ON YAQUB NIL-CLEAN RINGS

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A ring R is Yaqub nil-clean if $a+a^3$ or $a-a^3$ is nilpotent for all $a \in R$. We prove that a ring R is a Yaqub nil-clean ring if and only if $R \cong R_1, R_2, R_3, R_1 \times R_2$ or $R_1 \times R_3$, where $R_1/J(R_1)$ is Boolean, $R_2/J(R_2)$ is a Yaqub ring, $R_3/J(R_3) \cong \mathbb{Z}_5$ and each $J(R_i)$ is nil, if and only if $J(R)$ is nil and $R/J(R)$ is isomorphic to a Boolean ring R_1, a Yaqub ring R_2, $\mathbb{Z}_5$, $R_1 \times R_2$, or $R_1 \times \mathbb{Z}_5$, if and only if for any $a \in R$, there exists $e^3 = e$ such that $a - e$ or $a + 3e$ is nilpotent and $ae = ea$, if and only if R is an exchange Hirano ring. The structure of such rings is thereby completely determined.

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1. INTRODUCTION

Throughout, all rings are associative with an identity. A ring R is strongly nil-clean if $a-a^2$ is nilpotent for all $a \in R$. A ring R is strongly weakly nil-clean if $a + a^2$ or $a - a^2$ is nilpotent for all $a \in R$. Strongly (weakly) nil-clean rings are studied by many authors, e.g., [1, 2, 4, 6, 8] and [10, 12]. An element a in a ring is tripotent if $a^3 = a$. A ring is strongly 2-nil-clean if $a - a^3$ is nilpotent for all $a \in R$ (see [3]). It is proved that a ring R is strongly 2-nil-clean if for any $a \in R$ there exists a tripotent $e \in R$ such that $a - e \in R$ is nilpotent and $ae = ea$ (see [3, Theorem 2.8]).

We say that a ring R is Yaqub nil-clean if $a + a^3$ or $a - a^3$ is nilpotent for all $a \in R$. Clearly, strongly weakly nil-clean and strongly 2-nil-clean rings are Yaqub nil-clean, but the converse is not true, e.g., $\mathbb{Z}_5$. The motivation of this paper is to determine the structure of such rings.

A ring R is a Yaqub ring provided that it is a subdirect product of $\mathbb{Z}_3$'s (see [3]). We prove that a ring R is a Yaqub nil-clean ring if and only if $R \cong R_1, R_2, R_3, R_1 \times R_2$ or $R_1 \times R_3$, where $R_1/J(R_1)$ is Boolean, $R_2/J(R_2)$ is a Yaqub ring, $R_3/J(R_3) \cong \mathbb{Z}_5$ and each $J(R_i)$ is nil, if and only if $J(R)$ is nil and $R/J(R)$ is isomorphic to a Boolean ring R_1, a Yaqub ring R_2, $\mathbb{Z}_5$, $R_1 \times R_2$,

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or $R_1 \times \mathbb{Z}_5$, if and only if for any $a \in R$, there exists $e^3 = e$ such that $a - e$ or $a + 3e$ is nilpotent and $ae = ea$.

An element a in a ring R is (strongly) clean provided that it is the sum of an idempotent and a unit (that commute). A ring R is (strongly) clean in case every element in R is (strongly) clean. A ring R is an exchange ring provided that for any $a \in R$, there exists an idempotent $e \in R$ such that $e \in aR$ and $1 - e \in (1 - a)R$. Every (strongly) clean ring is an exchange ring, but the converse is not true (see [13, Proposition 1.8]). A ring R is a Hirano ring provided that for any $u \in U(R)$, $1 \pm u^2$ is nilpotent. Furthermore, we prove that a ring R is Yaqub nil-clean if and only if R is an exchange Hirano ring. The structure of such rings is thereby completely determined.

We use $N(R)$ to denote the set of all nilpotents in R and $J(R)$ the Jacobson radical of R. $\mathbb{N}$ stands for the set of all natural numbers. $a \pm b$ means that $a + b$ or $a - b$. $\mathbb{Z}[u] = \{f(u) \mid f(t) \text{ is a polynomial with integral coefficients}\}$.

2. ELEMENTARY CHARACTERIZATIONS

The aim of this section is to investigate elementary characterizations of Yaqub nil-clean rings. We begin with

Lemma 2.1. *Let R be a ring. Then the following are equivalent:*

(1) *R is Yaqub nil-clean.*

(2) *For any $a \in R$, $a^2 \in R$ is strongly weakly nil-clean.*

Proof. $\Longrightarrow$ Let $a \in R$. Then $a \pm a^3 \in N(R)$, and so $a^2 - a^4$ or $a^2 + a^4 \in N(R)$. Thus, $a^2 - a^4$ or $-a^2 - (-a^2)^2$ is nilpotent. That is, $a^2 \in R$ is weakly nil-clean.

$\Longleftarrow$ Suppose that a^2 is strongly weakly nil-clean. Then $a^2 - a^4 \in N(R)$ or $-a^2 - (-a^2)^2 \in N(R)$. This implies that $a(a - a^3)$ or $a(a + a^3)$ is nilpotent; hence, $(a - a^3)^2$ or $(a + a^3)^2$ is nilpotent. Therefore $a \pm a^3 \in N(R)$, as desired. □

Theorem 2.2. *Let R be a ring. Then R is Yaqub nil-clean if and only if*

(1) *$J(R)$ is nil;*

(2) *$R/J(R)$ has the identity $x^3 = \pm x$.*

Proof. $\Longrightarrow$ Let $x \in J(R)$. Then $x \pm x^3 \in N(R)$; hence, $x \in N(R)$. This shows that $J(R)$ is nil.

Let $a \in R$. Then $a \pm a^3 \in N(R)$, and so $a^3 \pm a^5 \in N(R)$. Thus, $a - a^5 = (a \pm a^3) + (\mp a^3 - a^5) \in N(R)$. In light of [15, Theorem 2.11], $R/J(R)$

has the identity $x^5 = x$; hence, it is commutative. We infer that $N(R) \subseteq J(R)$. This shows that $\overline{a^3} = \overline{\pm a}$ in $R/J(R)$.

$\Longleftarrow$ Let $a \in R$. Then $a^3 \pm a \in J(R) \subseteq N(R)$, as required. $\square$

LEMMA 2.3. *Every subring of any Yaqub nil-clean ring is Yaqub nil-clean.*

Proof. Let S be a subring of a Yaqub nil-clean ring R. For any $a \in S$, we see that $a \in R$, and so there exists some $n \in \mathbb{N}$ such that $(a \pm a^3)^n = 0$ in R; hence, $(a \pm a^3)^n = 0$ in S. This implies that S is Yaqub nil-clean. $\square$

As a consequence of Lemma 2.3, every center of a Yaqub nil-clean ring is Yaqub nil-clean. This generalizes [14, Theorem 2] as well.

PROPOSITION 2.4. *Every corner of any Yaqub nil-clean ring is Yaqub nil-clean.*

Proof. Let $e \in R$ be an idempotent. It will suffice to prove that eRe is Yaqub nil-clean. As eRe is a subring of R, we complete the proof by Lemma 2.3. $\square$

THEOREM 2.5. *Let $\{R_i \,|\, i \in I\}$ be a family of rings. Then the direct product $R = \prod_{i\in I} R_i$ of rings R_i is Yaqub nil-clean if and only if each R_i is Yaqub nil-clean and at most one is not strongly 2-nil-clean.*

Proof. $\Longrightarrow$ As homomorphic images of R, we see that all R_i are Yaqub nil-clean rings. Suppose that R_k and R_l $(k \neq l)$ are not strongly 2-nil-clean. Then we can find some $a \in R_k$ such that $a - a^3 \notin N(R_k)$ and $2 \notin N(R_l)$. Then $(a,1) \in R_k \times R_l$ and $(a,1) - (a,1)^3, (a,1) + (a,1)^3 \notin N(R_k \times R_l)$. Thus, $R_k \times R_l$ is not Yaqub nil-clean. This contradicts to the Yaqub nil-cleanness of R. Therefore, at most one R_i is not strongly 2-nil-clean.

$\Longleftarrow$ If each R_i is strongly 2-nil-clean, then so is R. If R_k is Yaqub nil-clean and each $R_i (i \neq k)$ is strongly 2-nil-clean. One easily checks that $R \cong \big(\prod_{i\neq k} R_i\big) \times R_k$ is Yaqub nil-clean, as asserted. $\square$

In particular, we have

COROLLARY 2.6. *Let $L = \prod_{i\in I} R_i$ be the direct product of rings $R_i \cong R$ and $|I| \geq 2$. Then L is Yaqub nil-clean if and only if R is strongly 2-nil-clean if and only if L is strongly 2-nil-clean.*

LEMMA 2.7. *Let I be a nil ideal of a ring R. Then R is Yaqub nil-clean if and only if R/I is Yaqub nil-clean.*

Proof. One direction is obvious. Conversely, assume that R/I is Yaqub nil-clean. Let $a \in R$. Then $\overline{a \pm a^3} \in N(R/I)$, and so $(a \pm a^3)^m \in I$ for some $m \in \mathbb{N}$. As I is nil, we have $n \in \mathbb{N}$ such that $(a \pm -a^3)^{mn} = 0$, i.e, $a \pm a^3 \in N(R)$. This completes the proof. $\square$

We use $T_n(R)$ to denote the ring of all $n \times n$ upper triangular matrices over a ring R. We have

THEOREM 2.8. *Let R be a ring, and let $n \geq 2$. Then the following are equivalent:*

(1) *$T_n(R)$ is Yaqub nil-clean.*

(2) *$T_n(R)$ is strongly 2-nil-clean.*

(3) *R is strongly 2-nil-clean.*

Proof. (1) $\Rightarrow$ (3) Let

$$I = \left\{ \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ & 0 & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & & & 0 \end{pmatrix} \in T_n(R) | \text{ each } a_{ij} \in R \right\}.$$

Then $T_n(R)/I \cong \prod\limits_{i \in I} R_i$, where each $R_i = R$. Clearly, $\prod\limits_{i \in I} R_i$ is Yaqub nil-clean. In light of Corollary 2.6, R is strongly 2-nil-clean.

(3) $\Rightarrow$ (2) This is proved in [3, Corollary 2.6].

(2) $\Rightarrow$ (1) This is obvious. $\square$

3. STRUCTURE THEOREMS

The aim of this section is to investigate the structure of Yaqub nil-clean rings. A ring R is periodic if for any $a \in R$ there exist distinct $m, n \in \mathbb{N}$ such that $a^m = a^n$. We now derive

THEOREM 3.1. *A ring R is Yaqub nil-clean if and only if*

$$R \cong R_1,\ R_2,\ R_3,\ R_1 \times R_2 \quad or \quad R_1 \times R_3,$$

where

(1) *$R_1/J(R_1)$ is Boolean and $J(R_1)$ is nil;*

(2) *$R_2/J(R_2)$ is a Yaqub ring and $J(R_2)$ is nil.*

(3) *$R_3/J(R_3) \cong \mathbb{Z}_5$ and $J(R_3)$ is nil.*

Proof. $\Longleftarrow$ In view of [10, Theorem 2.7], R_1 is strongly nil-clean. By virtue of [3, Theorem 4.2], R_2 is strongly 2-nil-clean. Hence, R_1, R_2 and $R_1 \times R_2$ are strongly 2-nil-clean, and then Yaqub nil-clean.

Let $a \in R_3$. Since $R_3/J(R_3) \cong \mathbb{Z}_5$, we easily check that $\overline{a \pm a^3} = \overline{0}$ in $R_3/J(R_3)$; and so $a \pm a^3 \in J(R_3) \subseteq N(R_3)$. Thus, R_3 is Yaqub nil-clean. Let $(a,b) \in R_1 \times R_3$. Then $b \pm b^3 \in N(R_3)$. Since $R_1/J(R_1)$ is Boolean, we see that $(a,b) \pm (a,b)^3 = (a - a^3, b \pm b^3) = (a - a^2 + a(a - a^2), b \pm b^3) \in N(R_1 \times R_3)$. Hence, $R_1 \times R_3$ is Yaqub nil-clean. Therefore, R is Yaqub nil-clean.

$\Longrightarrow$ Let $a \in R$. Then $a + a^3$ or $a - a^3$ is nilpotent. If $a + a^3 \in N(R)$, then $a - a^5 = (1 - a^2)(a + a^3) \in N(R)$. If $a - a^3 \in N(R)$, then $a - a^5 = (1 + a^2)(a - a^3) \in N(R)$. In any case, we have $a - a^5 \in N(R)$. In light of [11, Corollary 3.6], $R \cong R_1, R_2, R_3, R_1 \times R_2$ or $R_1 \times R_3$, where

(1) *$R_1/J(R_1)$ is Boolean and $J(R_1)$ is nil;*

(2) *$R_2/J(R_2)$ is a Yaqub ring and $J(R_2)$ is nil.*

(3) *$R_3/J(R_3)$ is the subdirect of some $\mathbb{Z}_5$ and $J(R_3)$ is nil.*

In $\mathbb{Z}_5 \times \mathbb{Z}_5$, we check that $(1,2) - (1,2)^3 = (0,4), (1,2) + (1,2)^3 = (2,0)$ are not nilpotent. This implies that $R_3/J(R_3) \cong \mathbb{Z}_5$, as asserted. $\square$

Corollary 3.2. *A ring R is Yaqub nil-clean if and only if*

(1) *$a - a^5 \in R$ is nilpotent for all $a \in R$;*

(2) *R has no homomorphic images $\mathbb{Z}_3 \times \mathbb{Z}_5, \mathbb{Z}_5 \times \mathbb{Z}_5$.*

Proof. $\Longrightarrow$ Let $a \in R$. Then $a \pm a^3 \in N(R)$, and so $a - a^5 = (a \pm a^3) \mp a^2(a \pm a^3) \in N(R)$. By virtue of Theorem 3.1, we easily see that R has no homomorphic images $\mathbb{Z}_3 \times \mathbb{Z}_5, \mathbb{Z}_5 \times \mathbb{Z}_5$.

$\Longleftarrow$ In light of [15, Theorem 2.1], $R \cong A, B, C$ or product of such rings, where $A/J(A)$ is Boolean with $J(A)$ is nil, $B/J(B)$ is a subdirect product of $\mathbb{Z}_3's$ with $J(B)$ is nil, and $C/J(C)$ is a subdirect product of $\mathbb{Z}_5's$ with $J(C)$ is nil. By hypothesis, we prove that R is Yaqub nil-clean, in terms of Theorem 3.1. $\square$

Corollary 3.3. *A ring R is strongly 2-nil-clean if and only if*

(1) *$6 \in R$ is nilpotent;*

(2) *R is Yaqub nil-clean.*

Proof. $\Longrightarrow$ In view of [3, Theorem 3.6], $6 \in N(R)$. (2) is obvious.

$\Longleftarrow$ Since $6 \in N(R)$, we see that $5 \in U(R)$. In view of Theorem 3.1, $R \cong R_1, R_2$ or $R_1 \times R_2$, where $R_1/J(R_1)$ is Boolean with $J(R_1)$ nil and $R_2/J(R_2)$ is a Yaqub ring with $J(R_2)$ nil. This completes the proof by [3, Theorem 4.5]. $\square$

Corollary 3.4. *A ring R is strongly nil-clean if and only if*

(1) *2 is nilpotent;*

(2) *R is a Yaqub nil-clean.*

Proof. $\Longrightarrow$ (1) is follows from [8, Proposition 3.14].
(2) This is obvious, by [10, Corollary 2.5].

$\Longleftarrow$ As R is Yaqub nil-clean and $6 \in N(R)$, R is strongly 2-nil-clean by Corollary 3.3. Since $2 \in N(R)$, it follows by [3, Theorem 2.11] that R is strongly nil-clean. □

Example 3.5. Let $R = \mathbb{Z}_n (n \geq 2)$. Then R is a Yaqub nil-clean ring if and only if $n = 2^k 3^l 5^s$ (k, l, s are nonnegative integers and $ls = 0$).

Proof. $\Longrightarrow$ Let p be a prime such that $p|n$. Then $n = pq$ with $(p, q) = 1$. Hence, $R \cong \mathbb{Z}_p \times \mathbb{Z}_q$. This shows that $\mathbb{Z}_p$ is a Yaqub nil-clean ring. Hence, $p = 2, 3$ or 5. If $kl \neq 0$, then $\mathbb{Z}_3 \times \mathbb{Z}_5$ is a Yaqub nil-clean, a contradiction. Therefore, $n = 2^k 3^l 5^s$ for some nonnegative integers k, l, s and $ls = 0$.

$\Longleftarrow$ Since $n = 2^k 3^l 5^s (ls = 0)$, we see that $R \cong \mathbb{Z}_{2^k} \times \mathbb{Z}_{3^l}$ or $\mathbb{Z}_{2^k} \times \mathbb{Z}_{5^l}$. Clearly, $J(\mathbb{Z}_{2^k}) = 2\mathbb{Z}_{2^k}, J(\mathbb{Z}_{3^l}) = 3\mathbb{Z}_{3^l}$ and $J(\mathbb{Z}_{5^s}) = 5\mathbb{Z}_{5^s}$ are all nil. Moreover,

$$\mathbb{Z}_{2^k}/J(\mathbb{Z}_{2^k}) \cong \mathbb{Z}_2, \mathbb{Z}_{3^l}/J(\mathbb{Z}_{3^l}) \cong \mathbb{Z}_3 \text{ and } \mathbb{Z}_{5^s}/J(\mathbb{Z}_{5^s}) \cong \mathbb{Z}_5.$$

According to Theorem 3.1, R is a Yaqub nil-clean ring. □

We are now ready to prove the following.

Theorem 3.6. *A ring R is Yaqub nil-clean if and only if*

(1) *$J(R)$ is nil;*

(2) *$R/J(R)$ is isomorphic to a Boolean ring R_1, a Yaqub ring R_2, $\mathbb{Z}_5$, $R_1 \times R_2$, or $R_1 \times \mathbb{Z}_5$.*

Proof. $\Longrightarrow$ In view of Theorem 3.1, $R \cong R_1, R_2, R_3, R_1 \times R_2$ or $R_1 \times R_3$, where

(i) *$R_1/J(R_1)$ is Boolean and $J(R_1)$ is nil;*

(ii) *$R_2/J(R_2)$ is a Yaqub ring and $J(R_2)$ is nil.*

(iii) *$R_3/J(R_3) \cong \mathbb{Z}_5$, $J(R_3)$ is nil.*

Therefore, $J(R)$ is nil and $R/J(R) \cong R_1/J(R_1), R_2/J(R_2), R_3/J(R_3)$, $R_1/J(R_1) \times R_2/J(R_2)$ or $R_1/J(R_1) \times R_3/J(R_3)$, as required.
$\Longleftarrow$ Let $a \in R$. By hypothesis, we easily check that $\overline{a \pm a^3} = \overline{0}$. As $J(R)$ is nil, $a \pm a^3 \in J(R) \subseteq N(R)$, as desired. □

Corollary 3.7. *A ring R is Yaqub nil-clean if and only if*

(1) *R is periodic;*

(2) *$R/J(R)$ is isomorphic to a Boolean ring R_1, a Yaqub ring R_2, $\mathbb{Z}_5$, $R_1 \times R_2$, or $R_1 \times \mathbb{Z}_5$.*

Proof. $\Longrightarrow$ As in the proof of [3, Proposition 3.5], R is periodic.
(2) follows by Theorem 3.6.

$\Longleftarrow$ Since R is periodic, we easily check that $J(R)$ is nil. This completes the proof by Theorem 3.6. □

Lemma 3.8. *Let R be a ring with $5 \in N(R)$, and let $a \in R$. Then the following are equivalent:*

(1) *$a + a^3 \in R$ is nilpotent.*

(2) *There exists $e \in \mathbb{Z}[a]$ such that $a - e \in N(R)$ and $e^3 = 4e$.*

Proof. (1) ⇒ (2) Suppose that $a + a^3 \in R$ is nilpotent. Set $x = 3a$. Then $x^3 - x = -30a + w$ for some $w \in N(R)$. This shows that $x^3 - x \in N(R)$. As $(5^n, 2) = 1$, we easily see that $2 \cdot 1_R \in U(R)$. In light of [10, Lemma 2.6], there exists $\theta \in \mathbb{Z}[x]$ such that $\theta^3 = \theta$ and $x - \theta \in N(R)$.

Take $\beta = 2(x - \theta) - 5a$. Then $\beta \in N(R)$. Further, we see that $\beta = a - 2\theta$. Set $e = 2\theta \in R$. Then $a - e \in N(R)$ and $e \in \mathbb{Z}[a]$. Moreover, $e^3 - 4e = 8\theta^3 - 8\theta = 0$, as desired.

(2) ⇒ (1) Let $a \in R$. Then we have $e \in \mathbb{Z}[a]$ such that $w := a - e \in N(R)$ and $e^3 = 4e$. Hence, $a + a^3 = (e + w) + (e^3 + 3e^2w + 3ew^2 + w^3) = 5e + (3e^2 + 3ew + w^2)w \in N(R)$, as required. □

Lemma 3.9. *Let R be a ring with $5 \in N(R)$, and let $a \in R$. Then the following are equivalent:*

(1) *$a + a^3 \in R$ is nilpotent.*

(2) *There exists $e^3 = e \in R$ such that $a + 3e \in N(R)$ and $ae = ea$.*

Proof. (1) ⇒ (2) In view of Lemma 3.8, there exists $f \in \mathbb{Z}[a]$ such that $a - f \in N(R)$ and $f^3 = 4f$. As $5 \in R$ is nilpotent, we see that $2 \in U(R)$. Set $e = \frac{f}{2}$. Then $e^3 = e$ and $a + 3e = (a - 2e) + 5e = (a - f) + 5e \in N(R)$, as desired.

(2) ⇒ (1) Let $a \in R$. Then we have $e^3 = e$ such that $w := a + 3e \in N(R)$ and $ae = ea$. This implies that $a + a^3 = (3e + w) + (27e^3 + 27e^2w + 9ew^2 + w^3) = 30e + (27e^2 + 9ew + w^2)w \in N(R)$, as needed. □

We come now to the demonstration for which this section has been developed.

Theorem 3.10. *Let R be a ring. Then the following are equivalent:*

(1) *R is Yaqub nil-clean.*

(2) *For any $a \in R$, there exists $e^3 = e$ such that $a - e$ or $a + 3e$ is nilpotent and $ae = ea$.*

Proof. (1) $\Rightarrow$ (2) In light of Theorem 3.1, $R \cong R_1, R_2, R_3, R_1 \times R_2$ or $R_1 \times R_3$, where

(i) *$R_1/J(R_1)$ is Boolean and $J(R_1)$ is nil;*

(ii) *$R_2/J(R_2)$ is a Yaqub ring and $J(R_2)$ is nil.*

(iii) *$R_3/J(R_3) \cong \mathbb{Z}_5$ and $J(R_3)$ is nil.*

Case I. $R \cong R_1, R_2$ or $R_1 \times R_2$. By virtue of [3, Theorem 4.5], R is strongly 2-nil-clean. Then for any $a \in R$, there exists $e^3 = e$ such that $a - e$ is nilpotent.

Case II. $R \cong R_1, R_3$ or $R_1 \times R_3$. Let $a \in R_1$. As R_1 is strongly nil-clean, there exists an idempotent $e \in R_1$ such that $a - e \in N(R_1)$ and $ae = ea$. Since $2 \in N(R_1)$, we see that $a + 3e = a - e + 4e \in N(R_1)$. Let $a \in R_3$. As $5 \in N(R_3)$, we see that $2 \in U(R_3)$. Let $a \in R_3$. Then $a - a^3 \in N(R_3)$ or $a + a^3 \in N(R_3)$. If $a - a^3 \in N(R_3)$, by [10, Lemma 2.6], there exists $e^3 = e \in R_3$ such that $a - e \in N(R_3)$ and $ae = ea$. If $a + a^3 \in N(R_3)$, it follows by Lemma 3.9 that there exists $e^3 = e \in R_3$ such that $a + 3e \in N(R_3)$. Therefore for any $x \in R_1 \times R_3$, we can find $f^3 = f \in R_1 \times R_3$ such that $x - f$ or $x + 3f$ is nilpotent in $R_1 \times R_3$ and $xf = fx$, as desired.

(2) $\Rightarrow$ (1) By hypothesis, there exists $e^3 = e$ such that $2 - e$ or $2 + 3e$ is nilpotent. Hence, $2^3 - 2$ or $2^3 - 2 \times 9$ is nilpotent. This shows that $2 \times 3 \in N(R)$ or $2 \times 5 \in N(R)$. We infer that $30 = 2 \times 3 \times 5 \in N(R)$.

Let $a \in R$. Then there exists $f^3 = f \in R$ such that $a - f$ or $a + 3f$ is nilpotent and $af = fa$. If $w := a - f \in N(R)$, then $a - a^3 = (f + w) - (f + w)^3 \in N(R)$. If $w := a + 3f \in N(R)$, then $a + a^3 = (-3f + w) + (-3f + w)^3 = -30f + (w^2 - 18f)w \in N(R)$, and so $a + a^3 \in N(R)$. Therefore, R is Yaqub nil-clean. $\square$

4. HIRANO RINGS

The goal of this section is to investigate elementary properties of Hirano rings which will be used in the sequel. We now derive

Proposition 4.1.

(1) *Every subring of any Hirano ring is a Hirano ring.*

(2) *If R is a Hirano ring, then eRe is a Hirano ring for all idempotents $e \in R$.*

Proof. (1) Let S be a subring of a Yaqub ring R, and $u \in U(S)$, so $u \in U(R)$ and $1_R \pm u^2 \in N(R)$, this implies that $\pm u^2 = 1_R + w$ for some nilpotent element $w \in R$. Thus, $\pm u^2 = \pm u^2 \times 1_S = 1_S + w \times 1_S$. As u^2 and 1_S are in S, then $w \times 1_S \in S$, and therefore S is a Hirano ring.

(2) This is obvious as eRe is a subring of R. □

We note that the finite direct product of Hirano rings may be not a Hirano ring.

Example 4.2. Let $R = \mathbb{Z}_5 \times \mathbb{Z}_5$. Then $\mathbb{Z}_5$ is a Hirano ring, while R is not.

Proof. Clearly, $\mathbb{Z}_5$ is a Hirano ring. Choose $u = (1, 2) \in R$. Then $u \in U(R)$. We see that $(1, 1) + u^2 = (2, 0)$ and $(1, 1) - u^2 = (0, 2)$; hence, $1_R + u^2$ and $1_R - u^2$ are not nilpotent. Thus, R is not a Hirano ring. □

Example 4.3. $R = \mathbb{Z}_{5^n}[x]$ is a Hirano ring, but it is not clean.

Proof. Let $f(x) = a_0 + a_1x + \cdots + a_nx^n \in U(R)$. Then $5 \nmid a_0$ and $5|a_i(i = 1, \cdots, a_n)$. Clearly, $\mathbb{Z}_{5^n}$ is a Hirano ring and $a_0 \in U(\mathbb{Z}_{5^n})$. Thus, $1 \pm a_0 \in N(\mathbb{Z}_{5^n})$, i.e., $5|(1 \pm a_0)$. This shows that $5|(1 \pm f(x))$, and so $1 \pm f(x) \in N(R)$. Therefore, R is a Hirano ring. But it is not clean, as $x \in R$ cannot be written as the sum of an idempotent and a unit in R. □

LEMMA 4.4. *Let I be a nil ideal of a ring R. Then R is a Hirano ring if and only if so is R/I.*

Proof. $\Longrightarrow$ This is obvious.

$\Longleftarrow$ Let $u \in U(R)$, so $\pm \bar{u}^2 = \bar{1} + \bar{w}$ for $\bar{w} \in N(R/I)$. Hence, $\pm u^2 = 1 + w + r$ for some $r \in I$. Here $w + r \in N(R)$. This yields the result. □

Recall that a ring R is a 2-UU ring if for any $u \in U(R)$, u^2 is a unipotent, i.e., $1 - u^2 \in N(R)$ [4]. We now derive

LEMMA 4.5. *Let $L = \prod_{i \in I} R_i$ be the direct product of rings $R_i \cong R$ and $|I| \geq 2$. Then L is a Hirano ring if and only if R is a 2-UU ring if and only if L is a 2-UU ring.*

Proof. In view of [4, Theorem 2.1], R is a 2-UU ring if and only if L is a 2-UU ring. If L is a 2-UU ring, we easily see that L is a Hirano ring.

Suppose that L is a Hirano ring. Then R is a Hirano ring as a subring of L. If R is not a 2-UU ring, we can find some $u \in U(R)$ such that $u^2 - 1 \notin N(R)$. Additionally, $2 \notin N(R)$. Choose $v := (u, 1, 1, \cdots) \in U(L)$. Then $v^2 - 1_L, v^2 + 1_L \notin N(L)$. This implies that L is not a Hirano ring, a contradiction. Therefore R is a 2-UU ring, as asserted. □

THEOREM 4.6. *Let R be a ring, and let $n \geq 2$. Then the following are equivalent:*

(1) *$T_n(R)$ is a Hirano ring.*

(2) *$T_n(R)$ is a 2-UU ring.*

(3) *R is a 2-UU ring.*

Proof. (1) $\Rightarrow$ (3) Choose I as in the proof of Theorem 2.8. Then I is a nil ideal of R. As $T_n(R)/I \cong \prod\limits_{i=1}^{n} R_i$ be the direct product of rings $R_i \cong R$, it follows by Lemma 4.4 that $\prod\limits_{i=1}^{n} R_i$ is a Hirano ring. In light of Lemma 4.5, R is a 2-UU ring, as required.

(3) $\Rightarrow$ (2) This is proved in [4, Theorem 2.1].

(2) $\Rightarrow$ (1) This is trivial. □

Example 4.7. The ring $M_2(\mathbb{Z}_2)$ is not a Hirano ring.

Proof. Choose $U = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. As $I_2 \pm U^2 = I_2 \pm \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, we see that $I_2 + U^2$ and $I_2 - U^2$ are not nilpotent, as required. □

5. EXCHANGE PROPERTIES

The class of exchange rings is very large. For instances, local rings, regular rings, π-regular rings, (strongly) clean rings and C^*-algebras with real rank one are all exchange rings. We now characterize Yaqub nil-clean rings by means of their exchange properties.

LEMMA 5.1. *Let R be an exchange ring. Then $-2 \in R$ is clean.*

Proof. See [4, Lemma 4.2]. □

LEMMA 5.2. *Let R be an exchange Hirano ring. Then $30 \in R$ is nilpotent.*

Proof. In view of Lemma 5.1, $-2 \in R$ is clean. Then $-2 = e + u$ for some idempotent e and unit u. As R is a Hirano ring, $1 \pm u^2 = w$ for some $w \in N(R)$.

Case I: $1 - u^2 = w$. Sine $-1 - e = u + 1$, then $1 + 3e = u^2 + 1 + 2u$, this implies that $3e - 2u = 1 - w = 1 + (-w) = 1 + v$ for some $v \in N(R)$. Hence $3e - 2(-2 - e) = 1 + v$, and so $5e = -3 + v$. We see that $5(-2 - u) = -3 + v$, i.e., $5u = -7+v$, and then $25u^2 = 49+v^2-14v = 49+v_1$ for some $v_1 \in N(R)$. Thus $24 \in N(R)$, and so $6 \in N(R)$ which implies $30 \in N(R)$.

Case II: $1+u^2 = w$. As $-2 = e+u$, then $1+3e = u^2+2u+1 = w+2u$, so $3e-2u = w-1$. Thus, $2u-3e = 1+(-w) = 1+w'$, $2(-2-e)-3e = 1+w'$, i.e., $-5-5e = w'$. This implies that $-5e = w'+5$, and then $-5(-2-u) = w'+5$. Hence, $5u = w\prime-5$, so $25u^2 = 25+w^{''}$, which implies that $25(w-1) = 25+w''$. We infer that $50 \in N(R)$, whence $2\times 5\times 5 \in N[R]$. Accordingly, $2\times 5 \in N(R)$, and therefore $30 \in N(R)$. □

Lemma 5.3. *Let R be an exchange Hirano ring. Then $J(R)$ is nil.*

Proof. In view of Lemma 5.2, $30 \in N(R)$. Write $30^n = 0 (n \in \mathbb{N})$. Then we can write $R = R_1 \times R_2 \times R_3$, where $R_1 \cong R/2^nR, R_2 \cong R/3^nR$ and $R_3 \cong R/5^nR$. As R is a Hirano ring, so is R_1 by Proposition 4.1, Then for any $u \in U(R_1), 1 \pm u^2 \in N(R_1)$, also $2 \in N(R_1)$. If $1+u^2 \in N(R_1)$ we can write $(u-1)^2 = 1+u^2-2u \in N(R_1)$ and so $1-u \in N(R_1)$, which implies R_1 is a UU ring. As in [5, Theorem 2.4], $J(R_1)$ is nil. If $1-u^2 \in N(R_1)$, then $-(1-u)^2 = -u^2-1+2u = 1-u^2-2(1-u) \in N(R_1)$, then $1-u \in N(R_1)$ and so $J(R_1)$ is nil. Let $x \in J(R_2)$, as R_2 is a Hirano ring, $\pm(1+x)^2 = 1+w$ for some $w \in N(R_2)$, hence $x(x+2)$ or $x(x+2)+2$ is nilpotent.

Case I. $w := x(x+2) \in N(R)$. As $3 \in N(R_2)$, we see that $2 \in U(R_2)$, and so $x+2 = 2^{-1}(1+2x) \in U(R_2)$. We infer that $x = (x+2)^{-1}w \in N(R_2)$.

Case II. $w := x(x+2)+2 \in N(R)$. Then $x(x+2) = w-2 \in U(R_2)$, and so $x \in U(R_2)$, a contradiction. This implies that $J(R_2)$ is nil. For R_3, as $5 \in N(R_3)$, we deduce that $2 \in U(R_3)$. Thus, by the similar route for R_2, we see that $J(R_3)$ is nil. Therefore $J(R)$ is nil, as asserted. □

We have accumulated all the information necessary to prove the following.

Theorem 5.4. *A ring R is Yaqub nil-clean if and only if R is a Hirano exchange ring.*

Proof. $\Longrightarrow$ By Corollary 3.7, R is periodic, and so it is an exchange ring. Let $u \in U(R)$. Then $u \pm u^3 \in N(R)$; hence, $1 \pm u^2 \in N(R)$. Therefore R is a Hirano ring, as desired.

$\Longleftarrow$ Let $0 \neq x \in N(R)$, we can assume that $x^2 = 0$. As R is an exchange ring with $J(R) = 0$, by [15, Lemma 2.7], we can find some idempotent $e \in R$ and some ring T, such that $eRe \cong M_2(T)$, but as we see in Example 4.5, $M_2(T)$ is not a Hirano ring, i.e, eRe is not a Hirano ring. This shows that R is not a Hirano by Proposition 4.1, a contradiction. So we deduce that $N(R) = 0$, and then R is a reduced ring. This implies that R is abelian. Since R is an exchange ring, it follows by [13, Proposition 1.8] that R is clean.

In light of Lemma 5.2, $30 \in N(R)$. Write $2^n \times 3^n \times 5^n = 0(n \in \mathbb{N}$. Then $R \cong R_1, R_2, R_3$ or products of these rings, where $R_1 = R/2^nR, R_2 = R/3^nR$ and $R_3 = R/5^nR$.

Case 1. $2 \in N(R_1)$. Let $a \in R_1$. Then we have a central idempotent $e \in R$ and a unit $u \in R$ such that $a = e + u$. As $1 \pm u^2 \in N(R_1)$, we see that $u \in 1 + N(R_1)$. Hence, $a^2 = e + 2eu + u^2$, and so $a - a^2 \in N(R_1)$. This implies that $a - a^3 = (a - a^2) + a(a - a^2) \in N(R_1)$, and so R_1 is Yaqub nil-clean.

Case 2. $3 \in N(R_2)$. Let $a \in R_2$. Then we have a central idempotent $e \in R$ and a unit $u \in R$ such that $a = e + u$. Hence, $a^3 = (e + u)^3 = e + 3eu + 3eu^2 + u^3$. If $1 + u^2 \in N(R_2)$, then $u + u^3 \in N(R_2)$, and so $a + a^3 \in N(R_2)$. If $-1 + u^2 \in N(R_2)$, then $u - u^3 \in N(R_2)$. Therefore $a - a^3 \in N(R_3)$. In any case, $a \pm a^3 \in N(R_2)$. This means that R_2 is Yaqub nil-clean.

Case 3. $5 \in N(R_3)$. Let $a \in R_3$. Then we have a central idempotent $e \in R_3$ and a unit $u \in R_3$ such that $a = e + u$. Then $1 \pm u^2 \in N(R_3)$, and so $u - u^5 \in N(R_3)$. Further, $a^5 = (e+u)^5 = e^5 + 5eu + 10u^2 + 10eu^3 + +5eu^4 + u^5$, whence, $a - a^5 \in N(R_3)$. Choose u is $(1, 2)$ in $\mathbb{Z}_3 \times \mathbb{Z}_5$ or $\mathbb{Z}_5 \times \mathbb{Z}_5$. Then $1 \pm u^2$ is not nilpotent. This implies that R_3 has no homomorphic images $\mathbb{Z}_3 \times \mathbb{Z}_5$ and $\mathbb{Z}_5 \times \mathbb{Z}_5$. According to Corollary 3.2, R_3 is Yaqub nil-clean.

Case 4. $R \cong R_1 \times R_2, R_1 \times R_3$. One easily checks that R is Yaqub nil-clean.

Case 5. $R \cong R_2 \times R_3, R_1 \times R_2 \times R_3$. But $R_2 \times R_3$ is not a Hirano ring, as $(1, 2) \in U(R_2 \times R_3)$ and $(1, 1) \pm (1, 2)^2 \notin N(R_2 \times R_3)$. Thus, this case cannot appear.

Therefore, R is Yaqub nil-clean. □

Corollary 5.5. *A ring R is Yaqub nil-clean if and only if R is a Hirano periodic ring.*

Proof. $\Longrightarrow$ This follows from Corollary 3.7 and Theorem 5.4.

$\Longleftarrow$ As every periodic ring is an exchange ring then, we get the result by Theorem 5.4. □

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