HYPERSURFACES WITH CONSTANT QUASI-GAUSS-KRONECKER CURVATURE IN $\mathbf{M}^{n+1}(c)$

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Let M^n be an n-dimensional ($n \geq 3$) complete smooth connected and oriented hypersurface in a real space form $\mathbf{M}^{n+1}(c)$ ($c = 0, 1, -1$) with constant quasi-Gauss-Kronecker curvature and two distinct principal curvatures. Denoting by H the mean curvature, $|A|^2$ the squared norm of the second fundamental form and K_q the quasi-Gauss-Kronecker curvature of M^n, we obtain some characterizations of $\mathbf{S}^k(a) \times \mathbf{R}^{n-k}$ or $\mathbf{S}^k(a) \times \mathbf{S}^{n-k}(\sqrt{1-a^2})$ or $\mathbf{S}^k(a) \times \mathbf{H}^{n-k}(-\sqrt{1+a^2})$ in terms of H, $|A|^2$ and K_q, where $1 \leq k \leq n-1$ and $\mathbf{S}^k(a)$ is the k-dimensional sphere with radius a.

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1. INTRODUCTION

Let M^n be an n-dimensional immersed hypersurface in a real space form $\mathbf{M}^{n+1}(c)$. If $c = 0$, $c > 0$ or $c < 0$, we call $\mathbf{M}^{n+1}(c)$ the Euclidean space, sphere space or hyperbolic space. Let (h_{ij}) be the second fundamental form. Denote by $H = \frac{1}{n}\sum_{i=1}^{n} h_{ii}$ the mean curvature, by $|A|^2 = \sum_{i,j=1}^{n} h_{ij}^2$ the squared norm of the second fundamental form, and by $K = \det(h_{ij})$ the Gauss-Kronecker curvature of M^n. We notice that if M^n is an n-dimensional immersed hypersurface in a real space form $\mathbf{M}^{n+1}(c)$ with constant mean curvature H or constant m-th mean curvature H_m and two distinct principal curvatures, there are many important characterization results of M^n (see [4, 6, 11–16]). If M^n is an n-dimensional immersed hypersurface in $\mathbf{M}^{n+1}(c)$ with constant squared norm of the second fundamental form $|A|^2$ and two distinct principal curvatures, or with constant Gauss-Kronecker curvature K and two distinct principal curvatures, the author and others also obtained some interesting characterization results of M^n (see [9, 10]). Putting $\mu_{ij} = h_{ij} - H\delta_{ij}$, we define the so-called quasi-Gauss-Kronecker curvature of M^n by $K_q = \det(\mu_{ij})$, which is a conformal

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invariant (see [5]). We notice that if M^n is a minimal hypersurface, then the quasi-Gauss-Kronecker curvature is exactly the Gauss-Kronecker curvature.

Since H, $|A|^2$, K and K_q are the important invariants of M^n under the isometric immersion, it is natural for us to ask the following question: if M^n has constant quasi-Gauss-Kronecker curvature and two distinct principal curvatures, what characterization results can we obtain?

In this article, we try to study such a problem and give some characterization results. We introduce the well-known standard models of complete hypersurfaces with constant quasi-Gauss-Kronecker curvature in $\mathbf{M}^{n+1}(c)$:

When $c = 0$, we consider the Riemannian product immersion

$$\mathbf{S}^k(a) \times \mathbf{R}^{n-k} \hookrightarrow \mathbf{M}^{n+1}(c),$$

where $a > 0$ and $1 \leq k \leq n-1$, then it has two distinct constant principal curvatures$\frac{1}{a}$ and 0 with multiplicities k and $n-k$, respectively. We easily see that $\mathbf{S}^k(a) \times \mathbf{R}^{n-k}$ has constant quasi-Gauss-Kronecker curvature

$$K_q = \Big(\frac{k}{k-n}\Big)^{n-k}\Big(\frac{n-k}{n}\Big)^n \frac{1}{a^n}.$$

If $k = n-1$, then $K_q = -\frac{n-1}{n^n}\frac{1}{a^n} < 0$. If $k = 1$, then $K_q = (-1)^n \frac{1-n}{n^n}\frac{1}{a^n}$. Thus $\frac{1}{a} = -n(\frac{K_q}{1-n})^{\frac{1}{n}} > 0$, this implies that it must have $K_q > 0$ and n is an odd number.

Putting $G = \frac{K_q}{1-n}$ and denote by $|A|^2$ and H the squared norm of the second fundamental form and the mean curvature of $\mathbf{S}^{n-1}(a) \times \mathbf{R}^1$ or $\mathbf{S}^1(a) \times \mathbf{R}^{n-1}$, then for $\mathbf{S}^{n-1}(a) \times \mathbf{R}^1$, where $a = \frac{1}{n}G^{-\frac{1}{n}}$ and $G > 0$ we have

$$|A|^2 = n^2(n-1)G^{\frac{2}{n}}, \quad H = n(n-1)G^{\frac{1}{n}},$$

for $\mathbf{S}^1(a) \times \mathbf{R}^{n-1}$, where $a = -\frac{1}{n}G^{-\frac{1}{n}}$, $G < 0$ and n is an odd number, we have

$$|A|^2 = n^2 G^{\frac{2}{n}}, \quad H = -nG^{\frac{1}{n}}.$$

When $c = 1$, we consider the standard immersions $\mathbf{S}^{n-k}(\sqrt{1-a^2}) \hookrightarrow \mathbf{R}^{n-k+1}$ and $\mathbf{S}^k(a) \hookrightarrow \mathbf{R}^{k+1}$, where $0 < a < 1$, $1 \leq k \leq n-1$, and take the Riemannian product immersion $\mathbf{S}^k(a) \times \mathbf{S}^{n-k}(\sqrt{1-a^2}) \hookrightarrow \mathbf{S}^{n+1}(c) \subset \mathbf{R}^{n+2}$, then it has two distinct constant principal curvatures

$$\lambda_1 = \cdots = \lambda_k = \frac{\sqrt{1-a^2}}{a}, \quad \lambda_{k+1} = \cdots = \lambda_n = -\frac{a}{\sqrt{1-a^2}},$$

respectively. We easily see that $\mathbf{S}^k(a) \times \mathbf{S}^{n-k}(\sqrt{1-a^2})$ has constant quasi-Gauss-Kronecker curvature

$$K_q = \Big(\frac{k}{k-n}\Big)^{n-k}\Big(\frac{n-k}{n}\Big)^n\Big(\frac{1}{a\sqrt{1-a^2}}\Big)^n.$$

If $k = n-1$, then $K_q = -\frac{n-1}{n^n}\frac{1}{(a\sqrt{1-a^2})^n} < 0$. If $k = 1$, then $K_q = (-1)^n\frac{1-n}{n^n}\frac{1}{(a\sqrt{1-a^2})^n}$. Thus $\frac{1}{a\sqrt{1-a^2}} = -n(\frac{K_q}{1-n})^{\frac{1}{n}} > 0$. This implies that we must have $K_q > 0$ and n is an odd number.

Putting $G = \frac{K_q}{1-n}$ and denoting by $|A|^2$ and H the squared norm of the second fundamental form and the mean curvature of $\mathbf{S}^{n-1}(a)\times\mathbf{S}^1(\sqrt{1-a^2})$ or $\mathbf{S}^1(a)\times\mathbf{S}^{n-1}(\sqrt{1-a^2})$, then for

$$\mathbf{S}^{n-1}(a)\times\mathbf{S}^1(\sqrt{1-a^2}),\ \text{where } a^2 = \frac{1}{2}\pm\frac{\sqrt{n^2G^{2/n}-4}}{2nG^{1/n}} \text{ and } G > (2/n)^n > 0,$$

or for

$$\mathbf{S}^1(a)\times\mathbf{S}^{n-1}(\sqrt{1-a^2}),\ \text{where } a^2 = \frac{1}{2}\mp\frac{\sqrt{n^2G^{2/n}-4}}{2nG^{1/n}}$$
$$\text{and } G < -(2/n)^n < 0,\ n \text{ being an odd number,}$$

we have

$$|A|^2 = \frac{n}{2}G^{\frac{1}{n}}\big[n^2G^{\frac{1}{n}} \pm (n-2)\sqrt{n^2G^{\frac{2}{n}}-4}\big] - n,$$
$$H = \frac{1}{2}\big[(n-2)G^{\frac{1}{n}} \mp \sqrt{n^2G^{\frac{2}{n}}-4}\big].$$

When $c = -1$, we consider the standard immersions $\mathbf{H}^{n-k}(-\sqrt{1+a^2}) \hookrightarrow \mathbf{R}_1^{n-k+1}$ and $\mathbf{S}^k(a) \hookrightarrow \mathbf{R}^{k+1}$, where $a > 0$, $1 \le k \le n-1$, and take the Riemannian product immersion $\mathbf{S}^k(a)\times\mathbf{H}^{n-k}(-\sqrt{1+a^2}) \hookrightarrow \mathbf{H}^{n+1}(c) \subset \mathbf{R}_1^{n+2}$, then it has two distinct constant principal curvatures

$$\lambda_1 = \cdots = \lambda_k = \frac{\sqrt{1+a^2}}{a},\quad \lambda_{k+1} = \cdots = \lambda_n = \frac{a}{\sqrt{1+a^2}},$$

respectively. We easily see that $\mathbf{S}^k(a)\times\mathbf{H}^{n-k}(-\sqrt{1+a^2})$ has constant quasi-Gauss-Kronecker curvature

$$K_q = \Big(\frac{k}{k-n}\Big)^{n-k}\Big(\frac{n-k}{n}\Big)^n\Big(\frac{1}{a\sqrt{1+a^2}}\Big)^n.$$

If $k = n-1$, then $K_q = -\frac{n-1}{n^n}\frac{1}{(a\sqrt{1+a^2})^n} < 0$. If $k = 1$, then $K_q = (-1)^n\frac{1-n}{n^n}\frac{1}{(a\sqrt{1+a^2})^n}$. Thus $\frac{1}{a\sqrt{1+a^2}} = -n(\frac{K_q}{1-n})^{\frac{1}{n}} > 0$, this implies that it must have $K_q > 0$ and n is an odd number.

Putting $G = \frac{K_q}{1-n}$ and denote by $|A|^2$ and H the squared norm of the second fundamental form and the mean curvature of $\mathbf{S}^{n-1}(a)\times\mathbf{H}^1(-\sqrt{1+a^2})$ or $\mathbf{S}^1(a)\times\mathbf{H}^{n-1}(-\sqrt{1+a^2})$, then for

$$\mathbf{S}^{n-1}(a)\times\mathbf{H}^1(-\sqrt{1+a^2}),\ \text{where } a^2 = -\frac{1}{2}+\frac{\sqrt{n^2G^{2/n}+4}}{2nG^{1/n}} \text{ and } G > 0,$$

or for

$$\mathbf{S}^1(a)\times\mathbf{H}^{n-1}(-\sqrt{1+a^2}),\ \text{where } a^2=-\frac{1}{2}-\frac{\sqrt{n^2G^{2/n}+4}}{2nG^{1/n}}$$

and $G<0$, n being an odd number,

we have

$$|A|^2=\frac{n}{2}G^{\frac{1}{n}}\big[n^2G^{\frac{1}{n}}+(n-2)\sqrt{n^2G^{\frac{2}{n}}+4}\big]+n,$$
$$H=\frac{1}{2}\big[(n-2)G^{\frac{1}{n}}+\sqrt{n^2G^{\frac{2}{n}}+4}\big].$$

If M^n has two distinct principal curvatures and the multiplicities of both principal curvatures are greater than 1, we obtain the following:

Theorem 1.1. *Let M^n be an n-dimensional, $n\geq 3$, complete smooth connected and oriented hypersurface in a real space form $\mathbf{M}^{n+1}(c)$ with constant quasi-Gauss-Kronecker curvature and two distinct principal curvatures. If the multiplicities of both principal curvatures are constant and greater than 1, then*

(1) for $c=0$, M^n is isometric to $\mathbf{S}^k(a)\times\mathbf{R}^{n-k}$, where $a=F$;

(2) for $c=1$ and $F^2\leq 1/4$, M^n is isometric to $\mathbf{S}^k(a)\times\mathbf{S}^{n-k}(\sqrt{1-a^2})$, where $a^2=\frac{1\pm\sqrt{1-4F^2}}{2}$;

(3) for $c=-1$, M^n is isometric to $\mathbf{S}^k(a)\times\mathbf{H}^{n-k}(-\sqrt{1+a^2})$, where $a^2=\frac{-1\pm\sqrt{1+4F^2}}{2}$.

In the above, $F=(\frac{k}{k-n})^{(n-k)/n}\frac{n-k}{n}K_q^{-1/n}$ and $1<k<n-1$.

If M^n has two distinct principal curvatures one of which is simple, in order to state Theorem 1.2 and Theorem 1.3 briefly, we denote

$$\delta^{\pm}(n,G)=\frac{n}{2}G^{\frac{1}{n}}\big[n^2G^{\frac{1}{n}}\pm(n-2)\sqrt{n^2G^{2/n}-4}\big]-n,$$
$$\epsilon^{+}(n,G)=\frac{n}{2}G^{\frac{1}{n}}\big[n^2G^{\frac{1}{n}}+(n-2)\sqrt{n^2G^{2/n}+4}\big]+n,$$
$$\alpha^{\pm}(n,G)=\frac{1}{2}\big[(n-2)G^{\frac{1}{n}}\pm\sqrt{n^2G^{\frac{2}{n}}-4}\big],$$
$$\beta^{+}(n,G)=\frac{1}{2}\big[(n-2)G^{\frac{1}{n}}+\sqrt{n^2G^{\frac{2}{n}}+4}\big],$$
$$b(n,G)^{\pm}=\frac{1}{2}\pm\frac{\sqrt{n^2G^{2/n}-4}}{2nG^{1/n}},$$
$$c(n,G)^{\pm}=-\frac{1}{2}\pm\frac{\sqrt{n^2G^{2/n}+4}}{2nG^{1/n}},$$

where $G=\frac{K_q}{1-n}$, and obtain the following:

THEOREM 1.2. *Let M^n be an n-dimensional with $n \geq 3$ complete smooth connected and oriented hypersurface in $\mathbf{M}^{n+1}(c)$ with constant quasi-Gauss-Kronecker curvature K_q and two distinct principal curvatures one of which is simple. Denote by $|A|^2$ the squared norm of the second fundamental form of M^n. Then $K_q \neq 0$.*

(I) *If $K_q < 0$, (i.e. $G > 0$), then*

(1) *for $c = 0$, if $|A|^2 \geq (n-1)n^2G^{\frac{2}{n}}$ or $|A|^2 \leq (n-1)n^2G^{\frac{2}{n}}$, then M^n is isometric to $\mathbf{S}^{n-1}(a) \times \mathbf{R}^1$, where $a = \frac{1}{n}G^{-\frac{1}{n}}$;*

(2) *for $c = 1$ and $K_q \leq -(\frac{1}{n-1})^{\frac{n-2}{2}}$, (i.e. $G \geq (\frac{1}{n-1})^{\frac{n}{2}}$),*

(i) *if $|A|^2 \geq \delta^+(n, G)$, then M^n is isometric to $\mathbf{S}^{n-1}(a) \times \mathbf{S}^1(\sqrt{1-a^2})$, where $a^2 = b(n, G)^+$;*

(ii) *if $|A|^2 \leq \delta^+(n, G)$, then M^n is isometric to $\mathbf{S}^{n-1}(a) \times \mathbf{S}^1(\sqrt{1-a^2})$, where $a^2 = b(n, G)^+$ or $a^2 = b(n, G)^-$;*

(iii) *if $|A|^2 \geq \delta^-(n, G)$, then M^n is isometric to $\mathbf{S}^{n-1}(a) \times \mathbf{S}^1(\sqrt{1-a^2})$, where $a^2 = b(n, G)^-$ or $a^2 = b(n, G)^+$;*

(iv) *if $|A|^2 \leq \delta^-(n, G)$, then M^n is isometric to $\mathbf{S}^{n-1}(a) \times \mathbf{S}^1(\sqrt{1-a^2})$, where $a^2 = b(n, G)^-$;*

(3) *for $c = -1$, if $|A|^2 \geq \epsilon^+(n, G)$ or $|A|^2 \leq \epsilon^+(n, G)$, then M^n is isometric to $\mathbf{S}^{n-1}(a) \times \mathbf{H}^1(-\sqrt{1+a^2})$, where $a^2 = c(n, G)^+$.*

(II) *If $K_q > 0$, (i.e. $G < 0$), and n is an odd number, then*

(1) *for $c = 0$, if $|A|^2 \geq n^2G^{\frac{2}{n}}$ or $|A|^2 \leq n^2G^{\frac{2}{n}}$, then M^n is isometric to $\mathbf{S}^1(a) \times \mathbf{R}^{n-1}$, where $a = -\frac{1}{n}G^{-\frac{1}{n}}$;*

(2) *for $c = 1$ and $K_q \geq (\frac{1}{n-1})^{\frac{n-2}{2}}$, (i.e. $G \leq -(\frac{1}{n-1})^{\frac{n}{2}}$),*

(i) *if $|A|^2 \geq \delta^+(n, G)$, then M^n is isometric to $\mathbf{S}^1(a) \times \mathbf{S}^{n-1}(\sqrt{1-a^2})$, where $a^2 = b(n, G)^-$ or $a^2 = b(n, G)^+$;*

(ii) *if $|A|^2 \leq \delta^+(n, G)$, then M^n is isometric to $\mathbf{S}^1(a) \times \mathbf{S}^{n-1}(\sqrt{1-a^2})$, where $a^2 = b(n, G)^-$;*

(iii) *if $|A|^2 \geq \delta^-(n, G)$, then M^n is isometric to $\mathbf{S}^1(a) \times \mathbf{S}^{n-1}(\sqrt{1-a^2})$, where $a^2 = b(n, G)^+$;*

(iv) *if $|A|^2 \leq \delta^-(n, G)$, then M^n is isometric to $\mathbf{S}^1(a) \times \mathbf{S}^{n-1}(\sqrt{1-a^2})$, where $a^2 = b(n, G)^-$ or $a^2 = b(n, G)^+$;*

(3) *for $c = -1$, if $|A|^2 \geq \epsilon^+(n, G)$ or $|A|^2 \leq \epsilon^+(n, G)$, then M^n is isometric to $\mathbf{S}^1(a) \times \mathbf{H}^{n-1}(-\sqrt{1+a^2})$, where $a^2 = c(n, G)^-$.*

THEOREM 1.3. *Let M^n be an n-dimensional with $n \geq 3$ complete smooth connected and oriented hypersurface in $\mathbf{M}^{n+1}(c)$ with constant quasi-Gauss-Kronecker curvature K_q and two distinct principal curvatures one of which is simple. Denote by H the mean curvature of M^n. Then $K_q \neq 0$.*

(I) *If $K_q < 0$, (i.e. $G > 0$), then*

(1) *for* $c=0$, *if* $H \geq n(n-1)G^{\frac{1}{n}}$, *or* $H \leq n(n-1)G^{\frac{1}{n}}$, *then* M^n *is isometric to* $\mathbf{S}^{n-1}(a) \times \mathbf{R}^1$, *where* $a = \frac{1}{n}G^{-\frac{1}{n}}$;

(2) *for* $c=1$ *and* $K_q \leq -(\frac{1}{n-1})^{\frac{n-2}{2}}$, *(i.e.* $G \geq (\frac{1}{n-1})^{\frac{n}{2}}$ *)*,

(i) *if* $H \geq \alpha^+(n,G)$, *then* M^n *is isometric to* $\mathbf{S}^{n-1}(a) \times \mathbf{S}^1(\sqrt{1-a^2})$, *where* $a^2 = b(n,G)^-$;

(ii) *if* $H \leq \alpha^+(n,G)$, *then* M^n *is isometric to* $\mathbf{S}^{n-1}(a) \times \mathbf{S}^1(\sqrt{1-a^2})$, *where* $a^2 = b(n,G)^-$ *or* $a^2 = b(n,G)^+$;

(iii) *if* $H \geq \alpha^-(n,G)$, *then* M^n *is isometric to* $\mathbf{S}^{n-1}(a) \times \mathbf{S}^1(\sqrt{1-a^2})$, *where* $a^2 = b(n,G)^-$ *or* $a^2 = b(n,G)^+$;

(iv) *if* $H \leq \alpha^-(n,G)$, *then* M^n *is isometric to* $\mathbf{S}^{n-1}(a) \times \mathbf{S}^1(\sqrt{1-a^2})$, *where* $a^2 = b(n,G)^+$;

(3) *for* $c=-1$, *if* $H \geq \beta^+(n,G)$ *or* $H \leq \beta^+(n,G)$, *then* M^n *is isometric to* $\mathbf{S}^{n-1}(a) \times \mathbf{H}^1(-\sqrt{1+a^2})$, *where* $a^2 = c(n,G)^+$.

(II) *If* $K_q > 0$, *(i.e.* $G < 0$*), and* n *is an odd number, then*

(1) *for* $c=0$, *if* $H \geq -n\, G^{\frac{1}{n}}$ *or* $H \leq -n\, G^{\frac{1}{n}}$, *then* M^n *is isometric to* $\mathbf{S}^1(a) \times \mathbf{R}^{n-1}$, *where* $a = -\frac{1}{n}G^{-\frac{1}{n}}$;

(2) *for* $c=1$ *and* $K_q \geq (\frac{1}{n-1})^{\frac{n-2}{2}}$, *(i.e.* $G \leq -(\frac{1}{n-1})^{\frac{n}{2}}$*)*,

(i) *if* $H \geq \alpha^+(n,G)$, *then* M^n *is isometric to* $\mathbf{S}^1(a) \times \mathbf{S}^{n-1}(\sqrt{1-a^2})$, *where* $a^2 = b(n,G)^+$;

(ii) *if* $H \leq \alpha^+(n,G)$, *then* M^n *is isometric to* $\mathbf{S}^1(a) \times \mathbf{S}^{n-1}(\sqrt{1-a^2})$, *where* $a^2 = b(n,G)^+$ *or* $a^2 = b(n,G)^-$;

(iii) *if* $H \geq \alpha^-(n,G)$, *then* M^n *is isometric to* $\mathbf{S}^1(a) \times \mathbf{S}^{n-1}(\sqrt{1-a^2})$, *where* $a^2 = b(n,G)^+$ *or* $a^2 = b(n,G)^-$;

(iv) *if* $H \leq \alpha^-(n,G)$, *then* M^n *is isometric to* $\mathbf{S}^1(a) \times \mathbf{S}^{n-1}(\sqrt{1-a^2})$, *where* $a^2 = b(n,G)^-$;

(3) *for* $c=-1$, *if* $H \geq \beta^+(n,G)$ *or* $H \leq \beta^+(n,G)$, *then* M^n *is isometric to* $\mathbf{S}^1(a) \times \mathbf{H}^{n-1}(-\sqrt{1+a^2})$, *where* $a^2 = c(n,G)^-$.

2. PRELIMINARIES

Let M^n be an n-dimensional complete smooth connected and oriented hypersurface in a real space form $\mathbf{M}^{n+1}(c)$. We choose a local orthonormal frame $e_1, \ldots, e_{n+1}$ in $\mathbf{M}^{n+1}(c)$ such that $e_1, \ldots, e_n$ are tangent to M^n. Let $\omega_1, \ldots, \omega_{n+1}$ be the dual coframe. We use the following convention on the range of indices:

$$1 \leq A, B, C, \ldots \leq n+1; \quad 1 \leq i, j, k, \ldots \leq n.$$

The structure equations of $\mathbf{M}^{n+1}(c)$ are given by

$$
\begin{aligned}
&\mathrm{d}\omega_A=\sum_B \omega_{AB}\wedge\omega_B, \quad \omega_{AB}+\omega_{BA}=0,\\
&\mathrm{d}\omega_{AB}=\sum_C \omega_{AC}\wedge\omega_{CB}+\Omega_{AB}, \quad \Omega_{AB}=-\frac{1}{2}\sum_{C,D}K_{ABCD}\omega_C\wedge\omega_D,\\
&K_{ABCD}=c(\delta_{AC}\delta_{BD}-\delta_{AD}\delta_{BC}),
\end{aligned}
$$

where Ω_{AB} and K_{ABCD} denote the curvature form and the components of the curvature tensor of $\mathbf{M}^{n+1}(c)$, respectively.

Restricting to M^n,

$$\omega_{n+1}=0, \tag{2.1}$$

$$\omega_{n+1i}=\sum_j h_{ij}\omega_j, \quad h_{ij}=h_{ji}, \tag{2.2}$$

where h_{ij} denotes the components of the second fundamental form of M^n. The structure equations of M^n are

$$\mathrm{d}\omega_i=\sum_j \omega_{ij}\wedge\omega_j, \quad \omega_{ij}+\omega_{ji}=0,$$

$$\mathrm{d}\omega_{ij}=\sum_k \omega_{ik}\wedge\omega_{kj}+\Omega_{ij}, \quad \Omega_{ij}=-\frac{1}{2}\sum_{k,l}R_{ijkl}\omega_k\wedge\omega_l, \tag{2.3}$$

$$R_{ijkl}=c(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})+(h_{ik}h_{jl}-h_{il}h_{jk}), \tag{2.4}$$

where Ω_{ij} and R_{ijkl} denote the curvature form and the components of the curvature tensor of M^n, respectively. From (2.4), we have

$$n(n-1)(r-c)=n^2H^2-|A|^2,$$

where $n(n-1)r=R$ is the scalar curvature, H is the mean curvature and $|A|^2$ is the squared norm of the second fundamental form of M^n.

Putting $\mu_{ij}=h_{ij}-H\delta_{ij}$, we call $K_q=\det(\mu_{ij})$ the quasi-Gauss-Kronecker curvature of M^n. We choose $e_1,\ldots,e_n$ such that $h_{ij}=\lambda_i\delta_{ij}$, then we see that

$$K_q=\det(\mu_{ij})=(\lambda_1-H)(\lambda_2-H)\cdots(\lambda_n-H). \tag{2.5}$$

From (2.2) we obtain

$$\omega_{n+1i}=\lambda_i\omega_i, \quad i=1,2,\ldots,n.$$

Hence, we get from the structure equations of M^n,

$$
\begin{aligned}
\mathrm{d}\omega_{n+1i}=&\mathrm{d}\lambda_i\wedge\omega_i+\lambda_i\mathrm{d}\omega_i\\
=&\mathrm{d}\lambda_i\wedge\omega_i+\lambda_i\sum_j\omega_{ij}\wedge\omega_j.
\end{aligned} \tag{2.6}
$$

On the other hand, we have on the curvature forms of $\mathbf{M}^{n+1}(c)$,

$$\begin{aligned}\Omega_{n+1i} =& -\frac{1}{2}\sum_{C,D} K_{n+1iCD}\omega_C \wedge \omega_D\\ =& -\frac{1}{2}\sum_{C,D} c(\delta_{n+1C}\delta_{iD} - \delta_{n+1D}\delta_{iC})\omega_C \wedge \omega_D\\ =& - c\omega_{n+1} \wedge \omega_i = 0.\end{aligned}$$

Therefore, from the structure equations of $M^{n+1}(c)$, we obtain

$$\begin{aligned}\mathrm{d}\omega_{n+1i} =& \sum_j \omega_{n+1j} \wedge \omega_{ji} + \omega_{n+1n+1} \wedge \omega_{n+1i} + \Omega_{n+1i}\\ =& \sum_j \lambda_j \omega_{ij} \wedge \omega_j.\end{aligned} \tag{2.7}$$

From (2.6) and (2.7), we get

$$\mathrm{d}\lambda_i \wedge \omega_i + \sum_j (\lambda_i - \lambda_j)\omega_{ij} \wedge \omega_j = 0. \tag{2.8}$$

Putting

$$\psi_{ij} = (\lambda_i - \lambda_j)\omega_{ij}, \tag{2.9}$$

we have $\psi_{ij} = \psi_{ji}$. Hence (2.8) can be written as

$$\sum_j (\psi_{ij} + \delta_{ij}\mathrm{d}\lambda_j) \wedge \omega_j = 0.$$

By E. Cartan's Lemma, we get

$$\psi_{ij} + \delta_{ij}\mathrm{d}\lambda_j = \sum_k Q_{ijk}\omega_k, \tag{2.10}$$

where Q_{ijk} are uniquely determined functions such that

$$Q_{ijk} = Q_{ikj}.$$

3. PROOFS OF THEOREMS

We have the following Proposition 3.1 (originally see Otsuki [7]):

PROPOSITION 3.1. *Let M^n be a hypersurface in a real space form $\mathbf{M}^{n+1}(c)$ such that the multiplicities of the principal curvatures are constant. Then the distribution of the space of the principal vectors corresponding to each principal curvature is completely integrable. In particular, if the multiplicity of a principal curvature is greater than 1, then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of the principal vectors.*

Proof of Theorem 1.1. If M^n has two distinct principal curvatures λ and λ' of multiplicities k and $n-k$, where $1<k<n-1$, from (2.5) and $\mu_{ij} = h_{ij} - H\delta_{ij}$, we get

$$K_q = \Big(\frac{k}{k-n}\Big)^{n-k}\Big(\frac{n-k}{n}\Big)^{n}(\lambda-\lambda')^n. \tag{3.1}$$

Denote by $\mathcal{D}_\lambda$ and $\mathcal{D}_{\lambda'}$ the integral submanifolds of the corresponding distribution of the space of principal vectors corresponding to the principal curvature λ and λ', respectively. From Proposition 3.1, we know that λ is constant on $\mathcal{D}_\lambda$. From (3.1), we infer that λ' is constant on $\mathcal{D}_\lambda$. By Proposition 3.1 again, we get that λ' is constant on $\mathcal{D}_{\lambda'}$. Thus, we see that λ' is constant on M^n. By the same assertion we know that λ is constant on M^n. Therefore, M^n is isoparametric. By the classical results of Segre [8] and Cartan [2, 1] (see also [3, pp. 238]), we know that M^n is isometric to one of $\mathbf{S}^k(a)\times\mathbf{R}^{n-k}$, $a=F$ for $c=0$, or $\mathbf{S}^k(a)\times\mathbf{S}^{n-k}(\sqrt{1-a^2})$, $a^2=\frac{1\pm\sqrt{1-4F^2}}{2}$ for $c=1$, or $\mathbf{S}^k(a)\times\mathbf{H}^{n-k}(-\sqrt{1+a^2})$, $a^2=\frac{-1\pm\sqrt{1+4F^2}}{2}$ for $c=-1$, where $F=(\frac{k}{k-n})^{(n-k)/n}\frac{n-k}{n}K_q^{-1/n}$ and $1<k<n-1$. This completes the proof of Theorem 1.1. $\square$

Now, we consider the case that M^n has two distinct principal curvatures one of which is simple. Let M^n be an n-dimensional complete smooth connected and oriented hypersurface with two distinct principal curvatures one of which is simple and $n\geq 3$, that is, without loss of generality, we may assume

$$\lambda_1=\lambda_2=\cdots=\lambda_{n-1}=\lambda,\qquad \lambda_n=\lambda',$$

where λ_i for $i=1,2,\ldots,n$ are the principal curvatures of M^n. Thus, we get

$$K_q=-\frac{n-1}{n^n}(\lambda-\lambda')^n.$$

Putting $G=\frac{K_q}{1-n}$, we get

$$0\neq\lambda-\lambda'=n\Big(\frac{K_q}{1-n}\Big)^{\frac{1}{n}}=nG^{\frac{1}{n}}, \tag{3.2}$$

this implies that $K_q\neq 0$ and there only exist two cases: $K_q<0$ or $K_q>0$ and n must be odd numbers. Hence, we obtain

$$\lambda'=\lambda-nG^{\frac{1}{n}}, \tag{3.3}$$

where $G>0$ or $G<0$ and n must be odd numbers.

We denote the integral submanifold through $x\in M^n$ corresponding to λ by $M_1^{n-1}(x)$. Putting

$$\mathrm{d}\lambda=\sum_{k=1}^n\lambda_{,k}\,\omega_k,\qquad \mathrm{d}\lambda'=\sum_{k=1}^n\lambda'_{,k}\,\omega_k,$$

from Proposition 3.1, we have

$$\lambda_{,1}=\lambda_{,2}=\cdots=\lambda_{,n-1}=0 \quad \text{on} \quad M_1^{n-1}(x). \tag{3.4}$$

From (3.3), we have

$$\mathrm{d}\lambda'=\mathrm{d}\lambda. \tag{3.5}$$

Thus, we also have

$$\lambda'_{,1}=\lambda'_{,2}=\cdots=\lambda'_{,n-1}=0 \quad \text{on} \quad M_1^{n-1}(x). \tag{3.6}$$

In this case, we may consider locally λ as a function of the arc length s of the integral curve of the principal vector field e_n corresponding to the principal curvature λ'. From (2.10) and (3.4), we have for $1\le j\le n-1$,

$$\lambda_{,n}\,\omega_n=\sum_{i=1}^{n}\lambda_{,i}\,\omega_i=\mathrm{d}\lambda=\mathrm{d}\lambda_j=\sum_{k=1}^{n}Q_{jjk}\omega_k=\sum_{k=1}^{n-1}Q_{jjk}\omega_k+Q_{jjn}\omega_n.$$

Therefore, we have

$$Q_{jjk}=0,\quad 1\le k\le n-1,\quad \text{and}\quad Q_{jjn}=\lambda_{,n}\,. \tag{3.7}$$

By (2.10) and (3.6), we have

$$\lambda'_{,n}\,\omega_n=\sum_{i=1}^{n}\lambda'_{,i}\,\omega_i=\mathrm{d}\lambda'=\mathrm{d}\lambda_n=\sum_{k=1}^{n}Q_{nnk}\omega_k=\sum_{k=1}^{n-1}Q_{nnk}\omega_k+Q_{nnn}\omega_n.$$

Hence, we obtain

$$Q_{nnk}=0,\quad 1\le k\le n-1,\quad \text{and}\quad Q_{nnn}=\lambda'_{,n}\,. \tag{3.8}$$

From (3.5), we get

$$Q_{nnn}=\lambda'_{,n}=\lambda_{,n}\,.$$

From the definition of ψ_{ij}, if $i\ne j$, we have $\psi_{ij}=0$ for $1\le i\le n-1$ and $1\le j\le n-1$. Therefore, from (2.10), if $i\ne j$ and $1\le i\le n-1$ and $1\le j\le n-1$ we have

$$Q_{ijk}=0,\quad \text{for any}\quad k. \tag{3.9}$$

By (2.10), (3.7), (3.8) and (3.9), for $j<n$, we get

$$\begin{aligned}\psi_{jn}&=\sum_{k=1}^{n}Q_{jnk}\omega_k\\&=Q_{jjn}\omega_j+Q_{jnn}\omega_n=\lambda_{,n}\,\omega_j.\end{aligned} \tag{3.10}$$

From (2.9), (3.2) and (3.10), for $j<n$, we have

$$\omega_{jn}=\frac{\psi_{jn}}{\lambda-\lambda'}=\frac{\lambda_{,n}}{nG^{\frac{1}{n}}}\omega_j.$$

Thus, from the structure equations of M^n we have

$$\mathrm{d}\omega_n = \sum_{k=1}^{n-1} \omega_k \wedge \omega_{kn} + \omega_{nn} \wedge \omega_n = 0.$$

Therefore, we may put $\omega_n = \mathrm{d}s$. By (3.4), we get

$$\mathrm{d}\lambda = \lambda_{,n}\, ds, \qquad \lambda_{,n} = \frac{\mathrm{d}\lambda}{\mathrm{d}s}.$$

Thus, we have

$$\omega_{jn} = \frac{\frac{\mathrm{d}\lambda}{\mathrm{d}s}}{nG^{\frac{1}{n}}}\omega_j = \frac{\mathrm{d}(\frac{1}{n}G^{-\frac{1}{n}}\lambda)}{\mathrm{d}s}\omega_j = \frac{\mathrm{d}(\ln e^{\frac{1}{n}G^{-\frac{1}{n}}\lambda})}{\mathrm{d}s}\omega_j. \tag{3.11}$$

From (3.11) and the structure equations of $\mathbf{M}^{n+1}(c)$, for $j < n$, we have

$$\begin{aligned}
\mathrm{d}\omega_{jn} &= \sum_{k=1}^{n-1} \omega_{jk} \wedge \omega_{kn} + \omega_{jn} \wedge \omega_{nn} + \omega_{jn+1} \wedge \omega_{n+1n} + \Omega_{jn} \\
&= \sum_{k=1}^{n-1} \omega_{jk} \wedge \omega_{kn} + \omega_{jn+1} \wedge \omega_{n+1n} - c\omega_j \wedge \omega_n \\
&= \frac{\mathrm{d}(\ln e^{\frac{1}{n}G^{-\frac{1}{n}}\lambda})}{\mathrm{d}s} \sum_{k=1}^{n-1} \omega_{jk} \wedge \omega_k - (\lambda\lambda' + c)\omega_j \wedge \mathrm{d}s.
\end{aligned}$$

Differentiating (3.11), we have

$$\begin{aligned}
\mathrm{d}\omega_{jn} =& \frac{\mathrm{d}^2(\ln e^{\frac{1}{n}G^{-\frac{1}{n}}\lambda})}{\mathrm{d}s^2}\mathrm{d}s \wedge \omega_j + \frac{\mathrm{d}(\ln e^{\frac{1}{n}G^{-\frac{1}{n}}\lambda})}{\mathrm{d}s}\mathrm{d}\omega_j \\
=& \frac{\mathrm{d}^2(\ln e^{\frac{1}{n}G^{-\frac{1}{n}}\lambda})}{\mathrm{d}s^2}\mathrm{d}s \wedge \omega_j + \frac{\mathrm{d}(\ln e^{\frac{1}{n}G^{-\frac{1}{n}}\lambda})}{\mathrm{d}s} \sum_{k=1}^{n} \omega_{jk} \wedge \omega_k \\
=& \Big\{ - \frac{\mathrm{d}^2(\ln e^{\frac{1}{n}G^{-\frac{1}{n}}\lambda})}{\mathrm{d}s^2} + \Big[\frac{\mathrm{d}(\ln e^{\frac{1}{n}G^{-\frac{1}{n}}\lambda})}{\mathrm{d}s}\Big]^2\Big\}\omega_j \wedge \mathrm{d}s \\
&+ \frac{\mathrm{d}(\ln e^{\frac{1}{n}G^{-\frac{1}{n}}\lambda})}{\mathrm{d}s} \sum_{k=1}^{n-1} \omega_{jk} \wedge \omega_k.
\end{aligned}$$

From the previous two equalities, we have

$$\frac{\mathrm{d}^2(\ln e^{\frac{1}{n}G^{-\frac{1}{n}}\lambda})}{\mathrm{d}s^2} - \Big\{\frac{\mathrm{d}(\ln e^{\frac{1}{n}G^{-\frac{1}{n}}\lambda})}{\mathrm{d}s}\Big\}^2 - (\lambda\lambda' + c) = 0. \tag{3.12}$$

Defining $\varpi = e^{-\frac{1}{n}G^{-\frac{1}{n}}\lambda}$, from (3.12), we get

$$\frac{\mathrm{d}^2\varpi}{\mathrm{d}s^2} + \varpi(\lambda\lambda' + c) = 0. \tag{3.13}$$

On the other hand, from (3.11), we have $\nabla_{e_n} e_n = \sum_{i=1}^n \omega_{ni}(e_n)e_i = 0$. By the definition of geodesic, we know that any integral curve of the principal vector field corresponding to the principal curvature λ' is a geodesic. Thus, we see that $\varpi(s)$ is a function defined in $(-\infty, +\infty)$ since M^n is complete and any integral curve of the principal vector field corresponding to λ' is a geodesic.

We can prove the following Lemma:

LEMMA 3.2. *The positive function ϖ is bounded from above.*

Proof. From (3.3) and (3.13), we get

$$\frac{\mathrm{d}^2\varpi}{\mathrm{d}s^2} + \varpi(\lambda^2 - nG^{\frac{1}{n}}\lambda + c) = 0, \tag{3.14}$$

that is

$$\frac{\mathrm{d}^2\varpi}{\mathrm{d}s^2} + \varpi\left[n^2G^{\frac{2}{n}}(\ln\varpi)^2 + n^2G^{\frac{2}{n}}\ln\varpi + c\right] = 0. \tag{3.15}$$

Multiplying (3.15) by $2\frac{d\varpi}{ds}$ and integrating, we get

$$\left(\frac{\mathrm{d}\varpi}{\mathrm{d}s}\right)^2 + c\varpi^2 + n^2G^{\frac{2}{n}}\varpi^2(\ln\varpi)^2 = C,$$

where C is a constant. Thus, we have

$$c + n^2G^{\frac{2}{n}}(\ln\varpi)^2 \le \frac{C}{\varpi^2}. \tag{3.16}$$

If the positive function ϖ is not bounded from above, that is, $\varpi \to +\infty$, from (3.16), we conclude a contradiction for all $c = 0, 1, -1$ since $G^{\frac{2}{n}} > 0$. Lemma 3.2 is proved. □

The following Lemma is obvious:

LEMMA 3.3. (1) *Let*

$$P(x) = x^2 - nG^{\frac{1}{n}}x + c.$$

Then $P(x)$ gets its minimum at $x_0 = \frac{1}{2}nG^{\frac{1}{n}}$, $x \ge x_0$ if and only if $P(x)$ is an increasing function, $x \le x_0$ if and only if $P(x)$ is a decreasing function and $P(x)$ has two real roots $x_1 = \frac{1}{2}(nG^{\frac{1}{n}} - \sqrt{n^2G^{\frac{2}{n}} - 4c})$, $x_2 = \frac{1}{2}(nG^{\frac{1}{n}} + \sqrt{n^2G^{\frac{2}{n}} - 4c})$, where $|G| \ge (2/n)^n$ if $c = 1$;

(2) *Let*

$$|A|^2(x) = nx^2 - 2nG^{\frac{1}{n}}x + n^2G^{\frac{2}{n}}$$

Then $|A|^2(x)$ gets its minimum at $x_0' = G^{\frac{1}{n}}$ and $x \ge x_0'$ if and only if $|A|^2(x)$ is an increasing function, $x \le x_0'$ if and only if $|A|^2(x)$ is a decreasing function.

(3) *Let*

$$H(x) = nx - nG^{\frac{1}{n}}.$$

Then $H(x)$ is a strictly increasing function of x.

Proof of Theorem 1.2. From (3.2), we know that $K_q \neq 0$ and there only exist two cases: $K_q < 0$ or $K_q > 0$ and n is an odd number. Since we denote $G = \frac{K_q}{1-n}$, these two cases are equivalent to $G > 0$ or $G < 0$ and n is an odd number.

(I) If $G > 0$, putting $x = \lambda$, from (3.3), we see that the squared norm of second fundamental form $|A|^2 = (n-1)\lambda^2 + (\lambda - nG^{\frac{1}{n}})^2 = n\lambda^2 - 2nG^{\frac{1}{n}}\lambda + n^2G^{\frac{2}{n}} = |A|^2(x)$. From (3.14), we have

$$\frac{\mathrm{d}^2\varpi}{\mathrm{d}s^2} + \varpi P(x) = 0. \tag{3.17}$$

(1) For $c = 0$ and $G > 0$,

(i) If $|A|^2 \geq n^2(n-1)G^{\frac{2}{n}}$, that is, $|A|^2(x) \geq |A|^2(x_2)$, where $x_2 = nG^{\frac{1}{n}}$, we consider two cases: $x \geq x_0'$ and $x < x_0'$, where $x_0' = G^{\frac{1}{n}}$ is the minimum point of $|A|^2(x)$.

Case (i). If $x \geq x_0'$, since $x_0' < x_0$, where $x_0 = \frac{1}{2}nG^{\frac{1}{n}}$ is the minimum point of $P(x)$, we consider two subcases: $x_0' \leq x < x_0$ and $x \geq x_0$.

If $x_0' \leq x < x_0$, from Lemma 3.3 and (3.17), we get $P(x) \leq P(x_0') = -(n-1)G^{\frac{2}{n}} < 0$ and $\frac{\mathrm{d}^2\varpi}{\mathrm{d}s^2} > 0$, this implies that $\frac{\mathrm{d}\varpi(s)}{\mathrm{d}s}$ is a strictly monotone increasing function of s and thus it has at most one zero point for $s \in (-\infty, +\infty)$. If $\frac{\mathrm{d}\varpi(s)}{\mathrm{d}s}$ has no zero point in $(-\infty, +\infty)$, then $\varpi(s)$ is a monotone function of s in $(-\infty, +\infty)$. If $\frac{\mathrm{d}\varpi(s)}{\mathrm{d}s}$ has exactly one zero point s_0 in $(-\infty, +\infty)$, then $\varpi(s)$ is a monotone function of s in both $(-\infty, s_0]$ and $[s_0, +\infty)$.

On the other hand, from Lemma 3.2, we know that $\varpi(s)$ is bounded. Since $\varpi(s)$ is bounded and monotonic when s tends to infinity, we know that both $\lim_{s\to-\infty}\varpi(s)$ and $\lim_{s\to+\infty}\varpi(s)$ exist and then we get

$$\lim_{s\to-\infty}\frac{\mathrm{d}\varpi(s)}{\mathrm{d}s} = \lim_{s\to+\infty}\frac{\mathrm{d}\varpi(s)}{\mathrm{d}s} = 0, \tag{3.18}$$

this is impossible because $\frac{\mathrm{d}\varpi(s)}{\mathrm{d}s}$ is a strictly monotone increasing function of s. Therefore, the subcase $x_0' \leq x < x_0$ does not occur, it follows that $x \geq x_0$.

If $x \geq x_0$, from Lemma 3.3 and (3.17), we obtain that $|A|^2(x) \geq |A|^2(x_2)$ holds if and only if $x \geq x_2$, if and only if $P(x) \geq P(x_2) = 0$ and if and only if $\frac{\mathrm{d}^2\varpi}{\mathrm{d}s^2} \leq 0$. Thus, $\frac{\mathrm{d}\varpi}{\mathrm{d}s}$ is a monotonic decreasing function of $s \in (-\infty, +\infty)$, this implies that $\frac{\mathrm{d}\varpi(s)}{\mathrm{d}s}$ has at most one zero point for $s \in (-\infty, +\infty)$. By the same arguments as in above, we know that (3.18) holds. From the monotonicity of $\frac{\mathrm{d}\varpi(s)}{\mathrm{d}s}$, we have $\frac{\mathrm{d}\varpi(s)}{\mathrm{d}s} \equiv 0$ and $\varpi(s) = $ constant. Combining $\varpi = e^{-\frac{1}{n}G^{-\frac{1}{n}}\lambda}$ and (3.3), we conclude that λ and λ' are constant, that is, M^n is isoparametric. From the classical result of Segre [8], we know that M^n is isometric to $\mathbf{S}^{n-1}(a) \times \mathbf{R}^1$, $a = \frac{1}{n}G^{-\frac{1}{n}}$.

Case (ii). If $x < x_0'$, we consider two subcases: $x < 0$ and $0 \le x < x_0'$.

If $x < 0$, since $0 < x_0' < x_0$, from Lemma 3.3 and (3.17), we get $P(x) > P(0) = 0$ and $\frac{d^2\varpi}{ds^2} < 0$, this implies that $\frac{d\varpi(s)}{ds}$ is a strictly monotone decreasing function of s. By the same arguments as in case (i), we get (3.18) holds, this is impossible because $\frac{d\varpi(s)}{ds}$ is a strictly monotone decreasing function of s. Therefore, we know that the subcase $x < 0$ does not occur, it follows that $0 \le x < x_0'$.

If $0 \le x < x_0'$, from Lemma 3.3, we get $|A|^2(x) \le |A|^2(0) = n^2G^{\frac{2}{n}}$, this contradicts the assumption $|A|^2 \ge (n-1)n^2G^{\frac{2}{n}}$, thus, the subcase $0 \le x < x_0'$ also does not occur.

(ii) If $|A|^2 \le n^2(n-1)G^{\frac{2}{n}}$, that is, $|A|^2(x) \le |A|^2(x_2)$, we also consider two cases: $x \ge x_0'$ and $x < x_0'$.

Case (i). If $x \ge x_0'$, since $x_0' < x_0$, we consider two subcases: $x_0' \le x < x_0$ and $x_0 \le x$,.

If $x_0' \le x < x_0$, by the same arguments as in case (i) of (i), we easily see that the subcase $x_0' \le x < x_0$ does not occur, it follows that $x_0 \le x$.

If $x_0 \le x$, since $x_0' < x_0$, from Lemma 3.3 and (3.17), we obtain that $|A|^2(x) \le |A|^2(x_2)$ holds if and only if $x \le x_2$, if and only if $P(x) \le P(x_2) = 0$ and if and only if $\frac{\mathrm{d}^2\varpi}{\mathrm{d}s^2} \ge 0$. By the same arguments as in case (i) of (i) and from Segre [8], we know that M^n is isometric to $\mathbf{S}^{n-1}(a) \times \mathbf{R}^1$, $a = \frac{1}{n}(\frac{1}{G})^{\frac{1}{n}}$.

Case (ii). If $x < x_0'$, we consider two subcases: $x < 0$ and $0 \le x < x_0'$.

If $x < 0$, by the same arguments as in case (ii) of (i), we see that this does not occur, it follows that $0 \le x < x_0'$.

If $0 \le x < x_0'$, from Lemma 3.3 and (3.17), we get $|A|^2(x) \le |A|^2(0) = n^2G^{\frac{2}{n}}$, $P(x) \le P(0) = 0$ and $\frac{\mathrm{d}^2\varpi}{\mathrm{d}s^2} \ge 0$. By the same arguments as in case (i) of (i) and from Segre [8], we see that M^n is isometric to $\mathbf{S}^1(a) \times \mathbf{R}^{n-1}$, $a = -\frac{1}{n}(\frac{1}{G})^{\frac{1}{n}}$ and $|A|^2 = n^2G^{\frac{2}{n}}$. Since $a > 0$, this implies $G < 0$ and n is an odd number, contradicts the assumption $G > 0$. Therefore, the subcase $0 \le x < x_0'$ does not occur.

(2) For $c = 1$ and $K_q \le -(\frac{1}{n-1})^{\frac{n-2}{2}}$, that is, $G \ge (\frac{1}{n-1})^{\frac{n}{2}}$,

(i) if $|A|^2 \ge \delta^+(n, G)$, that is, $|A|^2(x) \ge |A|^2(x_2)$, where $x_2 = \frac{1}{2}(nG^{\frac{1}{n}} + \sqrt{n^2G^{\frac{2}{n}} - 4})$, we consider two cases: $x \ge x_0'$ and $x < x_0'$.

Case (i). If $x \geq x_0'$, since $x_0' < x_0$, we consider two subcases: $x_0' \leq x < x_0$ and $x \geq x_0$.

If $x_0' \leq x < x_0$, by the same arguments as in (i) of (1), we see that $x_0' \leq x < x_0$ does not occur, it follows that $x \geq x_0$.

If $x \geq x_0$, from Lemma 3.3 and (3.17), we obtain that $|A|^2(x) \geq |A|^2(x_2)$ holds if and only if $x \geq x_2$, if and only if $P(x) \geq P(x_2) = 0$ and if and only if $\frac{\mathrm{d}^2\varpi}{\mathrm{d}s^2} \leq 0$. By the same arguments as in (i) of (1) and from Cartan [2], we know that M^n is isometric to $\mathbf{S}^{n-1}(a) \times \mathbf{S}^1(\sqrt{1-a^2})$, where $a^2 = b(n,G)^+$.

Case (ii). If $x < x_0'$, we consider two subcases: $x < 0$ and $0 \leq x < x_0'$.

If $x < 0$, the same arguments as in (i) of (1) implies $x < 0$ does not occur, it follows that $0 \leq x < x_0'$.

If $0 \leq x < x_0'$, since $G^{\frac{2}{n}} \geq \frac{1}{n-1} (> \frac{4}{n^2})$, we see that $x_1 \leq x_0'$, where $x_1 = \frac{1}{2}(nG^{\frac{1}{n}} - \sqrt{n^2 G^{\frac{2}{n}} - 4})$. Thus, we may consider two subcases: $0 \leq x < x_1$ and $x_1 \leq x \leq x_0'$.

If $0 \leq x < x_1$, since $x_1 \leq x_0' < x_0$, by the same arguments as in (i) of (1), we see that $0 \leq x < x_1$ does not occur, it follows that $x_1 \leq x \leq x_0'$.

If $x_1 \leq x \leq x_0'$, from Lemma 3.3, we get $|A|^2(x) \leq |A|^2(x_1) = \delta^-(n,G)$, this contradicts the assumption $|A|^2 \geq \delta^+(n,G)$, since $G^{\frac{2}{n}} \geq \frac{1}{n-1}$ implies $\delta^-(n,G) < \delta^+(n,G)$. Thus, the subcase $x_1 \leq x \leq x_0'$ also does not occur.

(ii) If $|A|^2 \leq \delta^+(n,G)$, that is, $|A|^2(x) \leq |A|^2(x_2)$, we also consider two cases: $x \geq x_0'$ and $x < x_0'$.

Case (i). If $x \geq x_0'$, since $x_0' < x_0$, we consider two subcases: $x_0' \leq x < x_0$ and $x_0 \leq x$,.

If $x_0' \leq x < x_0$, the same arguments as in (i) of (1) implies $x_0' \leq x < x_0$ does not occur, it follows that $x_0 \leq x$.

If $x_0 \leq x$, since $x_0' < x_0$, from Lemma 3.3 and (3.17), we obtain that $|A|^2(x) \leq |A|^2(x_2)$ holds if and only if $x \leq x_2$, if and only if $P(x) \leq P(x_2) = 0$ and if and only if $\frac{\mathrm{d}^2\varpi}{\mathrm{d}s^2} \geq 0$. By the same arguments as in (i) of (1) and from Cartan [2], we see that M^n is isometric to $\mathbf{S}^{n-1}(a) \times \mathbf{S}^1(\sqrt{1-a^2})$, where $a^2 = b(n,G)^+$.

Case (ii). If $x < x_0'$, we consider two subcases: $x < 0$ and $0 \leq x < x_0'$.

If $x < 0$, we easily see that this does not occur, it follows that $0 \leq x < x_0'$.

If $0 \leq x < x_0'$, since $x_1 \leq x_0'$, we may also consider two subcases: $0 \leq x < x_1$ and $x_1 \leq x < x_0'$.

If $0 \leq x < x_1$, we easily see that this does not occur, it follows that $x_1 \leq x < x_0'$.

If $x_1 \leq x < x_0'$, since $x_0' < x_0$, from Lemma 3.3 and (3.17), we get $|A|^2(x) \leq |A|^2(x_1) = \delta^-(n,G)$, $P(x) \leq P(x_1) = 0$ and $\frac{d^2\varpi}{ds^2} \geq 0$. By the same arguments as in (i) of (1) and from Cartan [2], we know that M^n is isometric to $\mathbf{S}^{n-1}(a) \times \mathbf{S}^1(\sqrt{1-a^2})$, $a^2 = b(n,G)^-$.

(iii) If $|A|^2 \geq \delta^-(n,G)$, that is, $|A|^2(x) \geq |A|^2(x_1)$, where $x_1 = \frac{1}{2}(nG^{\frac{1}{n}} - \sqrt{n^2G^{\frac{2}{n}} - 4})$, we consider two subcases: $x < x_0'$ and $x \geq x_0'$.

Case (i). If $x < x_0'$, we consider two subcases: $x < 0$ and $0 \leq x < x_0'$.

If $x < 0$, we easily see that this does not occur, it follows that $0 \leq x < x_0'$.

If $0 \leq x < x_0'$, since $x_0' < x_0$, from Lemma 3.3 and (3.17), we obtain that $|A|^2(x) \geq |A|^2(x_1)$ holds if and only if $x \leq x_1$, if and only if $P(x) \geq P(x_1) = 0$ and if and only if $\frac{\mathrm{d}^2\varpi}{\mathrm{d}s^2} \leq 0$. By the same arguments as in (i) of (1) and from Cartan [2], we see that M^n is isometric to $\mathbf{S}^{n-1}(a) \times \mathbf{S}^1(\sqrt{1-a^2})$, where $a^2 = b(n,G)^-$.

Case (ii). If $x \geq x_0'$, we consider two subcases: $x_0' \leq x < x_0$ and $x \geq x_0$.

If $x_0' \leq x < x_0$, we easily see that this does not occur, it follows that $x \geq x_0$.

If $x \geq x_0$, since $x_0 < x_2$, we also consider two subcases: $x_0 \leq x < x_2$ and $x \geq x_2$.

If $x_0 \leq x < x_2$, we easily see that this does not occur, it follows that $x \geq x_2$.

If $x \geq x_2$, since $x \geq x_2 > x_0 > x_0'$, from Lemma 3.3 and (3.17), we see that $|A|^2(x) \geq |A|^2(x_2) = \delta^+(n,G)$, $P(x) \geq P(x_2) = 0$ and $\frac{\mathrm{d}^2\varpi}{\mathrm{d}s^2} \leq 0$. By the same arguments as in (i) of (1) and from Cartan [2], we know that M^n is isometric to $\mathbf{S}^{n-1}(a) \times \mathbf{S}^1(\sqrt{1-a^2})$, $a^2 = b(n,G)^+$.

(iv) If $|A|^2 \leq \delta^-(n,G)$, that is, $|A|^2(x) \leq |A|^2(x_1)$, we consider two subcases: $x < x_0'$ and $x \geq x_0'$.

Case (i). If $x < x_0'$, we consider two subcases: $x < 0$ and $0 \leq x < x_0'$.

If $x < 0$, we easily see that this does not occur, it follows that $0 \leq x < x_0'$.

If $0 \leq x < x_0'$, since $x_1 \leq x_0'$, we may consider two subcases: $0 \leq x < x_1$ and $x_1 \leq x \leq x_0'$.

If $0 \leq x < x_1$, we easily see that this does not occur, it follows that $x_1 \leq x \leq x_0'$.

If $x_1 \leq x \leq x_0'$, from Lemma 3.3 and (3.17), we get $|A|^2 \leq \delta^-(n,G)$ holds if and only if $x \geq x_1$, if and only if $P(x) \leq P(x_1) = 0$ and if and only if $\frac{\mathrm{d}^2\varpi}{\mathrm{d}s^2} \geq 0$. By the same arguments as in (i) of (1) and from Cartan [2], we see that M^n is isometric to $\mathbf{S}^{n-1}(a) \times \mathbf{S}^1(\sqrt{1-a^2})$, where $a^2 = b(n,G)^-$.

Case (ii). If $x \geq x_0'$, we consider two subcases: $x_0' \leq x < x_0$ and $x \geq x_0$.

If $x_0' \leq x < x_0$, we easily see that this does not occur, it follows that $x \geq x_0$.

If $x \geq x_0$, since $x_0 < x_2$, we also consider two subcases: $x_0 \leq x < x_2$ and $x \geq x_2$.

If $x_0 \le x < x_2$, we easily see that this does not occur, it follows that $x \ge x_2$.

If $x \ge x_2$, since $x \ge x_2 > x_0 > x_0'$, from Lemma 3.3, we see that $|A|^2(x) \ge |A|^2(x_2) = \delta^+(n,G)$, this contradicts the assumption $|A|^2 \le \delta^-(n,G)$, thus, the subcase $x \ge x_2$ does not occur.

(3) For $c = -1$ and $G > 0$,

(i) if $|A|^2 \ge \epsilon^+(n,G)$, that is, $|A|^2(x) \ge |A|^2(x_2)$, where $x_2 = \frac{1}{2}(nG^{\frac{1}{n}} + \sqrt{n^2G^{\frac{2}{n}}+4})$, we consider two cases: $x \ge x_0'$ and $x < x_0'$.

Case (i). If $x \ge x_0'$, since $x_0' < x_0$, we consider two subcases: $x_0' \le x < x_0$ and $x \ge x_0$.

If $x_0' \le x < x_0$, we easily see that this does not occur, it follows that $x \ge x_0$.

If $x \ge x_0$, from Lemma 3.3 and (3.17), we get $|A|^2(x) \ge |A|^2(x_2)$ holds if and only if $x \ge x_2$, if and only if $P(x) \ge P(x_2) = 0$ and if and only if $\frac{\mathrm{d}^2\varpi}{\mathrm{d}s^2} \le 0$. By the same arguments as in (i) of (1) and from Cartan [1], we see that M^n is isometric to $\mathbf{S}^{n-1}(a) \times \mathbf{H}^1(-\sqrt{1+a^2})$, where $a^2 = c(n,G)^+$.

Case (ii). If $x < x_0'$, we consider two subcases: $x \le 0$ and $0 < x < x_0'$.

If $x \le 0$, since $x_1 = \frac{1}{2}(nG^{\frac{1}{n}} - \sqrt{n^2G^{\frac{2}{n}}+4}) < 0$, we also consider two subcases: $x < x_1$ and $x_1 \le x \le 0$.

If $x < x_1$, we easily see that this does not occur, it follows that $x_1 \le x \le 0$.

If $x_1 \le x \le 0$, since $G > 0$, we see that $x_1 < x_0' < x_0$. Thus, from Lemma 3.3, we see that $|A|^2(x) \le |A|^2(x_1) = \epsilon^-(n,G)$, where $\epsilon^-(n,G) = \frac{n}{2}G^{\frac{1}{n}}\left[n^2G^{\frac{1}{n}} - (n-2)\sqrt{n^2G^{2/n}+4}\right] + n$. This contradicts the assumption $|A|^2 \ge \epsilon^+(n,G)$, thus $x_1 \le x < 0$ does not occur.

If $0 < x < x_0'$, by the same arguments as in (i) of (1), we also easily see that this does not occur.

(ii) If $|A|^2 \le \epsilon^+(n,G)$, that is, $|A|^2(x) \le |A|^2(x_2)$, we also consider two cases: $x \ge x_0'$ and $x < x_0'$.

Case (i). If $x \ge x_0'$, since $x_0' < x_0$, we consider two subcases: $x_0' \le x < x_0$ and $x_0 \le x$,.

If $x_0' \le x < x_0$, we easily see that this does not occur, it follows that $x_0 \le x$.

If $x_0 \le x$, since $x_0' < x_0$, from Lemma 3.3 and (3.17), we obtain that $|A|^2(x) \le |A|^2(x_2)$ holds if and only if $x \le x_2$, if and only if $P(x) \le P(x_2) = 0$ and if and only if $\frac{\mathrm{d}^2\varpi}{\mathrm{d}s^2} \ge 0$. By the same arguments as in (i) of (1) and from Cartan [1], we know that M^n is isometric to $\mathbf{S}^{n-1}(a) \times \mathbf{H}^1(-\sqrt{1+a^2})$, where $a^2 = c(n,G)^+$.

Case (ii). If $x < x_0'$, we consider two subcases: $x \le 0$ and $0 < x < x_0'$.

If $x \le 0$, we also consider two subcases: $x < x_1$ and $x_1 \le x \le 0$.

If $x < x_1$, we easily see that this does not occur, it follows that $x_1 \le x \le 0$.

If $x_1 \le x \le 0$, from Lemma 3.3 and (3.17), we see that $|A|^2(x) \le |A|^2(x_1) = \epsilon^-(n, G)$, $P(x) \le P(x_1) = 0$ and $\frac{\mathrm{d}^2\varpi}{\mathrm{d}s^2} \ge 0$. By the same arguments as in (i) of (1) and from Cartan [1], we see that M^n is isometric to $\mathbf{S}^{n-1}(a) \times \mathbf{H}^1(-\sqrt{1+a^2})$, $a^2 = -\frac{1}{2} - \frac{\sqrt{n^2G^{2/n}+4}}{2nG^{1/n}}$ and $|A|^2 = \epsilon^-(n, G)$. Since $a^2 > 0$, this implies $G < 0$ and n is an odd number, it contradicts the assumption $G > 0$. Thus $x_1 \le x \le 0$ does not occur, it follows that $0 < x < x_0'$.

If $0 < x < x_0'$, we easily see that this does not occur.

(II) If $G < 0$ and n is an odd number, in this case, we notice that $x_0 < x_0' < 0$, $x_1 = \frac{1}{2}(nG^{\frac{1}{n}} - \sqrt{n^2G^{\frac{2}{n}} - 4c}) < 0$, $x_2 = \frac{1}{2}(nG^{\frac{1}{n}} + \sqrt{n^2G^{\frac{2}{n}} - 4c}) = 0$ if $c = 0$, $x_2 < 0$ if $c = 1$ and $x_2 > 0$ if $c = -1$. Combining Lemma 3.3 and (3.17), we see that the rest of the proof of (II) suffices to use the same method as in the proof of (I).

Theorem 1.2 is proved. $\square$

Proof of Theorem 1.3. Similar to the proof of Theorem 1.2, we may consider two cases: $G > 0$ or $G < 0$ and n is an odd number.

(I) If $G > 0$, putting $x = \lambda$, from (3.3), we see that the mean curvature $H = (n-1)\lambda + (\lambda - nG^{\frac{1}{n}}) = n\lambda - nG^{\frac{1}{n}} = H(x)$.

(1) For $c = 0$ and $G > 0$,

(i) If $H \ge (n-1)nG^{\frac{1}{n}}$, that is, $H(x) \ge H(x_2)$, where $x_2 = nG^{\frac{1}{n}}$, we consider two cases: $x \ge x_0$ and $x < x_0$, where $x_0 = \frac{1}{2}nG^{\frac{1}{n}}$ is the minimum point of $P(x)$.

Case (i). If $x \ge x_0$, since $x_0 < x_2$, from Lemma 3.3 and (3.17), we obtain that $H(x) \ge H(x_2)$ holds if and only if $x \ge x_2$, if and only if $P(x) \ge P(x_2) = 0$ and if and only if $\frac{\mathrm{d}^2\varpi}{\mathrm{d}s^2} \le 0$. By the same arguments as in the proof of (I) in Theorem 1.2, we know that M^n is isometric to $\mathbf{S}^{n-1}(a) \times \mathbf{R}^1$, $a = \frac{1}{n}G^{-\frac{1}{n}}$.

Case (ii). If $x < x_0$, since $x_1 < x_0$, where $x_1 = 0$, we consider two subcases: $x \le x_1$ and $x_1 < x < x_0$.

If $x \le x_1$, from Lemma 3.3, we get $H(x) \le H(x_1) = -nG^{\frac{1}{n}} < 0$, this contradicts the assumption $H \ge (n-1)nG^{\frac{1}{n}}$. Thus, $x \le x_1$ does not occur, it follows that $x_1 < x < x_0$.

If $x_1 < x < x_0$, from Lemma 3.3 and (3.17), we have $P(x) < P(x_1) = 0$ and $\frac{\mathrm{d}^2\varpi}{\mathrm{d}s^2} > 0$, by the same arguments as in the proof of (I) in Theorem 1.2, we know that $x_1 < x < x_0$ also does not occur.

(ii) If $H \leq (n-1)nG^{\frac{1}{n}}$, that is, $H(x) \leq H(x_2)$, we consider two cases: $x \geq x_0$ and $x < x_0$.

Case (i). If $x \geq x_0$, from Lemma 3.3 and (3.17), we obtain that $H(x) \leq H(x_2)$ holds if and only if $x \leq x_2$, if and only if $P(x) \leq P(x_2) = 0$ and if and only if $\frac{\mathrm{d}^2\varpi}{\mathrm{d}s^2} \geq 0$. By the same arguments as in the proof of Theorem 1.2, we see that M^n is isometric to $\mathbf{S}^{n-1}(a) \times \mathbf{R}^1$, $a = \frac{1}{n}G^{-\frac{1}{n}}$.

Case (ii). If $x < x_0$, we consider two subcases: $x \leq x_1$ and $x_1 < x < x_0$.

If $x \leq x_1$, from Lemma 3.3 and (3.17), we get $H(x) \leq H(x_1) = -nG^{\frac{1}{n}}$, $P(x) \geq P(x_1) = 0$ and $\frac{\mathrm{d}^2\varpi}{\mathrm{d}s^2} \leq 0$. By the same arguments as in the proof of Theorem 1.2, we know that M^n is isometric to $\mathbf{S}^1(a) \times \mathbf{R}^{n-1}$, $a = -\frac{1}{n}G^{-\frac{1}{n}}$. Since $a > 0$, this implies $G < 0$ and n is an odd number, it contradicts the assumption $G > 0$, thus $x \leq x_1$ does not occur, it follows $x_1 < x < x_0$.

If $x_1 < x < x_0$, we easily see that this also does not occur.

(2) For $c = 1$ and $G \geq (\frac{1}{n-1})^{\frac{n}{2}}$,

(i) if $H \geq \alpha^+(n, G)$, that is, $H(x) \geq H(x_2)$, where $x_2 = \frac{1}{2}(nG^{\frac{1}{n}} + \sqrt{n^2G^{\frac{2}{n}} - 4})$, we consider two cases: $x \geq x_0$ and $x < x_0$.

Case (i). If $x \geq x_0$, since $x_0 < x_2$, from Lemma 3.3 and (3.17), we obtain that $H(x) \geq H(x_2)$ holds if and only if $x \geq x_2$, if and only if $P(x) \geq P(x_2) = 0$ and if and only if $\frac{d^2\varpi}{ds^2} \leq 0$. By the same arguments as in the proof of Theorem 1.2, we see that M^n is isometric to $\mathbf{S}^{n-1}(a)\times\mathbf{S}^1(\sqrt{1-a^2})$, where $a^2 = b(n, G)^-$.

Case (ii). If $x < x_0$, since $x_1 < x_0$, where $x_1 = \frac{1}{2}(nG^{\frac{1}{n}} - \sqrt{n^2G^{\frac{2}{n}} - 4})$, we consider two subcases: $x \leq x_1$ and $x_1 < x < x_0$.

If $x \leq x_1$, from Lemma 3.3, we get $H(x) \leq H(x_1) = \alpha^-(n, G)$, this contradicts the assumption $H \geq \alpha^+(n, G)$. Thus, $x \leq x_1$ does not occur, it follows that $x_1 < x < x_0$.

If $x_1 < x < x_0$, we easily see that this also does not occur.

(ii) If $H \leq \alpha^+(n, G)$, that is, $H(x) \leq H(x_2)$, we consider two cases: $x \geq x_0$ and $x < x_0$.

Case (i). If $x \geq x_0$, from Lemma 3.3 and (3.17), we obtain that $H(x) \leq H(x_2)$ holds if and only if $x \leq x_2$, if and only if $P(x) \leq P(x_2) = 0$ and if and only if $\frac{\mathrm{d}^2\varpi}{\mathrm{d}s^2} \geq 0$. By the same arguments as in the proof of Theorem 1.2, we know that M^n is isometric to $\mathbf{S}^{n-1}(a) \times \mathbf{S}^1(\sqrt{1-a^2})$, where $a^2 = b(n, G)^-$.

Case (ii). If $x < x_0$, we consider two subcases: $x \leq x_1$ and $x_1 < x < x_0$.

If $x \leq x_1$, from Lemma 3.3 and (3.17), we get $H(x) \leq H(x_1) = \alpha^-(n, G)$, $P(x) \geq P(x_1) = 0$ and $\frac{\mathrm{d}^2\varpi}{\mathrm{d}s^2} \leq 0$. By the same arguments as in the proof of Theorem 1.2, we know that M^n is isometric to $\mathbf{S}^{n-1}(a) \times \mathbf{S}^1(\sqrt{1-a^2})$, where $a^2 = b(n, G)^+$.

If $x_1 < x < x_0$, we easily see that this also does not occur.

(iii) if $H \geq \alpha^-(n,G)$, that is, $H(x) \geq H(x_1)$, we consider two cases: $x \geq x_0$ and $x < x_0$.

Case (i). If $x \geq x_0$, since $x_0 < x_2$, we consider two subcases: $x_0 \leq x < x_2$ and $x \geq x_2$.

If $x_0 \leq x < x_2$, we easily see that this does not occur, it follows that $x \geq x_2$.

If $x \geq x_2$, from Lemma 3.3 and (3.17), we get $H(x) \geq H(x_2)$, $P(x) \geq P(x_2) = 0$ and $\frac{\mathrm{d}^2\varpi}{\mathrm{d}s^2} \leq 0$. By the same arguments as in the proof Theorem 1.2, we know that M^n is isometric to $\mathbf{S}^{n-1}(a) \times \mathbf{S}^1(\sqrt{1-a^2})$, where $a^2 = b(n,G)^-$.

Case (ii). If $x < x_0$, since $x_1 < x_0$, we consider two subcases: $x < x_1$ and $x_1 \leq x < x_0$.

If $x < x_1$, we easily see that this does not occur, it follows that $x_1 \leq x < x_0$.

If $x_1 \leq x < x_0$, from Lemma 3.3 and (3.17), we obtain that $H(x) \geq H(x_1)$ holds if and only if $x \geq x_1$, if and only if $P(x) \leq P(x_1) = 0$ and if and only if $\frac{\mathrm{d}^2\varpi}{\mathrm{d}s^2} \geq 0$. By the same arguments as in the proof of Theorem 1.2, we see that M^n is isometric to $\mathbf{S}^{n-1}(a) \times \mathbf{S}^1(\sqrt{1-a^2})$, where $a^2 = b(n,G)^+$.

(iv) If $H \leq \alpha^-(n,G)$, that is, $H(x) \leq H(x_1)$, we consider two cases: $x \geq x_0$ and $x < x_0$.

Case (i). If $x \geq x_0$, we consider two subcases: $x_0 \leq x < x_2$ and $x \geq x_2$.

If $x_0 \leq x < x_2$, we easily see that this does not occur, it follows that $x \geq x_2$.

If $x \geq x_2$, from Lemma 3.3, we get $H(x) \geq H(x_2) = \alpha^+(n,G)$, this contradicts the assumption $H \leq \alpha^-(n,G)$, thus $x \geq x_2$ does not occur.

Case (ii). If $x < x_0$, we consider two subcases: $x \leq x_1$ and $x_1 < x < x_0$.

If $x \leq x_1$, from Lemma 3.3 and (3.17), we get $H(x) \leq H(x_1)$ holds if and only if $x \leq x_1$, if and only if $P(x) \geq P(x_1) = 0$ and if and only if $\frac{\mathrm{d}^2\varpi}{\mathrm{d}s^2} \leq 0$. By the same arguments as in the proof of Theorem 1.2, we see that M^n is isometric to $\mathbf{S}^{n-1}(a) \times \mathbf{S}^1(\sqrt{1-a^2})$, where $a^2 = b(n,G)^+$.

If $x_1 < x < x_0$, we easily see that this does not occur.

(3) For $c = -1$ and $G > 0$,

(i) if $H \geq \beta^+(n,G)$, that is, $H(x) \geq H(x_2)$, where $x_2 = \frac{1}{2}(nG^{\frac{1}{n}} + \sqrt{n^2G^{\frac{2}{n}} + 4})$, we consider two cases: $x \geq x_0$ and $x < x_0$.

Case (i). If $x \geq x_0$, since $x_0 < x_2$, from Lemma 3.3 and (3.17), we obtain that $H(x) \geq H(x_2)$ holds if and only if $x \geq x_2$, if and only if $P(x) \geq P(x_2) = 0$ and if and only if $\frac{\mathrm{d}^2\varpi}{\mathrm{d}s^2} \leq 0$. By the same arguments as in the proof of Theorem 1.2, we know that M^n is isometric to $\mathbf{S}^{n-1}(a) \times \mathbf{H}^1(-\sqrt{1+a^2})$, where $a^2 =$

$c(n,G)^{+}$.

Case (ii). If $x < x_0$, since $x_1 < x_0$, where $x_1 = \frac{1}{2}(nG^{\frac{1}{n}} - \sqrt{n^2 G^{\frac{2}{n}} + 4})$, we consider two subcases: $x \le x_1$ and $x_1 < x < x_0$.

If $x \le x_1$, from Lemma 3.3, we get $H(x) \le H(x_1) = \beta^{-}(n,G)$, where $\beta^{-}(n,G) = \frac{1}{2}\big[(n-2)G^{\frac{1}{n}} - \sqrt{n^2 G^{\frac{2}{n}} + 4}\big]$, this contradicts the assumption $H \ge \beta^{+}(n,G)$. Thus, $x \le x_1$ does not occur, it follows that $x_1 < x < x_0$.

If $x_1 < x < x_0$, we easily see that this does not occur.

(ii) If $H \le \beta^{+}(n,G)$, that is, $H(x) \le H(x_2)$, we consider two cases: $x \ge x_0$ and $x < x_0$.

Case (i). If $x \ge x_0$, from Lemma 3.3 and (3.17), we obtain that $H(x) \le H(x_2)$ holds if and only if $x \le x_2$ if and only if $P(x) \le P(x_2) = 0$ and if and only if $\frac{\mathrm{d}^2\varpi}{\mathrm{d}s^2} \ge 0$. By the same arguments as in the proof of Theorem 1.2, we see that M^n is isometric to $\mathbf{S}^{n-1}(a) \times \mathbf{H}^1(-\sqrt{1+a^2})$, where $a^2 = c(n,G)^{+}$.

Case (ii). If $x < x_0$, we consider two subcases: $x \le x_1$ and $x_1 < x < x_0$.

If $x \le x_1$, from Lemma 3.3 and (3.17), we get $H(x) \le H(x_1) = \beta^{-}(n,G)$, $P(x) \ge P(x_1) = 0$ and $\frac{d^2\varpi}{ds^2} \le 0$. By the same arguments as in the proof of Theorem 1.2, we see that M^n is isometric to $\mathbf{S}^{n-1}(a) \times \mathbf{H}^1(-\sqrt{1+a^2})$, where $a^2 = -\frac{1}{2} - \frac{\sqrt{n^2G^{2/n}+4}}{2nG^{1/n}}$ and $H = \beta^{-}(n,G)$. Since $a^2 > 0$, this implies $G < 0$ and n is an odd number, it contradicts the assumption $G > 0$, thus $x \le x_1$ does not occur.

If $x_1 < x < x_0$, we easily see that this does not occur.

(II) If $G < 0$ and n is an odd number, it suffices to use the same method as in the proof of (I) in Theorem 1.3.

Theorem 1.3 is proved. $\square$

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