REGULARITY LIFTING OF WEAK SOLUTIONS FOR SUB-LAPLACE EQUATION ON HOMOGENEOUS GROUPS

XIAOJING FENG

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Let G be a homogeneous group and $X_1, X_2, \cdots, X_m$ be left invariant real vector fields satisfying Hörmander's rank condition on G. We also assume that $X_1, X_2, \cdots, X_m$ are homogeneous of degree one, by applying Moser iterations method, we lift the regularity of solutions for the following equation

$$-\sum_{j=1}^{m} X_j^2 u + \lambda u = |u|^{p-1}u,$$

where $\lambda > 0$, $1 < p < \frac{Q+2}{Q-2}$ and Q is the homogeneous dimension of G.

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1. INTRODUCTION AND MAIN RESULT

Let $X_1, X_2, \cdots, X_m$ form a system of C^∞ real vector fields defined in $\mathbb{R}^N (m \le N)$, and satisfy Hörmander's condition:

$$\operatorname{rank} \mathcal{L}(X_1, \dots, X_m)(x) = N, \ x \in \mathbb{R}^N,$$

where $\mathcal{L}(X_1, \dots, X_m)$ denotes the Lie algebra generated by $X_1, \dots, X_m$. We also assume that $X_1, \dots, X_m$ are left invariant with respect to the translations on the Lie group $G = (\mathbb{R}^N, \circ)$ and homogeneous with respect to the family of dilations $(\delta_\mu)_{\mu>0}$ on $\mathbb{R}^N$. More precisely, $X_1, \dots, X_m$ are homogeneous of degree one.

Our aim is to lift the regularity of solutions for the following equation

$$Lu + \lambda u = |u|^{p-1}u, \tag{1.1}$$

where $Lu = -\sum_{j=1}^{m} X_j^2 u$, $\lambda > 0$, $1 < p < \frac{Q+2}{Q-2}$ and Q is the homogeneous dimension of G (see (2.2)). In [8], by using the bootstrap approach introduced

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in [6], Feng and Niu proved that if u is a weak solution of (1.1) with critical term and singular term in a bounded domain $\Omega \subset G$, then $u \in L^q(\Omega)$ for any $q > 1$.

If we let $m = N,\ X_i = \partial_i,\ i = 1, \ldots, N$ and G be the Euclidean group $(\mathbb{R}^N, +)$, then (1.1) is of the form

$$-\Delta u + \lambda u = |u|^{p-1}u. \tag{1.2}$$

By using the mountain pass approach, Kurata, Shibata and Tada [13] obtained the existence of positive solution to (1.2) on unbounded domain.

When G is the Heisenberg group H^n and $N = 2n+1,\ m = 2n$, (1.1) can be written as

$$-\Delta_H u + \lambda u = |u|^{p-1}u, \tag{1.3}$$

where $1 < p < \frac{n+2}{n}$ and $-\Delta_H$ is the sub-Laplacian operator. After the works of Jerison and Lee [10, 11] on the Cauchy-Riemann (CR) Yamabe problem, the sub-Laplace equation arises as the Euler-Lagrange equation of a variational problem on CR manifolds. Several authors have investigated semilinear equations for sub-Laplacian operators with different techniques (see [2, 1, 14] and the references therein). Maad in [14] showed that there are infinitely many solutions by applying deformation Lemma. By developing the analogue of moving plane method for the Heisenberg group, Birindelli and Prajapat [2] derived some new nonexistence results for the equation (1.3) with $\lambda = 0$.

Motivated by above papers, the main purpose of this paper is to lift the regularity of nonnegative weak solutions for (1.1) by using Moser iterations method with a similar strategy [7, 12, 17]. More accurately, our main result reads as follows.

Theorem 1.1. *Let $u \in H^{1,2}(G)$ be a nonnegative weak solution of (1.1). Then $u \in L^\infty(G)$ and there exist $A,\ B > 0$, which depend only on Q and p, such that*

$$\|u\|_{L^\infty(G)} \le A(1 + \|u\|^B_{L^{2^*}(G)})\|u\|_{L^{2^*}(G)},$$

where $2^* = \frac{2Q}{Q-2}$.

The paper is organized as follows. In Section 2, we present some preliminaries and several Lemmas which will be used later. The proof of Theorem 1.1 is given in Section 3. As applications of our main result, Section 4 contains an example.

2. PRELIMINARY

In this section, some notations on homogeneous groups are given, which will be used throughout the article. By Stein [16], we call homogeneous group the space $\mathbb{R}^N$ equipped with a Lie group structure, together with a family of dilations that are group automorphisms. To be precise, let $\circ$ be a given group law on $\mathbb{R}^N$ and assume that the map $(x, y) \to y^{-1} \circ x$ is smooth, then $\mathbb{R}^N$ together with this mapping forms a Lie group. Next, there exists an N-tuple of positive real numbers $\omega_1 \leq \omega_2 \leq \ldots \leq \omega_N$, such that the dilations

$$\delta(\mu) : (x_1, \ldots, x_N) \mapsto (\mu^{\omega_1} x_1, \ldots, \mu^{\omega_N} x_N), \ \mu > 0$$

are group automorphisms. The space $\mathbb{R}^N$ with this structure of homogeneous group is denoted by G.

Hence, we can define a homogeneous norm $\|\cdot\|$ in G as follows. For $x = 0$, we set $\|x\| = 0$, while if $x \in G\backslash\{0\}$, set $\|x\| = \varrho \Leftrightarrow |\delta(1/\varrho)x| = 1$, where $|\cdot|$ denotes the Euclidean norm. By calculation, we have

(i) $\|\delta(\mu)x\| = \mu\|x\|$ for every $x \in G, \ \mu > 0$;

(ii) there exist $c_1, \ c_2 \geqslant 1$ such that for every $x, y \in G$,

$$\|x^{-1}\| \leq c_1\|x\| \quad \text{and} \quad \|x \circ y\| \leq c_2(\|x\| + \|y\|).$$

In view of the above properties, it is natural to define the quasidistance d by

$$d(x, y) = \|y^{-1} \circ x\|.$$

The ball with respect to d is denoted by

$$B_r(x) = B(x, r) = \{y \in G : d(x, y) < r\}.$$

Note that $B(0, r) = \delta(r)B(0, 1)$ and

$$|B(x, r)| = r^Q|B(0, 1)|, \tag{2.1}$$

where $x \in G$, $r > 0$, and

$$Q = \omega_1 + \cdots + \omega_N. \tag{2.2}$$

We call that Q is the homogeneous dimension of G. By (2.1), the Lebesgue measure $\mathrm{d}x$ is a doubling measure with respect to d, that is there exists a positive constant c such that

$$|B(x, 2r)| \leq c|B(x, r)|, \ x \in G, \ r > 0$$

and therefore $(G, \mathrm{d}x, d)$ is a space of homogenous type.

Throughout this paper, we use $\|\cdot\|_{L^p(\Omega)}$ for the usual norms of $L^p(\Omega)$, where $p \in [1, \infty]$ and Ω is an open subset of G. Let $p \in (1, \infty)$. In what follows, we present the notations:

$$Du = (X_1u, X_2u, \ldots, X_mu);$$

$$\|Du\|^2_{L^2(G)} = \sum_{j=1}^{m} \int_G |X_ju(x)|^2 \mathrm{d}x;$$

$$S^{1,2}(G)=\{u|\ u\in L^{2^*}(G), Du\in L^2(G)\},$$

where $2^*=\frac{2Q}{Q-2}$. Clearly, $S^{1,2}(G)$ is Hilbert space, its scalar products is given by

$$(u,v)_1=\int_G DuDv\mathrm{d}x,\ u,v\in S^{1,2}(G).$$

The norm of $S^{1,2}(G)$ is introduced by

$$\|u\|_1^2=(u,u)_1.$$

The space $H^{1,2}(G)$ is defined as the completion of $C_0^\infty(G)$ with respect to the norm

$$\|u\|_H^2=\int_{\mathbb{R}^3}(|Du|^2+u^2)\mathrm{d}x.$$

Definition 2.1. We say that $u\in H^{1,2}(G)$ is a weak solution of (1.1) provided

$$\sum_{j=1}^m\int_G X_ju(x)X_j\varphi(x)\mathrm{d}x+\lambda\int_G u(x)\varphi(x)\mathrm{d}x=\int_G |u(x)|^{p-1}u(x)\varphi(x)\mathrm{d}x,$$

for all $\varphi\in H^{1,2}(G)$.

LEMMA 2.1 ([9]). *There is a unique fundamental solution* Γ *for* L *such that:*

(a) $\Gamma\in C^\infty(G\backslash\{0\})$;
(b) Γ *is homogeneous of degree* $2-Q$;
(c) *for every distribution* τ,

$$L(\tau*\Gamma)=(L\tau)*\Gamma=\tau.$$

Remark 2.1. The transpose L^T of L is also left invariant and hypoelliptic (see [5]). From the remark of [9], it follows that the fundamental solution of L^T is

$$\Gamma^T(z)=\Gamma(z^{-1}).$$

LEMMA 2.2 ([9, 15]). *For every* $1<p<\frac{Q}{2}$ *and* $g\in L^p(G)$, *the function*

$$Tg(x)=\int_G\Gamma(x,y)g(y)\mathrm{d}y$$

is a.e. defined, and there exists $\Lambda>0$ *such that*

$$\|Tg\|_{L^q(G)}\leqslant\Lambda\|g\|_{L^p(G)},$$

for any q *verifying* $\frac{1}{p}=\frac{1}{q}+\frac{2}{Q}$.

LEMMA 2.3. *The embedding* $S^{1,2}(G)\hookrightarrow L^{2^*}(G)$ *is continuous.*

Proof. By Lemma 2.1, we have

$$u(x) = \int_G \Gamma(x^{-1} \circ y) Lu(y) dy$$

for any $u \in C_0^\infty(G)$. Keeping in mind that $X_j^* = -X_j$, $j = 1, \dots, m$, integrating by parts at right-hand side, we obtain

$$u(x) = \int_G (D\Gamma)(x^{-1} \circ y) Du(y) dy.$$

It follows from Lemma 2.2 that there exists a positive constant C such that

$$\|u\|_{L^{2^*}(G)} \leq C\|u\|_1.$$

Note that $C_0^\infty(G)$ is dense in $S^{1,2}(G)$, we can complete the proof. □

3. PROOF OF THE MAIN RESULT

In this section, we prove Theorem 1.1 by using the Moser iterations method as in [17]. Let $r_1 > r_2 > 0$. Given two balls B_{r_2}, B_{r_1} and a function $\varphi \in C_0^\infty(G)$, let us write $B_{r_2} \prec \varphi \prec B_{r_1}$ to mean that $0 \leq \varphi \leq 1$, $\varphi = 1$ on B_{r_2} and $\text{supp}\varphi \subseteq B_{r_1}$.

LEMMA 3.1 ([5]). *For any $r_1 > r_2 > 0$ and positive integer k, there exists $\varphi \in C_0^\infty(G)$ with $B_{r_2} \prec \varphi \prec B_{r_1}$ and*

$$|P^j \varphi| \leq \frac{c}{(r_1 - r_2)^j}, \ 1 \leq j \leq k,$$

where c is a positive constant, P^j is any left invariant differential monomial homogeneous of degree j.

In the following, we will adapt the ideas found in [7, 12, 17] to prove our main result.

Proof of Theorem 1.1. We first give two functions. For $\beta > 1$, $M > 1$, the function $h \in C^1(0, +\infty)$ is denoted by

$$h(s) = \begin{cases} s^\beta, & s \in [0, M], \\ \beta M^{\beta-1} s - (\beta - 1) M^\beta, & s \in (M, \infty). \end{cases}$$

And we define the function $g : (0, +\infty) \to (0, +\infty)$ by

$$g(s) = \int_0^s |h'(t)|^2 \mathrm{d}t = \begin{cases} \frac{\beta^2}{2\beta-1} s^{2\beta-1}, & s \in (0, M], \\ \beta^2 M^{2\beta-1} s - \frac{2\beta^2(\beta-1)}{2\beta-1} M^{2\beta-1}, & s \in (M, \infty). \end{cases}$$

Then, it is obvious that g, h are Lipschitz continuous in $[0, +\infty)$, and $g(u)$, $h(u) \in S^{1,2}(G)$ if $u \in S^{1,2}(G)$. We can verify that

$$0 \leq sg(s) \leq s^2 h'^2(s) \leq \beta^2 h^2(s). \tag{3.1}$$

Let $\bar\eta \in C_0^\infty(G)$, $0 < r_2 < r_1$ and $B_{r_2} \prec \bar\eta \prec B_{r_1}$. For each $y \in G$, setting $\eta(y) = \bar\eta(y^{-1} \circ x)$ and $\eta^2 g(u) \in H^{1,2}(G)$ with compact support in $B_{r_1}(y) \cap \{x \in G : u(x) \neq 0\}$. Combining (3.1) with the fact that u is the weak solution of (1.2), we conclude that

$$\begin{aligned}\int_G DuD(\eta^2 g(u))\mathrm{d}x + \lambda\int_G \eta^2 ug(u)\mathrm{d}x &= \int_G |u|^{p-1}\eta^2 ug(u)\mathrm{d}x \\ &\le \beta^2\int_G |u|^{p-1}\eta^2 h^2(u)\mathrm{d}x.\end{aligned} \tag{3.2}$$

It is easy to verify that

$$DuD(\eta^2 g(u)) = |Du|^2 h'^2(u)\eta^2 + 2DuD\eta g(u)\eta.$$

Moreover, we deduce from (3.1) that

$$\begin{aligned}|DuD\eta g(u)\eta| &\le \frac{1}{2}|\eta u^{-1/2} g^{1/2}(u) Du| \cdot 2|u^{1/2} g^{1/2}(u) D\eta| \\ &\le \frac{1}{4}|Du|^2 h'^2(u)\eta^2 + 4\beta^2|D\eta|^2 h^2(u).\end{aligned}$$

Hereby, we derive from (3.2) that

$$\begin{aligned}&\int_G |Du|^2 h'^2(u)\eta^2\mathrm{d}x \le \int_G |DuD(\eta^2 g(u))|\mathrm{d}x + 2\int_G |DuD\eta g(u)\eta|\mathrm{d}x \\ &\le \beta^2\int_G |u|^{p-1}\eta^2 h^2(u)\mathrm{d}x + \frac{1}{2}\int_G |Du|^2 h'^2(u)\eta^2 dx + 8\beta^2\int_G |D\eta|^2 h^2(u)\mathrm{d}x,\end{aligned}$$

which implies

$$\int_G |Du|^2 h'^2(u)\eta^2\mathrm{d}x \le 2\beta^2\int_G |u|^{p-1}\eta^2 h^2(u)\mathrm{d}x + 16\beta^2\int_G |D\eta|^2 h^2(u)\mathrm{d}x.$$

Note that $\beta > 1$, then making use of Hölder inequality ensures that

$$\begin{aligned}&\int_G |D(h(u)\eta)|^2\mathrm{d}x \le 2\int_G |D\eta|^2 h^2(u)\mathrm{d}x + 2\int_G |Du|^2|h'(u)|^2|\eta|^2\mathrm{d}x \\ &\le 4\beta^2\int_G |u|^{p-1}\eta^2 h^2(u)\mathrm{d}x + (32\beta^2+2)\int_G |D\eta|^2 h^2(u)\mathrm{d}x \\ &\le (32\beta^2+2)\int_G |D\eta|^2 h^2(u)\mathrm{d}x + 4\beta^2\|\eta h(u)\|^2_{L^{2^*}(G)}\|u\|^{p-1}_{L^{2^*}(G)}|B_{r_1}|^\gamma \\ &\le (32\beta^2+2)\int_G |D\eta|^2 h^2(u)\mathrm{d}x + 4\beta^2 C^2\|\eta h(u)\|^2_{L^{2^*}(G)}\|u\|^{p-1}_{L^{2^*}(G)}|B_{r_1}|^\gamma,\end{aligned} \tag{3.3}$$

where $\gamma = \frac{2^*-p-1}{2^*}$ and $|B_{r_1}|$ denotes the volume of B_{r_1}. Setting $\beta = \frac{2^*}{2} \triangleq \beta_0$ and

$$\begin{aligned}r_1 &= |B_1|^{-\frac{1}{Q}}(8\beta_0^2\|u\|^{p-1}_{L^{2^*}(G)}C^2+1)^{-\frac{1}{\gamma Q}} \\ &\le \min\{|B_1|^{-\frac{1}{Q}},\ |B_1|^{-\frac{1}{Q}}(8C^2\beta_0^2\|u\|^{p-1}_{L^{2^*}(G)})^{-\frac{1}{\gamma Q}}\},\end{aligned} \tag{3.4}$$

where C is the imbedding constant as in Lemma 2.3. It is easy to see that $4\beta_0^2\|u\|_{L^{2^*}(G)}^{p-1}|B_{r_1}|^{\gamma}C^2 \le \frac{1}{2}$ and $|B_{r_1}| \le 1$. Then taking $\beta = \beta_0$, (3.3) becomes

$$\int_G |D(h(u)\eta)|^2 \mathrm{d}x \le (64\beta_0^2 + 4)\int_G |D\eta|^2 h^2(u)\mathrm{d}x.$$

Hence, by Lemma 2.3 and using the fact that $\eta h(u) \in S^{1,2}(G)$, it follows

$$\begin{aligned}\|h(u)\|^2_{L^{2^*}(B_{r_2}(y))} &\le \left(\int_G |h(u)\eta|^{2^*}\mathrm{d}x\right)^{2/2^*} \le C^2\int_G |D(h(u)\eta)|^2\mathrm{d}x \\ &\le C^2(64\beta_0^2+4)\int_G |D\eta|^2 h^2(u)\mathrm{d}x \\ &\le \frac{c^2C^2(64\beta_0^2+4)}{(r_1-r_2)^2}\|h(u)\|^2_{L^2(B_{r_1}(y))}.\end{aligned} \tag{3.5}$$

By definition of h, there holds $h(s) = s^{\beta}$ if $M \to +\infty$. Noting that $\beta = \beta_0$ and $2\beta_0 = 2^*$, which together with (3.5) implies

$$\|u\|^{2\beta_0}_{L^{2\beta_0^2}(B_{r_2}(y))} \le \frac{c^2C^2(64\beta_0^2+4)}{(r_1-r_2)^2}\|u\|^{2\beta_0}_{L^{2^*}(B_{r_1}(y))}.$$

For each $i \ge 2$, let $r_i = \frac{2+2^{-i}}{4}r_1$. Let $\bar{\eta}_i \in C_0^\infty(G)$ and $B_{r_{i+1}} \prec \bar{\eta}_i \prec B_{r_i}$. Set $\rho = \frac{2\beta_0}{2\beta_0^2+1-p}$ and $\rho \in (0,1)$ by $p \in (1, \frac{Q+2}{Q-2})$. For $i \ge 2$, taking $\eta = \eta_i(x) = \bar{\eta}_i(y^{-1}\circ x)$ and $\beta = \beta_i \triangleq \rho^{-i} > 1$, then invoking Hölder inequality and using the fact that $|B_{r_i}| < |B_{r_1}| \le 1$, we have

$$\begin{aligned}\int_G |D(h(u)\eta_i)|^2\mathrm{d}x &\le 4\beta_i^2\int_G |u|^{p-1}\eta_i^2h^2(u)\mathrm{d}x + (32\beta_i^2+2)\int_G |D\eta_i|^2h^2(u)\mathrm{d}x \\ &\le \left(\frac{2}{r_i-r_{i+1}}\right)^2(32\beta_i^2+2)|B_{r_i}|^{\frac{2^*\rho-2}{^*\rho}}\|h(u)\|^2_{L^{2^*\rho}(B_{r_i}(y))} \\ &\quad + 4\beta_i^2\|h(u)\|^2_{L^{2^*\rho}(B_{r_i}(y))}\|u\|^{p-1}_{L^{2\beta_0^2}(B_{r_i}(y))} \\ &\le \left((32\beta_i^2+2)\left(\frac{2}{r_i-r_{i+1}}\right)^2 + 4\beta_i^2\|u\|^{p-1}_{L^{2\beta_0^2}(B_{r_i}(y))}\right)\|h(u)\|^2_{L^{2^*\rho}(B_{r_i}(y))} \\ &\le \left((32\beta_i^2+2)\left(\frac{2}{r_i-r_{i+1}}\right)^2 + \left(\frac{c^2C^2(64\beta_0^2+4)}{(r_1-r_2)^2}\right)^{\frac{(p-1)}{2\beta_0}}\|u\|^{p-1}_{L^{2\beta_0^2}(B_{r_i}(y))}\right) \\ &\quad \times \|h(u)\|^2_{L^{2^*\rho}(B_{r_i}(y))} \\ &\le \left(\frac{C_1\beta_i(1+\|u\|^{(p-1)/2}_{L^{2^*}(G)})}{r_i-r_{i+1}}\right)^2\|h(u)\|^2_{L^{2^*\rho}(B_{r_i}(y))},\end{aligned} \tag{3.6}$$

where C_1 is a constant depending only on N and p. In view of $\eta_i h(u) \in S^{1,2}(G)$, it follows from Lemma 2.3 that

$$\begin{aligned}\|h(u)\|^2_{L^{2^*}(B_{r_{i+1}}(y))} &\le C^2 \int_G |D(h(u)\eta_i)|^2 dx \\ &\le \left(\frac{C_2\beta_i(1+\|u\|^{(p-1)/2}_{L^{2^*}(G)})}{r_i - r_{i+1}}\right)^2 \|h(u)\|^2_{L^{2^*\rho}(B_{r_i}(y))},\end{aligned}$$

where $C_2 = CC_1$ depends only on Q and p. Let $M \to +\infty$, and $h(u) = u^{\beta_i}$, we obtain that

$$\|u\|_{L^{2^*\beta_i}(B_{r_{i+1}}(y))} \le \left(\frac{C_2\beta_i(1+\|u\|^{(p-1)/2}_{L^{2^*}(G)})}{r_i - r_{i+1}}\right)^{1/\beta_i} \|u\|_{L^{2^*\beta_{i-1}}(B_{r_i}(y))}.$$

Then combining with the Moser iterations in a standard way, it leads to

$$\begin{aligned}\|u\|_{L^{2^*\beta_i}(B_{r_{i+1}}(y))} &\le \Pi^i_{l=2}\left(\frac{C_2\beta_l(1+\|u\|^{(p-1)/2}_{L^{2^*}(G)})}{r_l - r_{l+1}}\right)^{1/\beta_l} \|u\|_{L^{2^*\beta_1}(B_{r_2}(y))} \\ &= \left(\frac{2}{\rho}\right)^{f(i)}\left(\frac{8C_2(1+\|u\|^{(p-1)/2}_{L^{2^*}(G)})}{r_1}\right)^{k(i)} \|u\|^2_{L^{2^*\beta_1}(B_{r_2}(y))},\end{aligned} \tag{3.7}$$

where $f(i) = \frac{2\rho^2}{1-\rho} + \frac{\rho^3(1-\rho^{i-2})}{(1-\rho)^2} - \frac{i\rho^{i+1}}{1-\rho} \to \frac{2\rho^2-\rho^3}{(1-\rho)^2}$ and $k(i) = \frac{\rho^2(1-\rho^{i-1})}{1-\rho} \to \frac{\rho^2}{1-\rho}$ as $i \to \infty$. Notice that $2^*\beta_1 = 2\beta_0^2 + 1 - p < 2\beta_0^2$, then by virtue of Hölder inequality, (3.4) and (3.6), there holds

$$\begin{aligned}\|u\|_{L^{2^*\beta_1}(B_{r_2}(y))} &\le |B_{r_2}|^{\frac{p-1}{2\beta_0^2 2^*\beta_1}} \|u\|_{L^{2\beta_0^2}(B_{r_2}(y))} \\ &\le \left(\frac{c^2C^2(64\beta_0^2+4)}{(r_1-r_2)^2}\right)^{\frac{1}{2\beta_0}} \|u\|_{L^{2^*}(G)}.\end{aligned}$$

We deduce from (3.7) that

$$\begin{aligned}&\|u\|_{L^{2^*\beta_i}(B_{r_1/2}(y))} \le \|u\|_{L^{2^*\beta_i}(B_{r_{i+1}}(y))} \\ &\le \left(\frac{2}{\rho}\right)^{f(i)}\left(\frac{8C_2(1+\|u\|^{(p-1)/2}_{L^{2^*}(G)})}{r_1}\right)^{k(i)}\left(\frac{(\frac{16}{7})^2c^2C^2(64\beta_0^2+4)}{r_1^2}\right)^{\frac{1}{2\beta_0}} \|u\|_{L^{2^*}(G)}.\end{aligned}$$

By taking the limit as $i \to \infty$, we know there exist positive constants $C_4(P, Q)$,

$C_5(P,Q)$ such that

$$\|u\|_{L^\infty(B_{r_1/2}(y))} \leq \left(\frac{2}{\rho}\right)^{\frac{2\rho^2-\rho^3}{(1-\rho)^2}} \left(\frac{8C_2(1+\|u\|_{L^{2*}(G)}^{(p-1)/2})}{r_1}\right)^{\frac{\rho^2}{1-\rho}}$$

$$\times \left(\frac{(\frac{16}{7})^2 c^2 C^2 (64\beta_0^2+4)}{r_1^2}\right)^{\frac{1}{2\beta_0}} \|u\|_{L^{2^*}(G)}$$

$$\leq \left(\frac{2}{\rho}\right)^{\frac{2\rho^2-\rho^3}{(1-\rho)^2}} \left(\frac{C_3(1+\|u\|_{L^{2^*}(G)}^{(p-1)/2})}{r_1}\right)^{\frac{\rho^2}{1-\rho}+\frac{1}{\beta_0}} \|u\|_{L^{2^*}(G)}$$

$$\leq \left(\frac{2}{\rho}\right)^{\frac{2\rho^2-\rho^3}{(1-\rho)^2}} \Big(|B_1|^{1/Q} C_3(1+\|u\|_{L^{2*}(G)}^{(p-1)/2})$$

$$\times (8\beta_0^2\|u\|_{L^{2^*}(G)}^{p-1}C^2+1)^{\frac{1}{\gamma Q}}\Big)^{\frac{\rho^2}{1-\rho}+\frac{1}{\beta_0}} \|u\|_{L^{2^*}(G)}$$

$$\leq C_4(p,Q)(1+\|u\|_{L^{2^*}(G)}^{C_5(p,Q)})\|u\|_{L^{2^*}(G)}.$$

Recalling that $y \in G$, and r_1 is fixed, we can derive that

$$\|u\|_{L^\infty(G)} \leq C_4(p,Q)(1+\|u\|_{L^{2^*}(G)}^{C_5(p,Q)})\|u\|_{L^{2^*}(G)}.$$

This completes the proof. □

4. APPLICATIONS

Example 4.1. Let $u \in H^{1,2}(H^n)$ be a weak solution of (1.4). Then by Theorem 1.1, it follows that u is in $L^q(H^n)$ for any $1 < q < \infty$. Moreover,

$$\|u\|_{L^\infty(G)} \leq C_6(p,n)(1+\|u\|_{L^{2^*}(G)}^{C_7(p,n)})\|u\|_{L^{2^*}(G)}.$$

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Shanxi University
School of Mathematical Sciences
Taiyuan 030006, People's Republic of China
fxj467@mail.nwpu.edu.cn