\mathrm{d}t &\le \sum_{i=1}^{\infty}\int_{A(M,R)\cap B_{r_0}(s_i)} a(t)|y_n(t)|^2\mathrm{d}t\\
&\le \varepsilon_R^{\frac12}\sum_{i=1}^{\infty}\left(\int_{A(M,R)\cap B_{r_0}(s_i)} a^2(t)|y_n(t)|^4\mathrm{d}t\right)^{\frac12}\\
&\le A\varepsilon_R^{\frac12}\sum_{i=1}^{\infty}\left(\int_{A(M,R)\cap B_{r_0}(s_i)} l^2(t)|y_n(t)|^4\mathrm{d}t\right)^{\frac12}\\
&\le AC\varepsilon_R^{\frac12}\sum_{i=1}^{\infty}\int_{B_{r_0}(s_i)} (|\dot y_n(t)|^2+(L(t)y_n(t).y_n(t)))\mathrm{d}t\\
&\le AC\varepsilon_R^{\frac12}||y_n||^2_{X^\alpha}
\end{aligned} \tag{2.7}$$

for some $C>0$.

It follows from (H5) that $\int_{A(M,R)} a(t)|y_n(t)|^2\mathrm{d}t \to 0$ as $R\to\infty$, which implies that

$$\begin{aligned}
\int_{B^c_R(0)} a(t)|y_n(t)|^2\mathrm{d}t &\le \int_{A(M,R)} a(t)|y_n(t)|^2\mathrm{d}t + \int_{B(M,R)} a(t)|y_n(t)|^2\mathrm{d}t\\
&\le \frac{2M_0}{M} + 2d_0C\varepsilon_R^{\frac12}||y_n||^2_{X^\alpha}\\
&\longrightarrow 0 \text{ as } \min\{M,R\}\longrightarrow\infty.
\end{aligned} \tag{2.8}$$

Then we can deduce that $y_n \to 0$ in $L^2_a(\mathbb{R},\mathbb{R}^N)$. □

From Lemma 2.2, we can easily deduce that there is a positive constant $K > 0$ such that the following inequality holds

$$||x||_{L^2_a} \leq K||x||_{X^\alpha}. \tag{2.9}$$

LEMMA 2.3. *Suppose that conditions (A5), (L1), (H1) and (H3) hold, then we have* $\nabla G(x_k) \longrightarrow \nabla G(x)$ *in* $L^2_a(\mathbb{R}, \mathbb{R}^N)$ *if* $x_k \longrightarrow x$ *in* X^α.

Proof. The proof is similar to Lemma 2.4 in [28]. □

Define the functional $I : X^\alpha \to \mathbb{R}$ by

$$\begin{aligned} I(x) &= \int_{\mathbb{R}} \left(\frac{1}{2}|_{-\infty}D^\alpha_t x(t)|^2 + \frac{1}{2}L(t)x(t).x(t) - W(t, x(t)) \right) \mathrm{d}t \\ &= \frac{1}{2}\|x\|^2_{X^\alpha} - \int_{\mathbb{R}} W(t, x(t))\mathrm{d}t. \end{aligned}$$

Under the conditions of our Theorems, we see that I is a continuously Fréchet-differentiable functional defined on X^α; i.e., $I \in C^1(X^\alpha, \mathbb{R})$. Moreover, we have

$$I'(x)y = \int_{\mathbb{R}} \left(({}_{-\infty}D^\alpha_t x(t). {}_{-\infty}D^\alpha_t y(t)) + L(t)x(t).y(t) - \nabla W(t, x(t)).y(t) \right) \mathrm{d}t, \tag{2.10}$$

for all $x, y \in X^\alpha$, which yields

$$I'(x)x = \|x\|^2_{X^\alpha} - \int_{\mathbb{R}} \nabla W(t, x(t)).x(t)\mathrm{d}t. \tag{2.11}$$

According to [27], we know that in order to find solutions of system (1.1) it is sufficient to obtain the critical points of I. For this purpose, we recall the following definitions and results (see [32]).

LEMMA 2.4 ([25], Mountain Pass Theorem). *Let E be a real Banach space and $\phi \in C^1(E, \mathbb{R})$ satisfying the Palais-Smale condition. If ϕ satisfies the following conditions:*

(i) $\phi(0) = 0$,

(ii) *there exist constants* $\rho, \gamma > 0$ *such that* $\phi_{/\partial B_\rho(0)} \geq \gamma$,

(iii) *there exists* $e \in E \backslash \overline{B}_\rho(0)$ *such that* $\phi(e) \leq 0$.

Then ϕ possesses a critical value $c \geq \gamma$ given by

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} \phi(g(s)),$$

where

$$\Gamma = \{g \in C([0, 1], E) : g(0) = 0, g(1) = e\}.$$

Lemma 2.5 ([25], Symmetric Mountain Pass Theorem). *Let E be a real Banach space and ϕ is even and $\in C^1(E, \mathbb{R})$ satisfying the Palais-Smale condition. If ϕ satisfies (i) and (ii) of Lemma 2.4 and the following conditions: (iii′) For each finite dimensional subspace $E_1 \subset E$, there is $r = r(E_1)$ such that $\phi(x) < 0$ for $x \in E_1 \backslash B_r(0)$. Then ϕ possesses an unbounded sequence of critical values.*

Remark 2.3. As shown in [6], a deformation Lemma can be proved with replacing the usual $(PS)-$condition with the $(C)-$condition introduced by Cerami, and it turns out that the Mountain Pass Theorem and the Symmetric Mountain Pass Theorem are true under the $(C)-$condition. Recall that a C^1 functional ϕ satisfies Cerami condition at level c ($(C)_c$ condition for short) if any sequence $(u_n) \subset E$ such that $\phi(u_n) \to c$ and $(1 + \|u_n\|) \|\phi'(u_n)\| \to 0$ has a convergent subsequence; such a sequence is then called a $(C)_c$ sequence.

3. PROOF OF THEOREM 1.1

The proof of Theorem 1.1 is divided into several lemmas.

Lemma 3.1. *Suppose that $\widetilde{H}(x) \geq 0$ and (H1) hold, then $H(x) \geq 0$ for all $x \in \mathbb{R}$.*

Proof. The proof of this lemma is similar to that of Lemma 7 in [29]. □

Lemma 3.2. *Suppose that the conditions of Theorem 1.1 hold, then I satisfies the $(C)-$condition.*

Proof. Assume that $\{x_n\} \subset X^\alpha$ is a sequence such that $\{I(x_n)\}$ is bounded and $\|I'(x_n)\| (1 + ||x_n||_{X^\alpha}) \to 0$ as $n \to \infty$. Then, for some $M_1 > 0$, it follows

$$|I(x_n)| \leq M_1, \; ||I'(x_n)x_n|| (1 + ||x_n||_{X^\alpha}) \leq M_1. \tag{3.1}$$

Next we show that $\{x_n\}$ is bounded in X^α. Assuming $||x_n||_{X^\alpha} \longrightarrow +\infty$ as $n \longrightarrow \infty$, set $z_n := \frac{x_n}{||x_n||_{X^\alpha}}$, then $||z_n||_{X^\alpha} = 1$, which implies that there exists a subsequence of $\{z_n\}$, still denoted by $\{z_n\}$, such that $z_n \rightharpoonup z_0$ in X^α. By (2.11) and (3.1), we get

$$\left| \int_{\mathbb{R}} \frac{W(t, x_n(t))}{||x_n||^2_{X^\alpha}} \mathrm{d}t - \frac{1}{2} \right| = \left| - \frac{I(x_n)}{||x_n||^2_{X^\alpha}} \right| \leq \frac{M_1}{||x_n||^2_{X^\alpha}}, \tag{3.2}$$

which implies that

$$\left| \int_{\mathbb{R}} \frac{W(t, x_n(t))}{||x_n||^2_{X^\alpha}} \mathrm{d}t \right| \leq 1, \tag{3.3}$$

for n large enough. The following discussion is divided into two cases.

Case 1: $z_0 \neq 0$, Let $\Omega := \{t \in \mathbb{R} : |z_0(t)| > 0\}$. We can see that $meas\,(\Omega) > 0$. Then there exists $\chi > 0$ such that, for $\Lambda := \Omega \cap B_\chi(0)$, $meas\,(\Lambda) > 0$. Since $||x_n||_{X^\alpha} \longrightarrow +\infty$ for a.e. $t \in \Lambda$.

Let $a_1 = \inf_{t\in B_\chi(0)} a(t) > 0$. By (H1), (H3) and Lemma 3.1 and Fatou's lemma, we can obtain

$$\lim_{n\longrightarrow\infty} \int_{\mathbb{R}} \frac{W(t, x_n(t))}{||x_n||^2_{X^\alpha}} \mathrm{d}t \geq a_1 \lim_{n\longrightarrow\infty} \int_{\Lambda} \frac{H(x_n(t))}{|x_n|^2} |z_n(t)|^2 \mathrm{d}t = \infty,$$

which contradicts (3.3).

Case 2: $z_0 = 0$. Set a sequence $\{T_n\} \subset [0, 1]$ such that $I(T_n x_n) = \max_{T\in[0,1]} I(T x_n)$. By Lemmas 2.2 and 3.1, and (H1), (H4), we obtain

$$\begin{aligned}
0 \leq & \int_{\mathbb{R}} a(t)H(4\sqrt{\theta M_1} z_n(t))\mathrm{d}t \\
& \leq d_1(16\theta M_1 \int_{\mathbb{R}} a(t)|z_n(t)|^2\mathrm{d}t + (4\sqrt{\theta M_1})^\zeta \int_{\mathbb{R}} a(t)|z_n(t)|^\zeta \mathrm{d}t) \\
& = d_1(16\theta M_1 + (4\sqrt{\theta M_1})^\zeta C_\infty^{\zeta-2}) \int_{\mathbb{R}} a(t)|z_n(t)|^2\mathrm{d}t \longrightarrow 0 \text{ as } n \longrightarrow \infty,
\end{aligned}$$

which implies that

$$\int_{\mathbb{R}} a(t)H(4\sqrt{\theta M_1} z_n(t))\mathrm{d}t \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{3.4}$$

By the definition of T_n and (3.4), for n large enough, we have

$$\begin{aligned}
I(T_n x_n) \geq I(\frac{4\sqrt{\theta M_1}}{||x_n||_{X^\alpha}} x_n) &= I(4\sqrt{\theta M_1} z_n) \\
&= \frac{1}{2}||4\sqrt{\theta M_1} z_n||^2_{X^\alpha} - \int_{\mathbb{R}} a(t)H(4\sqrt{\theta M_1} z_n(t))\mathrm{d}t \\
&= 8\theta M_1 - \int_{\mathbb{R}} a(t)H(4\sqrt{\theta M_1} z_n(t))\mathrm{d}t \\
&\geq 4\theta M_1.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
||T_n x_n||^2_{X^\alpha} - \int_{\mathbb{R}} a(t)(\nabla H(T_n x_n(t)).T_n x_n(t))\mathrm{d}t &= I'(T_n x_n(t))T_n x_n(t) \\
&= T_n \frac{\mathrm{d}I(T x_n(t))}{\mathrm{d}T}|_{T=T_n} = 0
\end{aligned} \tag{3.5}$$

Hence, it follows from (3.5) and (H3) that

$$\begin{aligned}\int_{\mathbb{R}} a(t)(\frac{1}{2}(\nabla H(x_n(t)).x_n(t) - H(x_n(t))))\mathrm{d}t &= \frac{1}{2}\int_{\mathbb{R}} a(t)\widetilde{H}(x_n(t))\mathrm{d}t \\ &\geq \frac{1}{2\theta}\int_{\mathbb{R}} a(t)\widetilde{H}(T_n x_n(t))\mathrm{d}t \\ &= \frac{1}{\theta}\int_{\mathbb{R}}(\frac{1}{2}a(t)(\nabla H(T_n x_n(t)).T_n x_n(t)) - a(t)H(T_n x_n(t)))\mathrm{d}t \\ &= \frac{1}{\theta}(\frac{1}{2}||T_n x_n||^2_{X^\alpha} - \int_{\mathbb{R}} a(t)H(T_n x_n(t))\mathrm{d}t) \\ &= \frac{1}{\theta}I(T_n x_n),\end{aligned}$$

which implies that

$$\int_{\mathbb{R}} a(t)(\frac{1}{2}(\nabla H(x_n(t)).x_n(t) - H(x_n(t))))\mathrm{d}t \geq 4M_1 \text{ for } n \text{ large enough.} \tag{3.6}$$

However, we can deduce from (3.1) that

$$\left|\int_{\mathbb{R}} a(t)(\frac{1}{2}(\nabla H(x_n(t)).x_n(t) - H(x_n(t))))\mathrm{d}t\right| = \left|2I(x_n) - I'(x_n).x_n\right| \leq 3M_1$$

for all $n \in \mathbb{N}$, which contradicts (3.6). Hence $\{x_n\}$ is bounded in X^α. Going if necessary to a subsequence, we can assume that $x_n \rightharpoonup x$ in X^α, which yields

$$(I'(x_n) - I'(x))(x_n - x) \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

and it follows from Lemma 2.2, Hölder's inequality, (2.9) and Lemma 2.3 that

$$\begin{aligned}&\left|\int_{\mathbb{R}} a(t)(\nabla H(x_n(t)) - \nabla H(x(t)))(x_n(t) - x(t))\mathrm{d}t\right| \\ &\qquad \leq \|\nabla H(x_n(t)) - \nabla H(x(t))\|_{L^2_a} \|x_n(t) - x(t)\|_{L^2_a} \longrightarrow 0 \text{ as } n \longrightarrow \infty.\end{aligned}$$

Hence, we conclude that

$$\begin{aligned}||x_n - x||^2_{X^\alpha} &= (I'(x_n) - I'(x))(x_n - x) \\ &\quad + \int_{\mathbb{R}} a(t)(\nabla H(x_n(t)) - \nabla H(x(t)))(x_n(t) - x(t))\mathrm{d}t \longrightarrow 0 \text{ as } n \longrightarrow \infty.\end{aligned}$$

□

Lemma 3.3. *Suppose that (A5), (L1), (H1), (H2) and (H5) hold, then there exist constants $\rho, \gamma > 0$ such that $I|_{\partial B_\rho(0)} \geq \gamma$.*

Proof. By (H1), for any $\varepsilon > 0$, there is $\sigma > 0$ such that

$$|H(x)| \leq \varepsilon |x|^2 \text{ for all } |x| \leq \sigma. \tag{3.7}$$

For $\varepsilon_1 = \frac{1}{4A}$, there exists $\delta_1 > 0$ such that (3.7) holds. Set $\rho := C_\infty^{-1}\delta_1, \gamma := \frac{1}{4}\rho^2$. By (H1), (H2), (H5) and (3.7), for any $||x||_{X^\alpha} = \rho$, we obtain

$$
\begin{aligned}
I(x) &= \frac{1}{2}||x||_{X^\alpha}^2 - \int_{\mathbb{R}} a(t)H(x_n(t))\mathrm{d}t \\
&\geq \frac{1}{2}||x||_{X^\alpha}^2 - \frac{1}{4A}\int_{\mathbb{R}} a(t)|x(t)|^2\mathrm{d}t \\
&\geq \frac{1}{2}||x||_{X^\alpha}^2 - \frac{1}{4}\int_{\mathbb{R}} l(t)|x(t)|^2\mathrm{d}t \\
&\geq \frac{1}{4}||x||_{X^\alpha}^2.
\end{aligned} \tag{3.8}
$$

By the definition of ρ and γ, (3.8) implies $I|_{\partial B_\rho(0)} \geq \gamma$. □

Lemma 3.4. *Suppose that (A5), (L1) and (H1)-(H3) hold, then there exists $e \in X^\alpha$ such that $||e||_X^\alpha > \rho$ and $I(e) \leq 0$.*

Proof. Set $e_0 \in C_0^\infty(-1, 1)$ with $||e||_X^\alpha = 1$. Let

$$a_2 := \min_{t\in B_1(0)} a(t),\ a_3 := \max_{t\in B_1(0)} a(t).$$

For $\beta > \frac{1}{2a_2\int_{-1}^1|e_0(t)|^2dt}$, it follows from (H3) that there exists $\xi > 0$ such that

$$H(x) \geq \beta|x|^2 \text{for all } |x| > \xi.$$

By Lemma 3.1, we have

$$H(x) \geq \beta(|x|^2 - \xi^2)\text{for all }\ x \in \mathbb{R}^N. \tag{3.9}$$

By (H1) and (3.9), for every $\eta \in \mathbb{R}\setminus\{0\}$, we have

$$
\begin{aligned}
I(\eta e_0) &= \frac{\eta^2}{2}||e_0||_{X^\alpha}^2 - \int_{-1}^1 a(t)H(\eta e_0(t))\mathrm{d}t \\
&\leq \frac{\eta^2}{2} - a_2\beta\eta^2\int_{-1}^1 |e_0(t)|^2\mathrm{d}t + 2a_3\beta\xi^2 \\
&= \left(\frac{1}{2} - a_2\beta\eta^2\int_{-1}^1 |e_0(t)|^2\mathrm{d}t\right)\eta^2 + 2a_3\beta\xi^2,
\end{aligned}
$$

which implies that

$$I(\eta e_0) \longrightarrow -\infty \text{ as } \eta \longrightarrow +\infty.$$

Then there exists $\eta_0 \in \mathbb{R}\setminus\{0\}$ such that $||\eta_0 e_0|| > \rho$ and $I(\eta_0 e_0) < 0$. Letting $e(t) := \eta_0 e_0(t)$, we finish the proof. □

From the above proofs and the mountain pass theorem, I possesses a critical point x_0 such that $I(x_0) = c$, which means system (1.1) has at least one nontrivial solution.

4. PROOF OF THEOREM 1.2

LEMMA 4.1. *Suppose that the conditions of Theorem 1.2 hold, then I satisfies the $(C)-$condition.*

Proof. Assume that $\{x_n\} \subset X^\alpha$ is a sequence such that $\{I(x_n)\}$ is bounded and $\|I'(x_n)\| (1 + ||x_n||_{X^\alpha}) \to 0$ as $n \to \infty$. Then, for some $M_1 > 0$, it follows

$$(4.1) \qquad |I(x_n)| \leq M_2, \; ||I'(x_n)x_n|| (1 + ||x_n||_{X^\alpha}) \leq M_2.$$

Next we show that $\{x_n\}$ is bounded in X^α. Assuming $||x_n||_{X^\alpha} \longrightarrow +\infty$ as $n \longrightarrow \infty$, set $z_n := \frac{x_n}{||x_n||_{X^\alpha}}$, then $||z_n||_{X^\alpha} = 1$, which implies that there exists a subsequence of $\{z_n\}$, still denoted by $\{z_n\}$, such that $z_n \rightharpoonup z_0$ in X^α. Similar to the proof of Lemma 3.2, we have

$$(4.2) \qquad \left| \int_{\mathbb{R}} \frac{W(t, x_n(t))}{||x_n||^2_{X^\alpha}} \mathrm{d}t \right| \longrightarrow \frac{1}{2}, \text{as } n \longrightarrow \infty.$$

The following discussion is divided into two cases.

Case 1: $z_0 \neq 0$. The proof is similar to the proof of Case 1 in Lemma 3.2.

Case 2: $z_0 = 0$. Let $\varepsilon := 1$, then there exists $\sigma_0 > 0$ such that (3.7) holds for all $|x| \leq \sigma_0$. By (H4'), we obtain that for any $B > 0$, there exists $r_\infty > 0$ such that, for all $|x| \geq r_\infty$, we have

$$(4.3) \qquad \frac{\widetilde{H}(x)}{H(x)} |x|^2 \geq B.$$

It follows from (4.3), (H2) and $\widetilde{H}(x) \geq 0$ that

$$\begin{aligned}
(4.4) \quad 0 &\leq \int_{\mathbb{R}} \frac{W(t, x_n(t))}{||x_n||^2_{X^\alpha}} \mathrm{d}t \\
&\leq \int_{\{t\in\mathbb{R}||x_n|>r_\infty\}} \frac{W(t, x_n(t))}{|x_n|^2} |z_n(t)|^2 \mathrm{d}t + \int_{\{t\in\mathbb{R}||x_n|\leq\sigma_0\}} \frac{W(t, x_n(t))}{||x_n||^2_{X^\alpha}} \mathrm{d}t \\
&+ \int_{\{t\in\mathbb{R}|\sigma_0\leq|x_n|\leq r_\infty\}} \frac{W(t, x_n(t))}{||x_n||^2_{X^\alpha}} \mathrm{d}t \\
&\leq ||z_n||^2_{L^\infty} \int_{\{t\in\mathbb{R}||x_n|>r_\infty\}} \frac{a(t)G(x_n)}{|x_n|^2} \mathrm{d}t + \int_{\{t\in\mathbb{R}||x_n|\leq\sigma_0\}} a(t)|x_n|^2 \mathrm{d}t \\
&+ \int_{\{t\in\mathbb{R}|\sigma_0\leq|x_n|\leq r_\infty\}} \frac{a(t)H(x_n(t))|x_n(t)|^2}{\sigma_0 ||x_n||^2_{X^\alpha}} \mathrm{d}t \\
&\leq \frac{||z_n||^2_{L^\infty}}{B} \int_{\{t\in\mathbb{R}||x_n|>r_\infty\}} a(t)(\nabla H(x_n).x_n - 2H(x_n)) \mathrm{d}t + ||z_n||^2_{L^2_a}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\max_{\sigma_0 \le |x| \le \rho_\infty} |H(x)|}{\sigma_0^2} \int_{\{t\in\mathbb{R} | \sigma_0 \le |x| \le \rho_\infty\}} a(t)|z_n|^2 \mathrm{d}t \\
&\le \frac{||z_n||^2_{L^\infty}}{B} \int_{\{t\in\mathbb{R} | |x_n| > r_\infty\}} a(t)(\nabla H(x_n).x_n - 2H(x_n)) \\
&+ (1 + \frac{\max_{\sigma_0 \le |x| \le \rho_\infty} |H(x)|}{\sigma_0^2})||z_n||^2_{L^2_a} \\
&\le \frac{||z_n||^2_{L^\infty}}{B}(2I(x_n) - I'(x_n).x_n) + (1 + \frac{\max_{\sigma_0 \le |x| \le \rho_\infty} |H(x)|}{\sigma_0^2})||z_n||^2_{L^2_a} \\
&\le \frac{3M_2C_\infty^2}{B} + (1 + \frac{\max_{\sigma_0 \le |x| \le \rho_\infty} |H(x)|}{\sigma_0^2})||z_n||^2_{L^2_a}.
\end{aligned}
$$

□

By the arbitraries of B and Lemma 3.1, we have

$$
\int_{\mathbb{R}} \frac{W(t, x_n(t))}{||x_n||^2_{X^\alpha}} \mathrm{d}t < \frac{1}{4} \text{ for } n \text{ large enough,} \tag{4.5}
$$

which contradicts (4.2). Hence $||x_n||_{X^\alpha}$ is bounded in X^α. Similar to the proof of Lemma 3.2, we see that I satisfies the $(C)-$condition.

5. PROOF OF THEOREMS 1.3 AND 1.4

LEMMA 5.1. *Suppose that (H1), (H3), (H5), (L1) and (A5) hold, then I satisfies (iii') of Lemma 2.5.*

Proof. Let $\widetilde{X}^\alpha \subset X^\alpha$ be a finite dimensional subspace. For any $x \in \widetilde{X}^\alpha \backslash \{0\}$ and $\upsilon > 0$, set

$$
\Gamma_\upsilon(x) := \{t \in \mathbb{R} : |x(t)| \ge \vartheta ||x||_{X^\alpha}\}
$$

Similar to Lemma 6.2 in [29], there exists $\upsilon_0 > 0$ such that

$$
meas\,(\Gamma_{\upsilon_0}(x)) \ge \upsilon_0 \tag{5.1}
$$

for all $x \in \widetilde{X}^\alpha \backslash \{0\}$. It is easy to see that there exists $\rho > 0$ such that $meas\,(\Gamma_{\upsilon_0}(x) \cap B_\rho(0)) > \frac{1}{2}\upsilon_0$, for any $x \in \widetilde{X}^\alpha \backslash \{0\}$. Set $a_4 := \inf_{|t| \le \rho} a(t) > 0$. By (H3), there exists $\gamma > 0$ such that

$$
H(x(t)) \ge \frac{1}{a_4 \upsilon_0}|x(t)|^2 \ge \frac{1}{a_4 \upsilon_0}||x||^2_{X^\alpha} \tag{5.2}
$$

for all $x \in \widetilde{X}^\alpha$ and $t \in \Gamma_{\upsilon_0}(x) \cap B_\rho(0)$ with $||x(t)||^2_{X^\alpha} \geq \gamma$. Then, for any $x \in \widetilde{X}^\alpha \backslash B_\gamma$, it follows from (H1), (H2) and (5.2) that

$$\begin{aligned} I(x) &= \frac{1}{2}||x||^2_{X^\alpha} - \int_{\mathbb{R}} a(t)H(x(t))\mathrm{d}t \\ &= \frac{1}{2}||x||^2_{X^\alpha} - \int_{\{\Gamma_{\upsilon_0}(x)\cap B_\rho(0)\}} a(t)H(x(t))\mathrm{d}t \\ &\quad - \int_{\mathbb{R}\backslash\{\Gamma_{\upsilon_0}(x)\cap B_\rho(0)\}} a(t)H(x(t))\mathrm{d}t \\ &= \frac{1}{2}||x||^2_{X^\alpha} - a_4 \int_{\{\Gamma_{\upsilon_0}(x)\cap B_\rho(0)\}} H(x(t))\mathrm{d}t \\ &\leq \frac{1}{2}||x||^2_{X^\alpha} - \frac{1}{\upsilon_0} meas\left(\Gamma_{\upsilon_0}(x) \cap B_\rho(0)\right)||x||^2_{X^\alpha} \\ &\leq -\frac{1}{2}||x||^2_{X^\alpha}. \end{aligned}$$

Then there exists $r > \gamma$ such that $I|_{\widetilde{X}^\alpha\backslash B_r} \leq 0$. □

Obviously that, under $H(-x) = H(x)$, I is even. By Lemmas 3.2 and 4.1, I satisfies the $(C)-$condition. Lemma 3.3 holds under the conditions of Theorems 1.3 and 1.4, respectively. Moreover, if we take $V = \{0\}$ and $X = X^\alpha$, we can see there are constants $\rho_1, \alpha_1 > 0$ such that $I|_{X\cap B_r} \geq \alpha_1$. From Lemma 5.1, I satisfies the condition (iii'). Hence, by Lemma 2.5, I possesses an unbounded sequence of critical values.

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