WHEN PRIME SUBMODULES ARE PRIME IDEALS?

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We study commutative ring extensions $R \subset S$ in which every ideal of S that is a prime R-submodule of S is prime.

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1. INTRODUCTION

Throughout this paper, all rings considered are integral domains, the dimension of a ring R, denoted $\dim R$, means its Krull dimension, and all module are unital. Let R be a ring and M be an R-module. A submodule N of M is called prime if $N \neq M$ and whenever $r \in R$ and $m \in M$ are such that $rm \in N$, then $rM \subseteq N$ or $m \in N$. Let R be a subring of a ring S. It is clear that the set of prime ideals of S is included in the set of ideals which are R-prime submodules of S and there are many examples of rings in which this inclusion is strict (see Section 2). In this article, we investigate rings in which there is equality between these two sets. We call each ideal of S that is R-prime submodule of S an R-prime ideal of S. We say that $R \subset S$ is a prime submodule ideal extension (for short PSI-extension) if each R-prime ideal of S is prime. We say that $R \subset S$ is a PSI-pair if for each intermediate ring T between R and S, $R \subset T$ is a PSI-extension.

In the second section, we investigate elementary properties of PSI-extensions and PSI-pairs. We compare them with extensions that are incomparable (under inclusion) (for short, INC), INC-pair and residually algebraic pair.

The third section is devoted to study polynomial/power series PSI-extensions. We show that if R is a quasi-local integrally closed domain, then $R[[X]] \subset T[[X]]$ is a PSI-extension for each overring T of R if and only if R is a valuation domain. Also, we show that if R is an integrally closed domain, then $R[X] \subset T[X]$ is a PSI-extension for each overring T of R if and only if R is a Prüfer domain.

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2. **PSI-EXTENSION**

Let S be an integral domain, R a subring of S and M an R-module. Recall that a proper submodule N of M is called a prime submodule of M if whenever $r \in R$ and $m \in M$ such that $rm \in N$, we have $rM \subseteq N$ or $m \in N$.

We call an R-prime ideal of S every ideal of S which is a prime R-submodule of S. In other words, a subset I of S is called R-prime ideal of S if:

1. I is a proper ideal of S.
2. For all $r \in R$ and $s \in S$: $[rs \in I \Rightarrow r \in I$ or $s \in I]$.

We have the following implications:

I is a prime ideal of $S \Rightarrow I$ is an R-prime ideal of $S \Rightarrow I$ is a prime R-submodule of S.

None of these implications is reversible: for the first one, an R-prime ideal of S may be not a prime ideal of S. For example, take X an indeterminate over an integral domain R, and let $S = R[X]$. Then the ideal X^2S of S is not prime but it is an R-prime ideal of S.

For the second one, a prime R-submodule of S may be not an R-prime ideal of S. For example, if $K \subset L$ is a field extension, then every proper subfield F of L containing K is a prime K-submodule of L but it is not an ideal of L. Another example: let R be an integral domain and X an indeterminate over R. Then $RX := \{rX, r \in R\}$ is a prime R-submodule of $R[X]$ but it is not an ideal of $R[X]$. Indeed: clearly, RX is a subgroup of the additive group $R[X]$. Since $X \in R[X]$ and $X \in RX$ but $XX = X^2 \notin RX$, RX is not an ideal of $R[X]$. Clearly, RX is a proper R-submodule of $R[X]$. Let $r \in R$ and $f = r_0 + r_1X + ... + r_nX^n \in R[X]$ such that $rf \in RX$. Thus $rf = aX$ for some $a \in R$ and so $rr_1 = a$ and $rr_i = 0$ for each $i \neq 1$. If $r = 0$, then $rR[X] = 0 \subseteq RX$. If $r \neq 0$, then $r_i = 0$ for each $i \neq 1$ and so $f = r_1X \in RX$. Hence, RX is a prime R-submodule of $R[X]$.

In order to study when the first implication is reversible, we give the following definitions:

Definitions 2.1. We say that $R \subset S$ is a prime submodule ideal ring extension (for short, PSI-extension) if each ideal that is a prime R-submodule of S is a prime ideal of S. We say that (R, S) is a PSI-pair if for each $T \in [R, S]$ (i.e., the set of intermediate rings between R and S), $R \subset T$ is a PSI-extension.

For each ideal I of R or of S and each multiplicative set N of R, the ideal of S denoted $\Omega_S(I, N) = \{x \in S$ such that $rx \in IS$ for some $r \in N\}$

plays an important role in PSI-extension. If P is a prime ideal of R, we denote $\Omega_S(P) := \Omega_S(P, R-P)$.

It is well known that in a ring, the intersection and union of elements of a chain (ordered by inclusion) of prime ideals are also prime ideals and every proper ideal has a minimal prime ideal ([5, Theorem 9 and Theorem 10]). With a similar proof, it is easy to see that if $(I_\lambda)_{\lambda\in\Lambda}$ is a chain of R-prime ideals of S, then both $\cup_{\lambda\in\Lambda}I_\lambda$ and $\cap_{\lambda\in\Lambda}I_\lambda$ are R-prime ideals of S. Indeed: since each I_λ is proper ideal of S, so is $\cup_{\lambda\in\Lambda}I_\lambda$ and $\cap_{\lambda\in\Lambda}I_\lambda$. Let $r \in R$ and $s \in S$ such that $rs \in \cup_{\lambda\in\Lambda}I_\lambda$. Thus $rs \in I_v$ for some $v \in \Lambda$ and so r or $s \in I_v$. Then r or $s \in \cup_{\lambda\in\Lambda}I_\lambda$. Hence $\cup_{\lambda\in\Lambda}I_\lambda$ is an R-prime ideal of S. Let $r \in R$ and $s \in S$ such that $rs \in \cap_{\lambda\in\Lambda}I_\lambda$. Thus $rs \in I_\lambda$ for each $\lambda \in \Lambda$. Assume that $r \notin \cap_{\lambda\in\Lambda}I_\lambda$. Thus $r \notin I_v$ for some $v \in \Lambda$. Thus $s \in I_v$. Let $\lambda \in \Lambda$. If $I_\lambda \subseteq I_v$, then $r \notin I_\lambda$ and so $s \in I_\lambda$. If $I_v \subset I_\lambda$, then $s \in I_\lambda$. So $s \in \cap_{\lambda\in\Lambda}I_\lambda$. Hence, $\cap_{\lambda\in\Lambda}I_\lambda$ is an R-prime ideal of S.

Consequently by Zorn Lemma, every proper ideal of S has a minimal R-prime ideal of S. Indeed: let J be a proper ideal of S and $\mathcal{F}$ be the set of R-prime ideals of S containing J. Since J is contained (at least) in a maximal (so prime) ideal of S, the set $\mathcal{F}$ is non-empty. Thus $(\mathcal{F}, \supseteq)$ is a partially ordered set. If $(I_\lambda)_{\lambda\in\Lambda}$ is a chain of elements of $(\mathcal{F}, \supseteq)$, then $\mathcal{F} \ni \cap_{\lambda\in\Lambda}I_\lambda \subseteq I_\lambda$. Zorn Lemma completes the proof.

We start with elementary results.

LEMMA 2.2. *Let S be an integral domain, R a subring of S and P a prime ideal of R.*

1. *Every R-prime ideal of S lies over a prime ideal of R.*
2. *An intersection of a family of R-prime ideals of S that lies over the same prime ideal of R is also an R-prime ideal of S.*
3. *Let $T \in [R, S]$. If $R \subset S$ is a PSI-extension, then so is $T \subset S$.*
4. *For every multiplicative set N of R, $R \subset R_N$ is a PSI-extension.*
5. *If $R \subset S$ is a PSI-extension, then so is $R/(Q \cap R) \subset S/Q$ for each prime ideal Q of S.*
6. *$\Omega_S(P) = S$ or $\Omega_S(P)$ is an R-prime ideal of S that lies over P.*
7. *If I is an R-prime ideal of S and $\mathcal{P}$ is a minimal prime over I, then $\mathcal{P} \cap R = I \cap R$.*
8. *If I is an ideal of S that lies over P, then $\Omega_S(I, R-P)$ is an R-prime ideal of S that lies over P.*
9. *If I is an ideal of S that lies over a maximal ideal of R, then I is an R-prime ideal of S.*

Proof. 1) By [4, Lemma 1].

2) Straightforward.

3) Because every T-prime ideal of S is an R-prime ideal of S.

4) If I is an ideal of R such that IR_N is an R-prime ideal of R_N, then $\Omega_R(I,N)$ is a prime ideal of R and $IR_N = \Omega_R(I,N)R_N$.

5) $R/(Q \cap R)$-prime ideals of S/Q are exactly J/Q where J ranges over R-prime ideals of S containing Q.

6) Straightforward.

7) Let $x \in \mathcal{P} \cap R$. Thus $x^n y \in I$ for some positive integer n and $y \in S - \mathcal{P}$. Then x^n or $y \in I$. Since $y \notin I$, $x \in I$.

8) Straightforward.

9) [4, Corollary 2]. $\square$

PSI is a local property as the following proposition shows.

Proposition 2.3. *The following statements are equivalent:*

1. *For every multiplicative set N of R, $R_N \subset S_N$ is a PSI-extension.*
2. *For every prime ideal P of R, $R_P \subset S_P$ is a PSI-extension.*
3. *For every maximal ideal M of R, $R_M \subset S_M$ is a PSI-extension.*
4. *$R \subset S$ is a PSI-extension.*

Proof. 1) $\Rightarrow$ 2) $\Rightarrow$ 3) are trivial.

3) $\Rightarrow$ 4) If I is an R-prime ideal of S and M is a maximal ideal of R such that $I \cap R \subseteq M$, then IS_M is an R_M-prime ideal of S_M. So $IS_M = QS_M$ for some prime ideal Q of S such that $Q \cap R \subseteq M$. If $a \in I$, then $sa \in Q$ for some $s \in R - M$ and so $a \in Q$. Thus $I \subseteq Q$. If $a \in Q$, then $sa \in I$ for some $s \in R - M$. Since I is an R-prime ideal of S and $s \notin I$, $a \in I$. Then $I = Q$.

4) $\Rightarrow$ 1) follows from the fact that R_N-prime ideals of S_N have the form JS_N where J is an R-prime ideal of S such that $J \cap N$ is empty. $\square$

Recall that $R \subset S$ is said to be an INC-extension (i.e., satisfy incomparability) if whenever two distinct prime ideals Q_1, Q_2 of S are such that $Q_1 \cap R = Q_2 \cap R$, then Q_1 and Q_2 are incomparable.

Theorem 2.4. *The following statements are equivalent:*

1. *$R \subset S$ is a PSI-extension.*
2. *$R \subset S$ is an INC-extension and $\Omega_S(P)$ is prime in S or $\Omega_S(P) = S$ for every prime ideal P of R.*
3. *$\Omega_S(P) = S$ or $\Omega_S(P)$ is the unique prime ideal of S that lies over P for every prime ideal P of R.*

Proof. 1) $\Rightarrow$ 2) The second fact follows from Lemma 2.2. For the first one: if not, then there exist two prime ideals $\mathcal{P} \subset \mathcal{P}'$ of S such that $\mathcal{P} \cap R = \mathcal{P}' \cap R =: P$. Let $x \in \mathcal{P}' - \mathcal{P}$. Note that $\Omega_S(\mathcal{P} + x^2S, R - P)$ is a proper ideal of S containing $\mathcal{P} + x^2S$ (and so containing P). Since $(\mathcal{P} + x^2S) \cap R = P$, $\Omega_S(\mathcal{P} + x^2S, R - P)$ is an R-prime ideal of S that lies over P in R by Lemma 2.2-(8). Thus $x \in \Omega_S(\mathcal{P} + x^2S, R - P)$ and so $rx - x^2y \in \mathcal{P}$ for some $r \in R - P$ and $y \in S$. Hence $r \in P$, a contradiction.

2) $\Rightarrow$ 3) A prime ideal of S that lies over P in R contains $\Omega_S(P)$.

3) $\Rightarrow$ 1) Let I be an R-prime ideal of S, $P = I \cap R$ and $\mathcal{P}$ be a minimal prime ideal of S over I. By Lemma 2.2-(1), P is a prime ideal of R. Then $\Omega_S(P) \subseteq I \subseteq \mathcal{P}$. Thus $\Omega_S(P) \neq S$ and so $\Omega_S(P)$ is the unique R-prime ideal of S that lies over P by assumption. By Lemma 2.2-(7), $\mathcal{P} \cap R = P$ and so $\Omega_S(P) = \mathcal{P}$. Hence $I = \mathcal{P}$ is a prime ideal of S. □

Corollary 2.5. *If $R \subset S$ is a PSI-extension and M is a maximal ideal of R such that $MS \neq S$, then MS is a maximal ideal of S.*

Proof. One can check easily that $MS = \Omega_S(M)$. By Lemma 2.2-(6), MS is an R-prime ideal of S and so MS is a prime ideal of S. By Theorem 2.4, $R \subset S$ is an INC-extension. Then MS is a maximal ideal of S. □

The following is an example of an INC-extension (in fact, an integral extension) which is not a PSI-extension.

Example 2.6. Let $K \subset L$ be a field extension and X an indeterminate over L. It is well known that $K[X] \subset L[X]$ is an INC-extension if and only if L is algebraic (so integral) over K. So $\mathbb{R}[X] \subset \mathbb{C}[X]$ is an INC-extension. The ideal of $\mathbb{C}[X]$ generated by $X^2 + 1$ is not prime but it is an $\mathbb{R}[X]$-prime ideal of $\mathbb{C}[X]$ (by Lemma 2.2-(9)) because it lies over the maximal ideal $(X^2 + 1)\mathbb{R}[X]$ of $\mathbb{R}[X]$. Then $\mathbb{R}[X] \subset \mathbb{C}[X]$ is not a PSI-extension. Later, we will show that $K[X] \subset L[X]$ is a PSI-extension if and only if $K = L$ (Lemma 3.1).

Corollary 2.7. *If $R \subset S$ have the same prime ideals, then $R \subset S$ is a PSI-extension.*

Proof. If the ideal of S, $\Omega_S(P)$ is proper, then it is included properly in R and so is a prime ideal of R. □

Examples of PSI-extensions:

Example 2.8. It is well known that if R is a pseudo-valuation domain (for short, PVD) with associated valuation domain V, then R and V have the same prime spectrum and so $R \subset V$ is a PSI-extension. If R is an almost pseudo-valuation domain (for short, APVD) with associated valuation domain V, then $R \subset V$ is a PSI-extension if and only if R is a PVD (see [6, Lemma 2.1]).

COROLLARY 2.9. *If $R \subset T$ is an integral PSI-extension and $T \subset S$ is a PSI-extension, then $R \subset S$ is a PSI-extension.*

Proof. By Theorem 2.4, $R \subset T$ and $T \subset S$ are INC-extensions and so is $R \subset S$ by [3, Proposition 6.53]. Let I be an R-prime ideal of S, $J = I \cap T$ and $P = I \cap R$. Since J is an R-prime ideal of T, J is a prime ideal of T.

CLAIM. $\Omega_S(J) \subseteq I$. Assume that there exists $x \in \Omega_S(J) - I$. Thus $tx \in JS$ for some $t \in T - J$. Since T is integral over R, $t^n + r_{n-1}t^{n-1} + ... + r_1 t + r_0 = 0$ for some positive integer n and $r_{n-1}, ..., r_0 \in R$. Thus $r_0 x \in JS \subseteq I$ and so $r_0 \in I$. Then $t(t^{n-1} + ... + r_1) = -r_0 \in J$ and so $t^{n-1} + ... + r_1 \in J$ because J is a prime ideal of T. Since $tx \in JS \subseteq I$, $r_1 x \in I$ and so $r_1 \in I$. We repeat the sketch: we will have each $r_i \in I$. Thus $t^n \in J$ and so $t \in J$, a contradiction. Therefore $\Omega_S(J) \subseteq I$.

Let $\mathcal{P}$ be a minimal prime ideal (of S) over I. By Lemma 2.2-(7), $\mathcal{P} \cap R = P$. Since $J \subseteq \mathcal{P} \cap T$ are prime ideals of T and each of them lies over P in R, $J = \mathcal{P} \cap T$ because $R \subset T$ is an INC-extension by Theorem 2.4. Since $T \subset S$ is a PSI-extension and $\Omega_S(J) \neq S$, $\Omega_S(J)$ is the unique prime ideal of S that lies over J in T by Theorem 2.4. Then $\Omega_S(J) = \mathcal{P}$. Since $\Omega_S(J) \subseteq I \subseteq \mathcal{P}$, $I = \mathcal{P}$ is a prime ideal of S. $\square$

The converse is false as the following shows:

Examples 2.10. 1. Let $\mathbb{Z}$ be the ring of integers, $\mathbb{Z}[i]$ the ring of Gaussian integers and $\mathbb{C}$ the field of complex numbers. Since zero is the only proper ideal of $\mathbb{C}$, $\mathbb{Z} \subset \mathbb{C}$ and $\mathbb{Z}[i] \subset \mathbb{C}$ are PSI-extensions. The ring $\mathbb{Z}[i]$ is integral over $\mathbb{Z}$ but the extension $\mathbb{Z} \subset \mathbb{Z}[i]$ is not PSI. Indeed: easily one can show that $\Omega_{\mathbb{Z}[i]}(2\mathbb{Z}) = 2\mathbb{Z}[i]$ which is a proper ideal of $\mathbb{Z}[i]$ but is not prime because: $(1+i)(1-i) = 2 \in 2\mathbb{Z}[i]$ but $1 \pm i \notin 2\mathbb{Z}[i]$. Then $\mathbb{Z} \subset \mathbb{Z}[i]$ is not PSI by Theorem 2.4.

2. Let $K \subset L$ be a proper field extension with finite degree (for instance, take $\mathbb{R}$ and $\mathbb{C}$) and let X, Y be two indeterminates over L. Take $R = K[X] + YL(X)[Y]_{(Y)}$, $T = L[X] + YL(X)[Y]_{(Y)}$ and $S = L(X)[Y]_{(Y)}$. Thus $R \subset S$ and $T \subset S$ are PSI-extensions (see Proposition 2.13) and it is clear that T is integral over R. By Lemma 3.1, $K[X] \subset L[X]$ is not PSI and nor is $R \subset T$ by Lemma 2.2-(5).

We say that $R \subset S$ is residually algebraic if for every prime ideal Q of S, S/Q is algebraic over $R/(Q \cap R)$ [3, page 213]. We say that (R, S) is an

INC-pair (respectively, residually algebraic pair) if for each ring $T \in [R, S]$, $R \subset T$ is an INC-extension (respectively, residually algebraic). We say that (R, S) is a normal pair if each ring $T \in [R, S]$ is integrally closed in S.

Corollary 2.11. *Suppose that R is integrally closed in S. The following statements are equivalent:*

1. *(R, S) is a PSI-pair.*
2. *$R \subset R[s]$ is a PSI-extension for all $s \in S$.*
3. *(R, S) is an INC-pair.*
4. *(R, S) is a residually algebraic pair.*
5. *(R, S) is a normal pair.*
6. *The prime ideals of each $T \in [R, S]$ are extensions of prime ideals of R.*

If moreover S is the quotient field of R, then each of (1)-(6) is equivalent to:

7. *R is a Prüfer domain.*

Proof. 1) $\Rightarrow$ 2) Trivial.

2) $\Rightarrow$ 3) By Theorem 2.4 and [1, Theorem 2.3].

3) $\Rightarrow$ 4) $\Rightarrow$ 5) $\Rightarrow$ 6) By [1, Theorem 2.3 and Theorem 2.10].

6) $\Rightarrow$ 1) Let $T \in [R, S]$ and I be an R-prime ideal of T. Let Q be a minimal prime ideal of T over I and $P = Q \cap R$. Thus $I \cap R = P$ and so $I = PT = Q$ is prime.

7) $\Leftrightarrow$ 1) By [1, Corollary 2.8]. □

Remarks 2.12. 1. $R \subset S$ is a PSI-pair if and only if $R \subset R[s, s']$ is a PSI-extension for all $s, s' \in S$. Indeed: let $T \in [R, S]$, I be an R-prime ideal of T that is not prime and $s, s' \in T - I$ such that $ss' \in I$. Then $I \cap R[s, s']$ is an R-prime ideal of $R[s, s']$ that is not prime.

2. There exists an intermediate ring T between R and S such that T is maximal (under inclusion) with respect to the following property: (R, T) is a PSI-pair. Indeed: let $\mathcal{F} = \{T \in [R, S]$ such that (R, T) is a PSI-pair$\}$. Then $R \in \mathcal{F}$. Let $(T_\lambda)_{\lambda \in \Lambda}$ be a chain of elements of $\mathcal{F}$ and $T = \cup_{\lambda \in \Lambda} T_\lambda$. For all $s, s' \in T$, $s, s' \in T_\lambda$ for some $\lambda \in \Lambda$. Thus (R, T) is a PSI-pair by (1) and so $T \in \mathcal{F}$. Hence, we may apply Zorn Lemma.

Proposition 2.13. *Let S be a quasi-local domain with maximal ideal M, residue field K, $\phi : S \longrightarrow K$ the natural surjection, and $R = \phi^{-1}(D)$ where D is a subring of K. Then:*

1. *$R \subset S$ is a PSI-extension.*
2. *(R, S) is a PSI-pair if and only if (D, K) is a PSI-pair.*

Proof. 1) If I is an R-prime ideal of S, then $I \subseteq M$ and so I is an R-prime ideal of R. Let $a, b \in S$ such that $ab \in I$ and $a \notin I$. If $a \in R$, then $b \in I$ because I is an R-prime ideal of S. If $a \notin R$, then $a \notin M$ (because $M \subset R$) and so a is an invertible element of S. Thus $b = a^{-1}ab \in I$.

2) Assume that (R, S) is a PSI-pair. Let $A \in [D, K]$ and $T = \phi^{-1}(A) \in [R, S]$. If J is a D-prime ideal of A, then $J = I/M$ for some R-prime ideal I of T. By assumption, I is a prime ideal of T and hence J is a prime ideal of A. For the converse, let $T \in [R, S]$ and I be an R-prime ideal of T. Suppose that $M \nsubseteq I$ and let $r \in M - I$. If $t, t' \in T$ such that $tt' \in I$ and $t' \notin I$, then $rtt' \in I$ and so $rt \in I$ (because $rt \in M \subset R$). Thus $t \in I$. Suppose that $M \subseteq I$. Then I/M is an R/M-prime ideal of T/M and so I is a prime ideal of T. $\square$

Example 2.14. If D is a Prüfer domain with quotient field K and X an indeterminate over K, then $(D+XK[[X]], K[[X]])$ and $(D+XK[X]_{(X)}, K[X]_{(X)})$ are PSI-pairs.

3. POLYNOMIAL/POWER SERIES PSI-EXTENSION

LEMMA 3.1. *Suppose that R is a field. Then $R[X] \subset S[X]$ is a PSI-extension if and only if $R = S$.*

Proof. Let $a \in S$. By Theorem 2.4 and [3, Lemma 6.6.1], S is algebraic over R and so S is a field. Thus $(X - a)S[X] \cap R[X] = \pi R[X]$ for some monic irreducible element π of $R[X]$. Since $\pi R[X]$ is a maximal ideal of $R[X], \pi S[X]$ is a maximal ideal of $S[X]$ by Corollary 2.5. Thus $\pi S[X] = (X - a)S[X]$ and so $\pi = X - a$. Hence $a \in R$. $\square$

If A is a ring and P is a prime ideal of A, then set $\mathbb{K}_A(P)$ to be the quotient field of A/P. Let α be a monic irreducible element of $\mathbb{K}_A(P)[X]$. Following McAdam [8], we will write $< P, \alpha >$ to denote the set $\{h \in A[X]$ such that α divides $\overline{h}\}$ where $\overline{h}$ is the result of reducing h modulo P.

THEOREM 3.2. *The following statements are equivalent:*

1. *$R[X] \subset S[X]$ is a PSI-extension.*
2. *$R \subset S$ is a PSI-extension and $\mathbb{K}_R(P) = \mathbb{K}_S(P')$ for all $P' \in Spec(S)$ and $P = P' \cap R$.*
3. *For any prime ideal P of R such that $\Omega_S(P)$ survives in S, $S/\Omega_S(P)$ is an overring of R/P.*

Proof. 1) ⇒ 2) The first one follows from the fact that for any R-prime ideal I of S, $I[X]$ is a $R[X]$-prime ideal of $S[X]$. The second one follows from Lemma 3.1 and the fact that $R_N \subset S_N$ and $R/(J \cap R) \subset S/J$ are PSI-extensions for each multiplicative set N of R and each prime ideal J of S.

2) ⇒ 3) By Theorem 2.4.

3) ⇒ 1) We will apply Theorem 2.4-(3). Let Q be a nonzero prime ideal of $R[X]$ such that $\Omega_{S[X]}(Q) \neq S[X]$. Let $P = Q \cap R$. Since $\Omega_S(P) \subseteq \Omega_{S[X]}(Q)$, $\Omega_S(P) \neq S$. Thus $S/\Omega_S(P)$ is an overring of R/P by assumption (so, in particular, $\Omega_S(P)$ is a prime ideal of S because $S/\Omega_S(P)$ is a domain).

Claim: There exists a unique prime ideal Q' of $S[X]$ that lies over Q in $R[X]$. Indeed: Denote $A = R/P$, $B = S/\Omega_S(P)$, $\overline{Q} = Q/P[X]$ which a prime ideal of $R[X]/P[X] \cong (R/P)[X] = A[X]$. Since $\overline{Q} \cap A = 0$, there exists a unique prime ideal of $B[X]$ that lies over $\overline{Q}$ by [7, Lemma 4] (the case $\overline{Q} = 0$ is trivial because B is an overring of (so algebraic over) A). Thus there exists a unique prime ideal Q' of $S[X]$ that lies over Q in $R[X]$.

Then $\Omega_{S[X]}(Q) \subseteq Q'$ and so it suffices to show that $\Omega_{S[X]}(Q) = Q'$. Set $P' = Q' \cap S$. Since $\Omega_S(P) \subseteq P'$ and $S/\Omega_S(P)$ is algebraic over R/P, $P' = \Omega_S(P)$ because $P' \cap R = P$. Since $\Omega_{S[X]}(P[X]) = \Omega_S(P)[X]$, we can assume that $Q =< P, f >$ for some monic irreducible element f in $\mathbb{K}_R(P)$ by [8, Theorem 1]. Thus $Q' =< P', g >$ for some monic irreducible element g in $\mathbb{K}_S(P') = \mathbb{K}_R(P)$ and g divides f by [8, Theorem 2]. So $g = f$. Thus $\overline{Q'} = Q'/P'[X] \subseteq \overline{Q}_{A-(0)}$ and $Q' \subseteq Q_{R-P}$. Hence $Q' \subseteq \Omega_{S[X]}(Q)$. □

Corollary 3.3. *If R is integrally closed in S such that $(R[X], S[X])$ is a PSI-pair, then $R = S$.*

Proof. S is integral over R by [2, Theorem 2.5]. □

Corollary 3.4. *Let R be an integrally closed domain with quotient field K. The following statements are equivalent:*

1. *$R[X] \subset T[X]$ is a PSI-extension for each overring T of R.*
2. *R is a Prüfer domain.*

Proof. 1) ⇒ 2) Let I be an R-prime ideal of T. Thus $I[X]$ is a $R[X]$-prime ideal of $T[X]$ and so it is prime. Then I is a prime ideal of T. Hence (R, K) is a PSI-pair and so R is a Prüfer domain.

2) ⇒ 1) Let T be an overring of R. By Corollary 2.11, $R \subset T$ is a PSI-extension. By Theorem 3.2, it suffices to show that $\mathbb{K}_R(P) = \mathbb{K}_T(PT)$ for all prime ideal P of R such that $PT \neq T$. But this follows from the fact that $I = (I \cap R)T$ for each ideal I of T (also from the fact that $R_P = T_{PT}$ for each prime ideal P of R such that $PT \neq T$ and the fact that $\mathbb{K}_R(P) \cong (R_P)/PR_P$). □

Theorem 3.5. *Let R be a quasi-local integrally closed domain with quotient field K. The following statements are equivalent:*

1. $R[[X]] \subset T[[X]]$ *is a PSI-extension for each overring T of R.*
2. R *is a valuation domain.*

Proof. 1) $\Rightarrow$ 2) Let T be an overring of R, I be an R-prime ideal of T and $P = I \cap R$. Thus $I[[X]]$ is a $R[[X]]$-prime ideal of $T[[X]]$. Then I is a prime ideal of T. Hence (R, K) is a PSI-pair and so R is a valuation domain.

2) $\Rightarrow$ 1) Let Q be a $R[[X]]$-prime ideal of $T[[X]]$. If $X \in Q$, then $Q = (Q \cap T) + XT[[X]]$ is a prime ideal of $T[[X]]$. Let P be the maximal ideal of T. Suppose that $Q \subseteq P[[X]]$ and let $f, g \in T[[X]]$ such that $fg \in Q$. Since f or $g \in P[[X]] \subset R[[X]]$, f or $g \in Q$ and so Q is a prime ideal of $T[[X]]$. Assume that $X \notin Q$ and $Q \nsubseteq P[[X]]$.

Claim: $P \nsubseteq Q$. If not: $P \subseteq Q$. Let $\mathcal{P}$ be a minimal prime ideal of $T[[X]]$ over Q. Thus $P \subseteq \mathcal{P} \nsubseteq P[[X]]$ and so $X \in \mathcal{P}$ (in fact $\mathcal{P} = P + XT[[X]]$ which is the maximal ideal of $T[[X]]$). Then $X^n u \in Q$ for some positive integer n and some unit u of $T[[X]]$. Thus $X^n \in Q$ and so $X \in Q$, a contradiction.

Let $a \in P - Q$. Note that $aT[[X]] \subseteq P[[X]] \subset R[[X]]$. Let $f, g \in T[[X]]$ such that $fg \in Q$ and $g \notin Q$. Since $af \in R[[X]]$ and $afg \in Q$, $af \in Q$ and so $f \in Q$ (because $a \in R$). Hence, Q is a prime ideal of $T[[X]]$. □

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