COFINITENESS AND ANNIHILATORS OF TOP LOCAL COHOMOLOGY MODULES

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Let R be a Noetherian ring and I be an ideal of R. Let M be a finitely generated R-module with $\operatorname{cd}(I, M) = t \geq 0$ and assume that L is the largest submodule of M such that $\operatorname{cd}(I, L) < \operatorname{cd}(I, M)$. It is shown that $\operatorname{Ann}_R H^t_I(M) = \operatorname{Ann}_R M/L$ in each of the following cases: (i) $\dim M/IM \leq 1$. (ii) $\dim R/I \leq 1$. (iii) The R-module $H^i_I(M)$ is Artinian for each $i \geq 2$. (iv) The R-module $H^i_I(R)$ is Artinian for each $i \geq 2$. (v) $\operatorname{cd}(I, M) \leq 1$. (vi) $\operatorname{cd}(I, R) \leq 1$. (vii) The R-module $H^t_I(M)$ is Artinian and I-cofinite. These assertions answer affirmatively a question raised by Atazadeh et al. in [2], in some special cases.

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1. INTRODUCTION

Throughout this paper, let R denote a commutative Noetherian ring (with identity) and I an ideal of R. The local cohomology modules $H^i_I(M)$, $i = 0, 1, 2, \ldots$, of an R-module M with respect to I were introduced by Grothendieck [13]. They arise as the derived functors of the left exact functor $\Gamma_I(-)$, where for an R-module M, $\Gamma_I(M)$ is the submodule of M consisting of all elements annihilated by some power of I, i.e., $\bigcup_{n=1}^{\infty}(0 :_M I^n)$. There is a natural isomorphism:

$$H^i_I(M) \cong \varinjlim_{n \geq 1} \operatorname{Ext}^i_R(R/I^n, M).$$

We refer the reader to [13] or [10] for more details about local cohomology.

The problem of finding annihilators of local cohomology modules have been studied by several authors; see, for example, [2, 4, 5, 7, 8, 15, 16, 17, 20, 21].

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Recall that, for an R-module M, the *cohomological dimension of M with respect to I* is defined as

$$\operatorname{cd}(I, M) := \sup\{i \in \mathbb{Z} : \ H_I^i(M) \neq 0\}.$$

In [8] it is shown that if $(R, \mathfrak{m})$ is a complete Noetherian local ring and M is a finitely generated R-module then $\operatorname{Ann}_R H_{\mathfrak{m}}^{\dim M}(M) = \operatorname{Ann}_R M/T_R(M)$, where $T_R(M)$ is the largest submodule of M such that $\dim T_R(M) < \dim M$. This result was later extended to non-complete Noetherian local rings by the present author in [4]. Also, for an ideal I in an arbitrary Noetherian ring R (not necessarily local), in [2] Atazadeh et al. defined the submodule $T_R(I, M)$ of M as the largest submodule of M such that $\operatorname{cd}(I, T_R(I, M)) < \operatorname{cd}(I, M)$. They observed that, in general there is an isomorphism of R-modules

$$H_I^{\operatorname{cd}(I,M)}(M) \simeq H_I^{\operatorname{cd}(I,M)}(M/T_R(I, M)).$$

Then, as a generalization of the main result of [8], they proved that if $\operatorname{cd}(I, M) = \dim M < \infty$, then $\operatorname{Ann}_R H_I^{\dim M}(M) = \operatorname{Ann}_R M/T_R(I, M)$.

Furthermore, in the same paper they asked about the similar result in general. In this paper, we prepare some partially affirmative answers for this question. More precisely, we prove the following result:

THEOREM 1. *Let R be a Noetherian ring and I be an ideal of R. Let M be a non-zero finitely generated R-module with* $\operatorname{cd}(I, M) = t \geq 0$. *Then* $\operatorname{Ann}_R H_I^t(M) = \operatorname{Ann}_R M/T_R(I, M)$ *in each of the following cases:*

(i) $\dim M/IM \leq 1$.

(ii) $\dim R/I \leq 1$.

(iii) *The R-module $H_I^i(M)$ is Artinian for each $i \geq 2$.*

(iv) *The R-module $H_I^i(R)$ is Artinian for each $i \geq 2$.*

(v) $\operatorname{cd}(I, M) \leq 1$.

(vi) $\operatorname{cd}(I, R) \leq 1$.

(vii) *The R-module $H_I^t(M)$ is Artinian and I-cofinite.*

Our main tools in the proof of Theorem 1 is the following theorem.

THEOREM 2. *Let R be a Noetherian ring and let I be an ideal of R. Let M be a non-zero finitely generated R-module such that* $\operatorname{cd}(I, M) = t \geq 0$ *and the R-module* $\operatorname{Hom}_R(R/I, H_I^t(M/N))$ *is finitely generated, for each submodule N of M. Then*

$$\operatorname{Ann}_R(H_I^t(M)) = \operatorname{Ann}_R M/T_R(I, M).$$

For each R-module L, we denote by $\operatorname{mAss}_R L$ the set of minimal elements of $\operatorname{Ass}_R L$ with respect to inclusion. Also, in this paper for any Noetherian local ring $(R, \mathfrak{m})$ and any R-module M, $E_R(M)$ denotes the injective envelope of M and $D(-)$ denotes the Matlis duality functor $\operatorname{Hom}_R(-, E_R(R/\mathfrak{m}))$. In addition, for any ideal I of R, $D_I(-)$ denotes the I-transform functor

$$D_I(-) = \varinjlim_{n\geq 1} \operatorname{Hom}_R(I^n, -).$$

For any prime ideal $\mathfrak{p}$ of R and any positive integer n we denote the n^{th} symbolic power of $\mathfrak{p}$ by $\mathfrak{p}^{(n)}$. Also, for any ideal I of R, we denote the set $\operatorname{Supp} R/I = \{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \supseteq I\}$ by $V(I)$. Furthermore, for any ideal I of R, *the radical of* I, denoted by $\operatorname{Rad}(I)$, is defined to be the set $\{x \in R : x^n \in I$ for some $n \in \mathbb{N}\}$. Finally, for any finitely generated R-module M, the notion $f_I(M)$, the *finiteness dimension* of M relative to I, is defined to be the least integer i such that $H^i_I(M)$ is not finitely generated, if there exist such i's and ∞ otherwise, i.e.

$$f_I(M) := \inf\{i \in \mathbb{N}_0 : H^i_I(M) \text{ is not finitely generated}\}.$$

For any unexplained notation and terminology, we refer the reader to [18] or [10].

2. THE RESULTS

Recall that following [14], for a given ideal I of a Noetherian ring R, an arbitrary R-module M is said to be *I-cofinite*, if $\operatorname{Supp} M \subseteq V(I)$ and the R-module $\operatorname{Ext}^i_R(R/I, M)$ is finitely generated for each integer $i \geq 0$. For any R-module N, the second author of the present paper defined $\widetilde{q}(I, N)$ as the greatest integer i such that $H^i_I(N)$ is not an Artinian I-cofinite module if there exist such i's and $-\infty$ otherwise (see [6, Definition 2.4]).

Let R be a Noetherian ring and $\mathscr{C}(R)$ denote the category of all R-modules and R-homomorphisms. We say that a functor $T : \mathscr{C}(R) \longrightarrow \mathscr{C}(R)$ is additive if $T(g + h) = T(g) + T(h)$, for each pair of homomorphisms of R-modules $g : M \longrightarrow N$ and $h : M \longrightarrow N$. Also, we say that T is R-linear precisely when it is additive and $T(rf) = rT(f)$ for all $r \in R$ and all homomorphisms f of R-modules. The following theorem, which is needed in the proofs of Lemmata 2.2 and 2.5, is a generalization of [5, Lemma 2.3].

THEOREM 2.1. *Let $(R, \mathfrak{m})$ be a Noetherian complete local ring and I be an ideal of R. Let M be a non-zero finitely generated R-module such that $\operatorname{cd}(I, M) = t \geq 0$ and $\widetilde{q}(I, M) < \operatorname{cd}(I, M)$. Then*

$$\operatorname{Att}_R H^t_I(M) = \{\mathfrak{q} \in \operatorname{mAss}_R M : \dim R/\mathfrak{q} = t \text{ and } \operatorname{Rad}(I + \mathfrak{q}) = \mathfrak{m}\}.$$

Proof. Let $\Psi := \{\mathfrak{q} \in \operatorname{mAss}_R M : \dim R/\mathfrak{q} = t$ and $\operatorname{Rad}(I + \mathfrak{q}) = \mathfrak{m}\}$ and let $\mathfrak{q} \in \operatorname{Att} H^t_I(M)$. Then, it follows from the definition of the attached prime ideal that $\operatorname{Ann}_R H^t_I(M)/\mathfrak{q} H^t_I(M) = \mathfrak{q}$ and hence $H^t_I(M)/\mathfrak{q} H^t_I(M) \neq 0$. Assume that $a \in \operatorname{Ann}_R M$. Since by [10, Theorem 1.2.2(ii)], for each $i \geq 0$, the functor $H^i_I(-)$ is R-linear, one sees that

$$a\mathrm{Id}_{H^i_I(M)} = aH^i_I(\mathrm{Id}_M) = H^i_I(a\mathrm{Id}_M) = H^i_I(0) = 0,$$

where "Id" denotes the identity map. Therefore, $a \in \operatorname{Ann}_R H^i_I(M)$ and hence

$$\operatorname{Ann}_R M \subseteq \operatorname{Ann}_R H^i_I(M).$$

Now, since

$$\operatorname{Ann}_R M \subseteq \operatorname{Ann}_R H^t_I(M) \subseteq \operatorname{Ann}_R H^t_I(M)/\mathfrak{q} H^t_I(M) = \mathfrak{q},$$

it follows that $\mathfrak{q} \in \operatorname{Supp} M$ and hence $\operatorname{Supp} R/\mathfrak{q} = \operatorname{Supp} M/\mathfrak{q} M \subseteq \operatorname{Supp} M$. So, using the fact that $\operatorname{Supp} R/\operatorname{Ann}_R M = \operatorname{Supp} M$, by [12, Theorem 2.2] one has

$$\operatorname{cd}(I, R/\operatorname{Ann}_R M) = \operatorname{cd}(I, M) = t,$$

and

$$\operatorname{cd}(I, R/\mathfrak{q}) = \operatorname{cd}(I, M/\mathfrak{q} M) \leq \operatorname{cd}(I, M) = t.$$

Now, it follows from [10, Exercise 6.1.9] and Independence Theorem that

$$\begin{aligned}
H^t_I(M/\mathfrak{q} M) &\simeq H^t_{(I+\operatorname{Ann}_R M)/\operatorname{Ann}_R M}(R/\operatorname{Ann}_R M \otimes_R M/\mathfrak{q} M)\\
&\simeq H^t_{(I+\operatorname{Ann}_R M)/\operatorname{Ann}_R M}(R/\operatorname{Ann}_R M) \otimes_R M/\mathfrak{q} M\\
&\simeq H^t_{(I+\operatorname{Ann}_R M)/\operatorname{Ann}_R M}(R/\operatorname{Ann}_R M) \otimes_R \big(M \otimes_R R/\mathfrak{q}\big)\\
&\simeq \Big(H^t_{(I+\operatorname{Ann}_R M)/\operatorname{Ann}_R M}(R/\operatorname{Ann}_R M) \otimes_R M\Big) \otimes_R R/\mathfrak{q}\\
&\simeq H^t_{(I+\operatorname{Ann}_R M)/\operatorname{Ann}_R M}(M) \otimes_R R/\mathfrak{q}\\
&\simeq H^t_I(M) \otimes_R R/\mathfrak{q}\\
&\simeq H^t_I(M)/\mathfrak{q} H^t_I(M) \neq 0.
\end{aligned}$$

Therefore, $\operatorname{cd}(I, M/\mathfrak{q} M) \geq t$, which implies that

$$\operatorname{cd}(I, R/\mathfrak{q}) = \operatorname{cd}(I, M/\mathfrak{q} M) = t.$$

Moreover, by [6, Theorem 2.6 and Lemma 5.1] the R-module

$$H^t_I(R/\mathfrak{q}) \simeq H^t_{(I+\mathfrak{q})/\mathfrak{q}}(R/\mathfrak{q})$$

is Artinian and $(I + \mathfrak{q})/\mathfrak{q}$-cofinite. Hence, [5, Lemma 2.3] implies that

$$\operatorname{Att}_R H^t_I(R/\mathfrak{q}) = \{\mathfrak{q}\},\ \dim R/\mathfrak{q} = t \text{ and } \operatorname{Rad}(I + \mathfrak{q}) = \mathfrak{m}.$$

In view of [6, Theorem 2.6], the R-module $H^t_I(R/\operatorname{Ann}_R M)$ is Artinian and I-cofinite. Also, the exact sequence

$$0 \longrightarrow \mathfrak{q}/\operatorname{Ann}_R M \longrightarrow R/\operatorname{Ann}_R M \longrightarrow R/\mathfrak{q} \longrightarrow 0,$$

induces an exact sequence

$$H^t_I(R/\operatorname{Ann}_R M) \longrightarrow H^t_I(R/\mathfrak{q}) \longrightarrow H^{t+1}_I(\mathfrak{q}/\operatorname{Ann}_R M).$$

But, $H^{t+1}_I(\mathfrak{q}/\operatorname{Ann}_R M) = 0$ by [12, Theorem 2.2]. So, we get the exact sequence

$$H^t_I(R/\operatorname{Ann}_R M) \longrightarrow H^t_I(R/\mathfrak{q}) \longrightarrow 0,$$

which using [5, Lemma 2.3] implies that

$$\mathfrak{q} \in \operatorname{Att}_R H^t_I(R/\operatorname{Ann}_R M) \subseteq \operatorname{mAss}_R R/\operatorname{Ann}_R M = \operatorname{mAss}_R M.$$

Thus, $\mathfrak{q} \in \Psi$. Hence, $\operatorname{Att}_R H^t_I(M) \subseteq \Psi$.

Now, let $\mathfrak{q} \in \Psi$. Then, $\operatorname{Supp} M/\mathfrak{q} M = V(\mathfrak{q})$ and so $\dim M/\mathfrak{q} M = t$. Therefore, using the Independence Theorem we have

$$H^t_I(M/\mathfrak{q} M) \simeq H^t_{(I+\mathfrak{q})/\mathfrak{q}}(M/\mathfrak{q} M) = H^t_{\mathfrak{m}/\mathfrak{q}}(M/\mathfrak{q} M) \simeq H^t_{\mathfrak{m}}(M/\mathfrak{q} M)$$

and hence, [10, Theorem 7.3.2] implies that

$$\operatorname{Att}_R H^t_I(M/\mathfrak{q} M) = \operatorname{Att}_R H^t_{\mathfrak{m}}(M/\mathfrak{q} M) = \{\mathfrak{q}\}.$$

Also, the exact sequence

$$0 \longrightarrow \mathfrak{q} M \longrightarrow M \longrightarrow M/\mathfrak{q} M \longrightarrow 0$$

induces the exact sequence

$$H^t_I(M) \longrightarrow H^t_I(M/\mathfrak{q} M) \longrightarrow H^{t+1}_I(\mathfrak{q} M).$$

But, in view of [12, Theorem 2.2], we have $H^{t+1}_I(\mathfrak{q} M) = 0$. Hence, we get the exact sequence

$$H^t_I(M) \longrightarrow H^t_I(M/\mathfrak{q} M) \longrightarrow 0,$$

which yields that

$$\{\mathfrak{q}\} = \operatorname{Att}_R H^t_I(M/\mathfrak{q} M) \subseteq \operatorname{Att}_R H^t_I(M).$$

Therefore, $\mathfrak{q} \in \operatorname{Att}_R H^t_I(M)$. So, we have $\Psi \subseteq \operatorname{Att}_R H^t_I(M)$. $\square$

Let R be a Noetherian ring, I be an ideal of R and M be a non-zero finitely generated R-module with $\operatorname{cd}(I, M) \geq 0$. Following [2], the submodule $T_R(I, M)$ of M is defined as:

$$T_R(I, M) := \cup\{N \ : \ N \leq M \text{ and } \operatorname{cd}(I, N) < \operatorname{cd}(I, M)\}.$$

The following consequence of Theorem 2.1 will be useful in the proof of Lemma 2.6.

Lemma 2.2. *Let $(R, \mathfrak{m})$ be a Noetherian complete local ring and I be an ideal of R. Let M be a non-zero finitely generated R-module such that $\operatorname{cd}(I, M) = t \geq 0$ and $\widetilde{q}(I, M) < \operatorname{cd}(I, M)$. Set*

$$\mathfrak{A} := \operatorname{Att}_R H^t_I(M) = \{\mathfrak{q}_1, ..., \mathfrak{q}_n\} \text{ and } \mathfrak{B} := \operatorname{Ass}_R M \setminus \operatorname{Att}_R H^t_I(M) = \{\mathfrak{p}_1, ..., \mathfrak{p}_k\}.$$

Let

$$0=\left(\bigcap_{i=1}^{n} K_i\right)\bigcap\left(\bigcap_{j=1}^{k} L_j\right)$$

be a minimal primary decomposition of the zero submodule of M*, where* K_i *is* $\mathfrak{q}_i$*-primary for each* $1\leq i\leq n$ *and* L_j *is* $\mathfrak{p}_j$*-primary for each* $1\leq j\leq k$*. Then the following statements hold:*

i) $\bigcap_{i=1}^{n} K_i = T_R(I,M)$.

ii) $\operatorname{Ann}_R H_I^t(M)=\operatorname{Ann}_R M/T_R(I,M)$.

Proof. (i) Set $K:=\bigcap_{i=1}^{n} K_i$ and $L:=\bigcap_{j=1}^{k} L_j$. Then we have $K\cap L=0$ and so there is an exact sequence $0\longrightarrow K\longrightarrow M/L$. In particular, we have

$$\operatorname{Supp}(K)\subseteq \operatorname{Supp}(M/L)\subseteq \operatorname{Supp} M$$

and therefore, it follows from [12, Theorem 2.2] and [6, Theorem 2.6] that

$$\operatorname{cd}(I,K)\leq \operatorname{cd}(I,M/L)\leq \operatorname{cd}(I,M)=t$$

and

$$\widetilde{q}(I,M/L)\leq \widetilde{q}(I,M)<\operatorname{cd}(I,M)=t.$$

We claim that $\operatorname{cd}(I,M/L)<t$. Assume the opposite. Then we have $\operatorname{cd}(I,M/L)=t$ and so it follows from Theorem 2.1 that

$$\emptyset\neq \operatorname{Att}_R H_I^t(M/L)=\{\mathfrak{q}\in \operatorname{mAss}_R M/L\ :\ \dim R/\mathfrak{q}=t \text{ and } \operatorname{Rad}(I+\mathfrak{q})=\mathfrak{m}\}.$$

Let $\mathfrak{q}\in \operatorname{Att}_R H_I^t(M/L)$. Then

$$\mathfrak{q}\in \operatorname{mAss}_R M/L\subseteq \operatorname{Ass}_R M/L=\mathfrak{B}=\operatorname{Ass}_R M\setminus \operatorname{Att}_R H_I^t(M).$$

Then we claim $\mathfrak{q}\in \operatorname{mAss}_R M$. Assume the opposite. Then there is an element $\mathfrak{q}_1\in \operatorname{mAss}_R M$ such that $\mathfrak{q}_1\subset\mathfrak{q}$. Since $\mathfrak{q}$ is a minimal element of $\mathfrak{B}$ and $\operatorname{Ass}_R M=\mathfrak{A}\cup\mathfrak{B}$ it follows that $\mathfrak{q}_1\in\mathfrak{A}=\operatorname{Att}_R H_I^t(M)$. Then by Theorem 2.1 we have

$$\dim R/\mathfrak{q}_1=t=\dim R/\mathfrak{q},$$

which is a contradiction. So, $\mathfrak{q}\in \operatorname{mAss}_R M$ and hence, by Theorem 2.1 we have $\mathfrak{q}\in\mathfrak{A}=\operatorname{Att}_R H_I^t(M)$. Therefore, $\mathfrak{q}\in\big(\mathfrak{A}\cap\mathfrak{B}\big)=\emptyset$, which is a contradiction. So,

$$\operatorname{cd}(I,K)\leq \operatorname{cd}(I,M/L)<t.$$

Therefore, by the definition we have $K\subseteq T_R(I,M)$. On the other hand, using the fact that $\operatorname{Ass}_R M/K=\mathfrak{A}$ it follows that for any non-zero submodule U of M/K we have $\emptyset\neq \operatorname{Ass}_R U\subseteq\mathfrak{A}$. Therefore, there exists $\mathfrak{q}\in \operatorname{Ass}_R U$ such that $\mathfrak{q}\in\mathfrak{A}$. Applying the method used in the proof of Theorem 2.1, it is easy to see that $\operatorname{cd}(I,R/\mathfrak{q})=t$. So, as

$$\operatorname{Supp} R/\mathfrak{q}\subseteq \operatorname{Supp} U\subseteq \operatorname{Supp} M$$

it follows from [12, Theorem 2.2] that $\operatorname{cd}(I, U) = t$. On the other hand, for the submodule $T_R(I, M)/K$ of M/K by [12, Theorem 2.2], one has

$$\operatorname{cd}(I, T_R(I, M)/K) \leq \operatorname{cd}(I, T_R(I, M)) < t$$

and so $T_R(I, M)/K = 0$. Thus, $T_R(I, M) = K = \bigcap_{i=1}^n K_i$.

(ii) Let K denote the same R-module as in the proof of (i). The exact sequence

$$0 \longrightarrow K \longrightarrow M \longrightarrow M/K \longrightarrow 0$$

yields the isomorphism $H_I^t(M) \simeq H_I^t(M/K)$. Also, using the fact that

$$\operatorname{Ass}_R M/K = \mathfrak{A},$$

it follows from Theorem 2.1 that M/K is a non-zero finitely generated R-module of dimension t such that $\operatorname{Rad}(I + \operatorname{Ann}_R M/K) = \mathfrak{m}$. Hence, using the Grothendieck's Non-vanishing Theorem and Independence Theorem we have

$$\begin{aligned} H_I^t(M/K) &\simeq H^t_{(I+\operatorname{Ann}_R M/K)/\operatorname{Ann}_R M/K}(M/K) \\ &= H^t_{\mathfrak{m}/\operatorname{Ann}_R M/K}(M/K) \\ &\simeq H^t_{\mathfrak{m}}(M/K). \end{aligned}$$

Now, it follows from [8, Theorem 2.6] that

$$\operatorname{Ann}_R H_I^t(M/K) = \operatorname{Ann}_R H_{\mathfrak{m}}^t(M/K) = \operatorname{Ann}_R M/K = \operatorname{Ann}_R M/T_R(I, M).$$

□

The following easy lemma is needed in the proofs of Lemma 2.4 and Theorem 2.8.

LEMMA 2.3. *Let R be a Noetherian local ring and M be a non-zero finitely generated R-module. If $\operatorname{Ann}_R M = J$, then $\operatorname{Ann}_T M \otimes_R T = JT$, for any flat R-algebra T.*

Proof. By the hypothesis M is a finitely generated R-module and so there are elements $x_1, ..., x_n \in M$ such that $M = Rx_1 + \cdots + Rx_n$. We define $f : R \longrightarrow \bigoplus_{i=1}^n M$ by $f(r) = (rx_1, ..., rx_n)$. Then it is easy to see that $\ker f = \operatorname{Ann}_R M = J$. Whence, we get an exact sequence

$$0 \longrightarrow R/J \longrightarrow \bigoplus_{i=1}^{n} M.$$

Effecting the exact functor $- \otimes_R T$ to this exact sequence we get the exact sequence

$$0 \longrightarrow T/JT \longrightarrow \bigoplus_{i=1}^{n} M \otimes_R T,$$

which implies that

$$\mathrm{Ann}_T\, M \otimes_R T \subseteq \mathrm{Ann}_T\, T/JT = JT \subseteq \mathrm{Ann}_T\, M \otimes_R T$$

and so $\mathrm{Ann}_T\, M \otimes_R T = JT$. □

The following lemma plays a key role in the proof of Lemma 2.5.

LEMMA 2.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be a non-zero finitely generated R-module. If $\mathfrak{Q} \in \mathrm{mAss}_{\widehat{R}}\, M \otimes_R \widehat{R}$, then $(\mathfrak{Q} \cap R) \in \mathrm{mAss}_R\, M$, where $\widehat{R}$ denotes the $\mathfrak{m}$-adic completion of R.*

Proof. Set $J := \mathrm{Ann}_R\, M$. Since, $\widehat{R}$ is a flat R-algebra, it follows from Lemma 2.3 that $\mathrm{Ann}_{\widehat{R}}\, M \otimes_R \widehat{R} = J\widehat{R}$. From the hypothesis $\mathfrak{Q} \in \mathrm{mAss}_{\widehat{R}}\, M \otimes_R \widehat{R}$. It follows that $\mathfrak{Q} \in \mathrm{mAss}_{\widehat{R}}\, \widehat{R}/J\widehat{R}$ and hence, $J \subseteq \mathfrak{Q} \cap R$. Therefore, we have $(\mathfrak{Q} \cap R) \in \mathrm{Supp}\, M$. Now, in order to prove the lemma, assume the opposite and set $\mathfrak{q} = \mathfrak{Q} \cap R$. Then there exists $\mathfrak{q}_1 \in \mathrm{mAss}_R\, M$ such that $\mathfrak{q}_1 \subset \mathfrak{q}$. Since $\widehat{R}/J\widehat{R}$ is a flat R/J-algebra it follows from [18, Theorem 9.5] that the going-down theorem holds between R/J and $\widehat{R}/J\widehat{R}$. So, there exists a prime ideal $\mathfrak{P}/J\widehat{R}$ of $\widehat{R}/J\widehat{R}$ such that $\mathfrak{P} \subset \mathfrak{Q}$ and $\mathfrak{P} \cap R = \mathfrak{q}_1$, which is a contradiction because, by the hypothesis, we have $\mathfrak{Q} \in \mathrm{mAss}_{\widehat{R}}\, \widehat{R}/J\widehat{R}$. □

The following lemma is crucial for us in the proofs of Theorems 2.10 and 2.11.

LEMMA 2.5. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and I be an ideal of R. Let M be a non-zero finitely generated R-module such that $\mathrm{cd}(I, M) = t \geq 0$ and $\widetilde{q}(I, M) < \mathrm{cd}(I, M)$. Then*

$$\mathrm{Att}_R\, H^t_I(M) = \{\mathfrak{q} \in \mathrm{mAss}_R\, M \ : \ \mathrm{cd}(I, R/\mathfrak{q}) = t\}.$$

In particular, $\mathrm{Ann}_R\, H^t_I(M) \subseteq \bigcup_{\mathfrak{p} \in \mathrm{mAss}_R M} \mathfrak{p}$.

Proof. Let $\mathfrak{q}$ be an arbitrary element of $\mathrm{Att}_R\, H^t_I(M)$. Then by [10, Exercise 8.2.4]

$$H^t_I(M) \simeq H^t_I(M) \otimes_R \widehat{R} \simeq H^t_{I\widehat{R}}(M \otimes_R \widehat{R})$$

has an $\widehat{R}$-module structure. Therefore, it follows from [10, Exercise 8.2.5] that $\mathfrak{Q} \cap R = \mathfrak{q}$, for some $\mathfrak{Q} \in \mathrm{Att}_{\widehat{R}}\, H^t_{I\widehat{R}}(M \otimes_R \widehat{R})$. Since $\widehat{R}$ is a faithfully flat R-algebra, it follows that $\mathrm{cd}(I\widehat{R}, M \otimes_R \widehat{R}) = t$ and the $\widehat{R}$-module $H^t_{I\widehat{R}}(M \otimes_R \widehat{R})$ is Artinian and $I\widehat{R}$-cofinite. So, it follows from Theorem 2.1 that

$$\mathrm{Att}_{\widehat{R}}\, H^t_I(M) \subseteq \mathrm{mAss}_{\widehat{R}}\, M \otimes_R \widehat{R}.$$

Hence, $\mathfrak{Q} \in \mathrm{mAss}_{\widehat{R}}\, M \otimes_R \widehat{R}$ and so, by Lemma 2.4 one has $\mathfrak{q} \in \mathrm{mAss}_R\, M$. Also, by the method used in the proof of Theorem 2.1 we have $\mathrm{cd}(I, R/\mathfrak{q}) = t$. Therefore,

$$\mathrm{Att}_R\, H_I^t(M) \subseteq \{\mathfrak{q} \in \mathrm{mAss}_R\, M \;:\; \mathrm{cd}(I, R/\mathfrak{q}) = t\}.$$

On the other hand, let $\mathfrak{q}$ be an arbitrary element of the set

$$\{\mathfrak{q} \in \mathrm{mAss}_R\, M \;:\; \mathrm{cd}(I, R/\mathfrak{q}) = t\}.$$

Then by [6, Theorem 2.6] and [12, Theorem 2.2] we have

$$\widetilde{q}(I, M/\mathfrak{q}M) < \mathrm{cd}(I, M/\mathfrak{q}M) = \mathrm{cd}(I, R/\mathfrak{q}) = t.$$

So, by the first part of the proof, we have

$$\emptyset \neq \mathrm{Att}_R\, H_I^t(M/\mathfrak{q}M) \subseteq \mathrm{mAss}_R\, M/\mathfrak{q}M = \{\mathfrak{q}\},$$

which implies that $\mathrm{Att}_R\, H_I^t(M/\mathfrak{q}M) = \{\mathfrak{q}\}$. The exact sequence

$$0 \longrightarrow \mathfrak{q}M \longrightarrow M \longrightarrow M/\mathfrak{q}M \longrightarrow 0$$

induces the exact sequence

$$H_I^t(M) \longrightarrow H_I^t(M/\mathfrak{q}M) \longrightarrow H_I^{t+1}(\mathfrak{q}M).$$

But [12, Theorem 2.2], implies that $H_I^{t+1}(\mathfrak{q}M) = 0$. Hence, we get the exact sequence

$$H_I^t(M) \longrightarrow H_I^t(M/\mathfrak{q}M) \longrightarrow 0,$$

which yields that

$$\{\mathfrak{q}\} = \mathrm{Att}_R\, H_I^t(M/\mathfrak{q}M) \subseteq \mathrm{Att}_R\, H_I^t(M)$$

and hence $\mathfrak{q} \in \mathrm{Att}_R\, H_I^t(M)$. Therefore,

$$\{\mathfrak{q} \in \mathrm{mAss}_R\, M \;:\; \mathrm{cd}(I, R/\mathfrak{q}) = t\} \subseteq \mathrm{Att}_R\, H_I^t(M).$$

Now, we are ready to deduce that

$$\mathrm{Att}_R\, H_I^t(M) = \{\mathfrak{q} \in \mathrm{mAss}_R\, M \;:\; \mathrm{cd}(I, R/\mathfrak{q}) = t\}.$$

Finally, it is clear that

$$\mathrm{Ann}_R\, H_I^t(M) \subseteq \left(\bigcap_{\mathfrak{q} \in \mathrm{Att}_R\, H_I^t(M)} \mathfrak{q} \right) \subseteq \left(\bigcup_{\mathfrak{p} \in \mathrm{mAss}_R\, M} \mathfrak{p} \right).$$

□

The proof of the following result is quite similar to the proof of [1, Theorem 3.5].

Lemma 2.6. *Let $(R,\mathfrak{m})$ be a Noetherian local ring and I be an ideal of R. Let M be a non-zero finitely generated R-module such that $\operatorname{cd}(I,M)=t\geq 0$ and $\widetilde{q}(I,M)<\operatorname{cd}(I,M)$. Then*

$$\operatorname{Ann}_R H^t_I(M)=\operatorname{Ann}_R M/T_R(I,M).$$

Proof. The exact sequence

$$0\longrightarrow T_R(I,M)\longrightarrow M\longrightarrow M/T_R(I,M)\longrightarrow 0$$

yields the isomorphism $H^t_I(M)\simeq H^t_I(M/T_R(I,M))$.

So, we may assume that $T_R(I,M)=0$. Now, as $\operatorname{Ann}_R M\subseteq \operatorname{Ann}_R H^t_I(M)$, it is enough to show that $\operatorname{Ann}_R H^t_I(M)\subseteq \operatorname{Ann}_R M$.

To this end, let $x\in \operatorname{Ann}_R H^t_I(M)$ and we show that $xM=0$. Suppose the contrary, that $xM\neq 0$. Then, as $T_R(I,M)=0$, it follows that $\operatorname{cd}(I,xM)=t$. Hence $\operatorname{cd}(I\widehat{R},x\widehat{M})=t$, and so $xH^t_{I\widehat{R}}(\widehat{M})\neq 0$. Because, if $xH^t_{I\widehat{R}}(\widehat{M})=0$, then $x\widehat{R}\subseteq \operatorname{Ann}_{\widehat{R}} H^t_{I\widehat{R}}(\widehat{M})$. Hence, in view of Lemma 2.2, $x\widehat{R}\subseteq \operatorname{Ann}_{\widehat{R}}\widehat{M}/T_{\widehat{R}}(I\widehat{R},\widehat{M})$, and so $x\widehat{M}\subseteq T_{\widehat{R}}(I\widehat{R},\widehat{M})$. Therefore, $\operatorname{cd}(I\widehat{R},x\widehat{M})<t$, which is a contradiction. Consequently, $xH^t_{I\widehat{R}}(\widehat{M})\neq 0$ and hence $x(H^t_I(M)\otimes_R\widehat{R})\neq 0$. Therefore, $x\notin \operatorname{Ann}_R H^t_I(M)$, which is a contradiction. □

The next easy lemma is needed in the proofs of Theorems 2.8 and 2.11.

Lemma 2.7. *Let $(R,\mathfrak{m})$ be a Noetherian local ring and I be an ideal of R. Let M be a non-zero finitely generated R-module such that $\operatorname{cd}(I,M)=t\geq 0$. Assume that $\operatorname{Supp} H^t_I(M)\subseteq\{\mathfrak{m}\}$ and the R-module $\operatorname{Hom}_R(R/I,H^t_I(M))$ is finitely generated. Then the R-module $H^t_I(M)$ is Artinian and I-cofinite and hence $\widetilde{q}(I,M)<\operatorname{cd}(I,M)$.*

Proof. Since, by the hypothesis, we have $\operatorname{Supp} H^t_I(M)\subseteq\{\mathfrak{m}\}$ it follows that

$$\operatorname{Supp}\operatorname{Hom}_R(R/I,H^t_I(M))\subseteq\{\mathfrak{m}\}.$$

Hence, the finitely generated R-module $\operatorname{Hom}_R(R/I,H^t_I(M))$ is of finite length. So, in view of [19, Proposition 4.1] the R-module $H^t_I(M)$ is Artinian and I-cofinite. Now the remainder part of the assertion is clear. □

The following result plays a key role in the proof of our first main result, Theorem 2.9.

Theorem 2.8. *Let R be a Noetherian ring and let I be an ideal of R. Let M be a non-zero finitely generated R-module such that $\operatorname{cd}(I,M)=t\geq 0$ and the R-module $\operatorname{Hom}_R(R/I,H^t_I(M/N))$ is finitely generated, for each submodule N of M. Then*

$$\operatorname{Ann}_R(H^t_I(M))=\operatorname{Ann}_R M/T_R(I,M).$$

Proof. By the proof of Lemma 2.5, we may assume that $T_R(I,M)=0$ and with this assumption our aim is to show that $\operatorname{Ann}_R H_I^t(M)=\operatorname{Ann}_R M$. To this end, as $\operatorname{Ann}_R M \subseteq \operatorname{Ann}_R H_I^t(M)$, it is enough for us to prove that $\operatorname{Ann}_{R/\operatorname{Ann}_R M} H_I^t(M)=0$. So, it is enough for us to show that

$$\operatorname{Ann}_{R/\operatorname{Ann}_R M} H^t_{(I+\operatorname{Ann}_R M)/\operatorname{Ann}_R M}(M)=0.$$

Replacing R by $R/\operatorname{Ann}_R M$ and replacing I by $(I+\operatorname{Ann}_R M)/\operatorname{Ann}_R M$, we may assume that M is a faithful R-module such that $T_R(I,M)=0$, $\operatorname{cd}(I,M)=t$ and the R-module $\operatorname{Hom}_R(R/I,H_I^t(M/N))$ is finitely generated, for each submodule N of M. Let $\operatorname{Ass}_R M=\{\mathfrak{q}_1,\dots,\mathfrak{q}_n\}$ and assume that $0=\bigcap_{i=1}^n K_i$ is a minimal primary decomposition of the zero submodule of M, where K_i is a $\mathfrak{q}_i$-primary submodule of M for each $1\leq i\leq n$.

Henceforth, in order to prove the relation $\operatorname{Ann}_R H_I^t(M)=0$, our main strategy is to show $\operatorname{Ann}_R H_I^t(M)\subseteq \operatorname{Ann}_R M/K_i$, for each $1\leq i\leq n$. Assume that $1\leq i\leq n$ and set $K:=K_i$ and $\mathfrak{q}:=\mathfrak{q}_i$. Since, $\mathfrak{q}\in \operatorname{Ass}_R M/K\cap \operatorname{Ass}_R M$ and $T_R(I,M)=0$ it follows from [12, Theorem 2.2] that

$$t=\operatorname{cd}(I,R/\mathfrak{q})\leq \operatorname{cd}(I,M/K)\leq \operatorname{cd}(I,M)=t$$

and hence, $\operatorname{cd}(I,M/K)=t$. The exact sequence

$$0\longrightarrow K\longrightarrow M\longrightarrow M/K\longrightarrow 0$$

induces the exact sequence

$$H_I^t(M)\longrightarrow H_I^t(M/K)\longrightarrow H_I^{t+1}(K).$$

But, in view of [12, Theorem 2.2], we have $H_I^{t+1}(K)=0$. Hence, we have the following exact sequence

$$H_I^t(M)\longrightarrow H_I^t(M/K)\longrightarrow 0,$$

which yields that

$$\operatorname{Ann}_R H_I^t(M)\subseteq \operatorname{Ann}_R H_I^t(M/K).$$

By the hypothesis, the R-module $\operatorname{Hom}_R(R/I,H_I^t(M/K))$ is finitely generated and so by Lemma 2.7, for each $\mathfrak{p}\in \operatorname{mAss}_R H_I^t(M/K)$, the $R_\mathfrak{p}$-module $H^t_{IR_\mathfrak{p}}((M/K)_\mathfrak{p})$ is Artinian and $IR_\mathfrak{p}$-cofinite.

On the other hand, using the fact that the set $\operatorname{Ass}_R M/K$ has exactly one element, it is straightforward to see that $T_{R_\mathfrak{p}}(IR_\mathfrak{p},(M/K)_\mathfrak{p})=0$. So, it follows from Lemma 2.6, that

$$\operatorname{Ann}_{R_\mathfrak{p}} H^t_{IR_\mathfrak{p}}((M/K)_\mathfrak{p})=\operatorname{Ann}_{R_\mathfrak{p}}(M/K)_\mathfrak{p},$$

which using the fact that $R_\mathfrak{p}$ is a flat R-algebra, Lemma 2.2, implies that

$$(\operatorname{Ann}_R H_I^t(M))R_\mathfrak{p} \quad\subseteq\quad \operatorname{Ann}_{R_\mathfrak{p}} H^t_{IR_\mathfrak{p}}((M/K)_\mathfrak{p})$$

$$\begin{aligned} &= \operatorname{Ann}_{R_{\mathfrak{p}}}(M/K)_{\mathfrak{p}} \\ &= (\operatorname{Ann}_R M/K)R_{\mathfrak{p}}. \end{aligned}$$

Set $J := \operatorname{Ann}_R H^t_I(M)$ and $Q := \operatorname{Ann}_R M/K$. Then,

$$\big((J+Q)/Q\big)_{\mathfrak{p}} \simeq (JR_{\mathfrak{p}} + QR_{\mathfrak{p}})/QR_{\mathfrak{p}} = 0$$

and hence

$$\mathfrak{p} \notin \operatorname{Supp}(J+Q)/Q = V(\operatorname{Ann}_R(J+Q)/Q).$$

Therefore, there exists an element

$$s \in \big(\operatorname{Ann}_R (J+Q)/Q\big) \setminus \mathfrak{p}\,.$$

So, $sJ \subseteq Q$ and $s \notin \mathfrak{p}$. But,

$$\mathfrak{p} \in \operatorname{mAss}_R H^t_I(M/K) \subseteq V(\operatorname{Ann}_R H^t_I(M/K)) \subseteq V(\operatorname{Ann}_R M/K) = V(\mathfrak{q}).$$

So, $s \notin \mathfrak{q}$ and $sJ \subseteq Q$. By the proof of Lemma 2.2, for some integer $n \geq 1$, there exists an exact sequence

$$0 \longrightarrow R/Q \longrightarrow \bigoplus_{i=1}^{n} M/K,$$

which implies that $\operatorname{Ass}_R R/Q = \{\mathfrak{q}\}$ and hence Q is a $\mathfrak{q}$-primary ideal of R. Now, since $s \notin \mathfrak{q}$ and $sJ \subseteq Q$, it follows that

$$\operatorname{Ann}_R H^t_I(M) = J \subseteq Q = \operatorname{Ann}_R M/K.$$

So, we have

$$J = \operatorname{Ann} H^t_I(M) \subseteq \bigcap_{i=1}^{n} \operatorname{Ann}_R M/K_i = \operatorname{Ann}_R \bigoplus_{i=1}^{n} M/K_i.$$

Furthermore, since by the hypothesis $\bigcap_{i=1}^{n} K_i = 0$, we have an exact sequence

$$0 \longrightarrow M \longrightarrow \bigoplus_{i=1}^{n} M/K_i,$$

which implies that $\operatorname{Ann}_R \bigoplus_{i=1}^{n} M/K_i \subseteq \operatorname{Ann}_R M = 0$. Therefore,

$$J = \operatorname{Ann} H^t_I(M) = 0.$$

□

Let R be a Noetherian ring, I be an ideal of R and let N be an R-module. Recall that $q(I,N)$ is defined as the greatest integer i such that $H^i_I(N)$ is not an Artinian module if there exist such i's and $-\infty$ otherwise.

Now, we are ready to state and prove the first main result of this paper.

Theorem 2.9. *Let R be a Noetherian ring and I be an ideal of R. Let M be a non-zero finitely generated R-module with* $\operatorname{cd}(I,M)=t\geq 0$. *Then,* $\operatorname{Ann}_R H_I^t(M)=\operatorname{Ann}_R M/T_R(I,M)$ *in each of the following cases:*

i) $\dim M/IM\leq 1$.

ii) $\dim R/I\leq 1$.

iii) $q(I,M)\leq 1$.

iv) $q(I,R)\leq 1$.

v) $\operatorname{cd}(I,M)\leq 1$, *(see [3, Corollary 2.16] for the local case).*

vi) $\operatorname{cd}(I,R)\leq 1$.

vii) $\widetilde{q}(I,M)<\operatorname{cd}(I,M)$.

Proof. (i) Let N be an arbitrary submodule of M and set $K:=M/N$. Then, $\operatorname{Supp} K/IK\subseteq \operatorname{Supp} M/IM$ and hence $\dim K/IK\leq \dim M/IM\leq 1$. So, by [9, Corollary 2.7] the R-module $H_I^t(K)=H_I^t(M/N)$ is I-cofinite. Therefore, the R-module $\operatorname{Hom}_R(R/I,H_I^t(M/N))$ is finitely generated, for each submodule N of M. Hence, the assertion follows from Theorem 2.8.

(ii) Follows from (i).

(iii) Let N be an arbitrary submodule of M and set $K:=M/N$. Then, $\operatorname{Supp} K\subseteq \operatorname{Supp} M$ and hence in view of [11, Theorem 3.2], there is an inequality $q(I,K)\leq q(I,M)\leq 1$ and hence by [6, Theorem 4.9] the R-module $H_I^t(K)=H_I^t(M/N)$ is I-cofinite. So, the R-module $\operatorname{Hom}_R(R/I,H_I^t(M/N))$ is finitely generated, for each submodule N of M. Hence, the assertion follows from Theorem 2.8.

(iv) In view of [6, Theorem 2.6], it follows from the hypothesis $q(I,R)\leq 1$ that $q(I,M)\leq 1$ and hence the assertion follows from (iii).

(v) Using the inequalities $q(I,M)\leq \operatorname{cd}(I,M)\leq 1$, the assertion follows from (iii).

(vi) Applying [12, Theorem 2.2], the assertion follows from (v).

(vii) Let N be an arbitrary submodule of M and set $K:=M/N$. Then, $\operatorname{Supp} K\subseteq \operatorname{Supp} M$ and hence in view of [6, Theorem 2.6], the R-module $H_I^t(K)=H_I^t(M/N)$ is Artinian and I-cofinite. Thus, the R-module

$$\operatorname{Hom}_R(R/I,H_I^t(M/N))$$

is finitely generated, for each submodule N of M. So, the assertion follows from Theorem 2.8. □

The following theorem is the second main result of this paper.

THEOREM 2.10. *Let R be a Noetherian ring and let I be an ideal of R. Let M be a non-zero finitely generated R-module such that* $\mathrm{cd}(I, M) = t \geq 0$ *and* $\widetilde{q}(I, M) < \mathrm{cd}(I, M)$. *Then*

$$\mathrm{Att}_R\, H_I^t(M) = \{\mathfrak{q} \in \mathrm{mAss}_R\, M \ : \ \mathrm{cd}(I, R/\,\mathfrak{q}) = t\}.$$

In particular, $\mathrm{Ann}_R\, H_I^t(M) \subseteq \bigcup_{\mathfrak{p}\in \mathrm{mAss}_R M} \mathfrak{p}$.

Proof. By the definition of $\widetilde{q}(I, M)$, the non-zero R-module $H_I^t(M)$ is Artinian and I-cofinite. So the R-module $H_I^t(M)$ has finite support contained in $\mathrm{Max}(R)$. Assume that

$$\mathrm{Supp}\, H_I^t(M) = \{\mathfrak{n}_1, ..., \mathfrak{n}_k\}.$$

Set $L_j := \Gamma_{\mathfrak{n}_j}(H_I^t(M))$ for $j = 1, \ldots, k$ and put $L'_j = \sum_{i\in(\{1,\ldots,k\}\setminus\{j\})} L_i$ for $j = 1, \ldots, k$. Then it is clear that $\mathrm{Supp}\, L_j \cap L'_j \subseteq \{\mathfrak{n}_j\} \cap (\{\mathfrak{n}_1, ..., \mathfrak{n}_k\} \setminus \{\mathfrak{n}_j\}) = \emptyset$, for each $1 \leq j \leq k$. Therefore, $L_j \cap L'_j = 0$, for each $1 \leq j \leq k$. Hence, $\sum_{j=1}^k L_j \simeq \bigoplus_{j=1}^k L_j$. Also, for each $1 \leq j \leq k$ one has

$$\mathfrak{n}_j \notin \mathrm{Ass}_R\, H_I^t(M)/L_j = \mathrm{Supp}\, H_I^t(M)/L_j,$$

which means that $\mathrm{Supp}\, H_I^t(M)/(\sum_{j=1}^k L_j) = \emptyset$ and hence $H_I^t(M) = \sum_{j=1}^k L_j$. So, there is an isomorphism

$$H_I^t(M) \simeq \bigoplus_{j=1}^k L_j.$$

Furthermore, one sees that for each $1 \leq j \leq k$,

$$H^t_{IR_{\mathfrak{n}_j}}(M_{\mathfrak{n}_j}) \simeq (H_I^t(M))_{\mathfrak{n}_j} \simeq \left(\sum_{j=1}^k L_j\right)_{\mathfrak{n}_j} \simeq (L_j)_{\mathfrak{n}_j} \simeq L_j.$$

Consequently, there is an isomorphism

$$H_I^t(M) \simeq \bigoplus_{j=1}^k H^t_{IR_{\mathfrak{n}_j}}(M_{\mathfrak{n}_j}).$$

Moreover, it is clear that for each $1 \leq j \leq k$ the $R_{\mathfrak{n}_j}$-module $H^t_{IR_{\mathfrak{n}_j}}(M_{\mathfrak{n}_j})$ is Artinian and $IR_{\mathfrak{n}_j}$-cofinite. So, for each $1 \leq j \leq k$ we have $\widetilde{q}(IR_{\mathfrak{n}_j}, M_{\mathfrak{n}_j}) < \mathrm{cd}(IR_{\mathfrak{n}_j}, M_{\mathfrak{n}_j}) = t$. Now, since

$$\mathrm{Att}_R\, H_I^t(M) = \bigcup_{j=1}^k \mathrm{Att}_R\, H^t_{IR_{\mathfrak{n}_j}}(M_{\mathfrak{n}_j})$$

and

$$\operatorname{Att}_R H^t_{IR_{\mathfrak{n}_j}}(M_{\mathfrak{n}_j}) = \{\mathfrak{p} \in \operatorname{Spec} R \ : \ \mathfrak{p} R_{\mathfrak{n}_j} \in \operatorname{Att}_{R_{\mathfrak{n}_j}} H^t_{IR_{\mathfrak{n}_j}}(M_{\mathfrak{n}_j})\},$$

the assertion follows from Lemma 2.5. □

THEOREM 2.11. *Let R be a Noetherian ring and I be an ideal of R. Let M be a non-zero finitely generated R-module such that $\operatorname{cd}(I, M) = t$ and the R-module $\operatorname{Hom}_R(R/I, H^t_I(M))$ is finitely generated. Then for each $\mathfrak{p} \in \operatorname{mAss}_R H^t_I(M)$, the $R_{\mathfrak{p}}$-module $H^t_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}})$ is Artinian and $IR_{\mathfrak{p}}$-cofinite and $\operatorname{Att}_{R_{\mathfrak{p}}} H^t_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}}) \subseteq \operatorname{mAss}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$. In particular, we have*

$$\operatorname{Ann}_R H^t_I(M) \subseteq \bigcup_{\mathfrak{p} \in \operatorname{mAss}_R M} \mathfrak{p} .$$

Proof. The assertion follows from Lemma 2.5 and Lemma 2.7, using the localization. □

THEOREM 2.12. *Let R be a Noetherian ring and I be an ideal of R with $\operatorname{cd}(I, R) = t \geq 0$. Assume that there exists a non-zero finitely generated R-module M such that $\operatorname{cd}(I, M) = \operatorname{cd}(I, R)$ and $\widetilde{q}(I, M) < \operatorname{cd}(I, M)$. Then $\operatorname{Ann}_R H^t_I(R) \subseteq \bigcup_{\mathfrak{p} \in \operatorname{mAss}_R R} \mathfrak{p}$.*

Proof. Set $T := \bigoplus_{\mathfrak{p} \in \operatorname{mAss}_R M} R/\mathfrak{p}$. Then T is a finitely generated R-module with $\operatorname{Supp} T = \operatorname{Supp} M$. So, using [12, Theorem 2.2] we can deduce that $\operatorname{cd}(I, T) = \operatorname{cd}(I, M) = t$. But we have

$$\operatorname{cd}(I, T) = \max\{\operatorname{cd}(I, R/\mathfrak{p}) \ : \ \mathfrak{p} \in \operatorname{mAss}_R M\}.$$

So, there exists an element $\mathfrak{q} \in \operatorname{mAss}_R M$ such that $\operatorname{cd}(I, R/\mathfrak{q}) = \operatorname{cd}(I, R) = t$. Furthermore, since $\operatorname{Supp} R/\mathfrak{q} \subseteq \operatorname{Supp} M$ it follows from [6, Theorem 2.6] that

$$\widetilde{q}(I, R/\mathfrak{q}) \leq \widetilde{q}(I, M) < \operatorname{cd}(I, M) = t = \operatorname{cd}(I, R/\mathfrak{q}).$$

Since $\mathfrak{q}$ is a prime ideal of R, it contains a minimal prime ideal of R. So, there exists $\mathfrak{q}_1 \in \operatorname{mAss}_R R$ such that $\mathfrak{q}_1 \subseteq \mathfrak{q}$. The exact sequence

$$0 \longrightarrow \mathfrak{q}_1 \longrightarrow R \longrightarrow R/\mathfrak{q}_1 \longrightarrow 0$$

induces the exact sequence

$$H^t_I(R) \longrightarrow H^t_I(R/\mathfrak{q}_1) \longrightarrow H^{t+1}_I(\mathfrak{q}_1).$$

But, in view of [12, Theorem 2.2], we have $H^{t+1}_I(\mathfrak{q}_1) = 0$. Hence we have the following exact sequence

$$H^t_I(R) \longrightarrow H^t_I(R/\mathfrak{q}_1) \longrightarrow 0,$$

which yields that $\operatorname{Ann}_R H^t_I(R) \subseteq \operatorname{Ann}_R H^t_I(R/\mathfrak{q}_1)$. So, it is enough to prove $\operatorname{Ann}_R H^t_I(R/\mathfrak{q}_1) = \mathfrak{q}_1$. Since

$$\operatorname{Supp} R/\mathfrak{q} \subseteq \operatorname{Supp} R/\mathfrak{q}_1 \subseteq \operatorname{Spec} R = \operatorname{Supp} R,$$

it follows from [12, Theorem 2.2] that

$$\operatorname{cd}(I,R) = \operatorname{cd}(I,R/\mathfrak{q}) \leq \operatorname{cd}(I,R/\mathfrak{q}_1) \leq \operatorname{cd}(I,R)$$

and hence $\operatorname{cd}(I,R/\mathfrak{q}_1) = \operatorname{cd}(I,R) = t$. So, using Independence Theorem and replacing R by $R/\mathfrak{q}_1$, without loss of generality we may assume that R is a domain, I is an ideal of R and $\mathfrak{q}$ is a prime ideal with

$$\widetilde{q}(I,R/\mathfrak{q}) < \operatorname{cd}(I,R/\mathfrak{q}) = \operatorname{cd}(I,R) = t.$$

Then it is enough to prove that $\operatorname{Ann}_R H^t_I(R) = 0$. Since for each integer $n \geq 1$ we have $\operatorname{Supp} R/\mathfrak{q}^{(n)} = \operatorname{Supp} R/\mathfrak{q}$ it follows from [6, Theorem 2.6] and [12, Theorem 2.2] that

$$\widetilde{q}(I,R/\mathfrak{q}^{(n)}) = \widetilde{q}(I,R/\mathfrak{q}) < \operatorname{cd}(I,R/\mathfrak{q}) = \operatorname{cd}(I,R/\mathfrak{q}^{(n)}) = t.$$

Whence, by Theorem 2.9(iii), for each integer $n \geq 1$ we have

$$\operatorname{Ann}_R H^t_I(R/\mathfrak{q}^{(n)}) = \mathfrak{q}^{(n)}.$$

On the other hand, the exact sequence

$$0 \longrightarrow \mathfrak{q}^{(n)} \longrightarrow R \longrightarrow R/\mathfrak{q}^{(n)} \longrightarrow 0$$

induces the following exact sequence

$$H^t_I(R) \longrightarrow H^t_I(R/\mathfrak{q}^{(n)}) \longrightarrow H^{t+1}_I(\mathfrak{q}^{(n)}).$$

But, in view of [12, Theorem 2.2], we have

$$H^{t+1}_I(\mathfrak{q}^{(n)}) = 0.$$

Hence, we have the following exact sequence

$$H^t_I(R) \longrightarrow H^t_I(R/\mathfrak{q}^{(n)}) \longrightarrow 0,$$

which implies that

$$\operatorname{Ann}_R H^t_I(R) \subseteq \operatorname{Ann} H^t_I(R/\mathfrak{q}^{(n)}) = \mathfrak{q}^{(n)}.$$

So, we have

$$\operatorname{Ann}_R H^t_I(R) \subseteq \bigcap_{n=1}^{\infty} \mathfrak{q}^{(n)}.$$

Let $\varphi : R \longrightarrow R_\mathfrak{q}$ be the natural homomorphism. Then, since for each positive integer n by the definition we have $\mathfrak{q}^{(n)} = \varphi^{-1}(\mathfrak{q}^n R_\mathfrak{q})$ and by Krull's Intersection Theorem we have $\bigcap_{n=1}^{\infty} \mathfrak{q}^n R_\mathfrak{q} = 0$ it follows that $\varphi(\bigcap_{n=1}^{\infty} \mathfrak{q}^{(n)}) = 0$. So, as the ideal $J := \bigcap_{n=1}^{\infty} \mathfrak{q}^{(n)}$ is finitely generated, it is straightforward and so left to reader, that $sJ = 0$ for some element $s \in (R \setminus \mathfrak{q})$. As R is a domain it follows that $J = 0$. Hence, $\operatorname{Ann}_R H^t_I(R) = 0$. This completes the proof. □

The following lemma will be useful in the proof of Theorem 2.14.

Lemma 2.13. *Let R be a Noetherian ring and I be an ideal of R. Assume that M is a non-zero finitely generated R-module such that $f_I(M) = 1$. Then*

$$\operatorname{Ann}_R H^1_I(M) \subseteq \bigcup_{\mathfrak{p}\in(\operatorname{Ass}_R M\setminus V(I))} \mathfrak{p}\,.$$

Proof. Assume the opposite. Then there is an element $x \in \operatorname{Ann}_R H^1_I(M)$ such that

$$x \notin \left(\bigcup_{\mathfrak{p}\in(\operatorname{Ass}_R M\setminus V(I))} \mathfrak{p}\right)$$

and so

$$x \notin \left(\bigcup_{\mathfrak{p}\in\operatorname{Ass}_R M/\Gamma_I(M)} \mathfrak{p}\right).$$

By [10, Remark 2.2.7], there is an exact sequence

$$0 \longrightarrow M/\Gamma_I(M) \longrightarrow D_I(M) \longrightarrow H^1_I(M) \longrightarrow 0,$$

which using the Snake Lemma induces an exact sequence

$$(0 :_{D_I(M)} x) \longrightarrow (0 :_{H^1_I(M)} x) \longrightarrow M/(xM + \Gamma_I(M)).$$

In view of [7, Lemma 3.7], we have $\operatorname{Ass}_R D_I(M) = \operatorname{Ass}_R M/\Gamma_I(M)$ and hence it follows from the hypothesis that

$$x \notin \left(\bigcup_{\mathfrak{p}\in\operatorname{Ass}_R D_I(M)} \mathfrak{p}\right).$$

So, we have $(0 :_{D_I(M)} x) = 0$ and $(0 :_{H^1_I(M)} x) = H^1_I(M)$. Hence from the last exact sequence we get the following exact sequence

$$0 \longrightarrow H^1_I(M) \longrightarrow M/(xM + \Gamma_I(M)),$$

that means the R-module $H^1_I(M)$ is finitely generated. This is a contradiction, because $f_I(M) = 1$. □

Theorem 2.14. *Let R be a Noetherian ring and I be an ideal of R. Assume that M is a non-zero finitely generated R-module with $\operatorname{cd}(I, M) \geq 1$. Then $f_I(M) = 1$ if and only if $\operatorname{Ann}_R H^1_I(M) \subseteq \bigcup_{\mathfrak{p}\in(\operatorname{Ass}_R M\setminus V(I))} \mathfrak{p}$.*

Proof. By Lemma 2.13 it is enough to prove that if

$$\operatorname{Ann}_R H^1_I(M) \subseteq \bigcup_{\mathfrak{p}\in(\operatorname{Ass}_R M\setminus V(I))} \mathfrak{p}$$

then $f_I(M) = 1$. Assume that $\operatorname{Ann}_R H^1_I(M) \subseteq \bigcup_{\mathfrak{p}\in(\operatorname{Ass}_R M\setminus V(I))} \mathfrak{p}$ and that $f_I(M) \neq 1$. Then as $\operatorname{cd}(I, M) \geq 1$ we can conclude that $f_I(M) \geq 2$. So, the I-torsion R-module $H^1_I(M)$ is finitely generated. Hence, there exists a positive integer n such that $I^n H^1_I(M) = 0$ and so that

$$I^n \subseteq \operatorname{Ann}_R H^1_I(M) \subseteq \bigcup_{\mathfrak{p}\in(\operatorname{Ass}_R M\setminus V(I))} \mathfrak{p}\,.$$

Therefore, there is an element $\mathfrak{p} \in \big(\operatorname{Ass}_R M \setminus V(I)\big)$ such that $I \subseteq \mathfrak{p}$, which is a contradiction. □

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